A Note on Futures-Tradable Processes

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1 Introduction

The purpose of this note is to provide formal definitions of a *tradable process* and *futures-tradable process*. The reasons for doing so may initially not be clear: most of us already have some idea of what tradable processes should represent and how to use them. For instance, they could be the price process X of a non-dividend paying stock; or the process B describing the value of a discount factor with a given maturity. If we apply the strategy θ in X using B as numeraire, we then believe the associated wealth process π follows the SDE:[1]

$$d\pi_t = \theta_t dX_t + \frac{1}{B_t}(\pi_t - \theta_t X_t)dB_t$$

Similarly, we believe that a futures-tradable process F simply represents the value of a futures contract over time. If we apply the strategy θ in F using B as numeraire, the associated wealth process is now given by the SDE:[2]

$$d\pi_t = \theta_t dF_t + \frac{\pi_t}{B_t}dB_t$$

So why should there be a need to have anything more formal?

The answer lies with *Algorithmic Trading*: more specifically the business of offering over-the-counter (OTC) derivatives transactions on newly created indices, which mimic the performance of trading algorithms themselves designed to achieve certain investment objectives. There are typically two types of indices: a *total return index* can be used as a fund for a customer to invest in, who will then benefit from the index performance. Alternatively, rather than commit any cash, a customer may just pay the funding cost associated with an investment in exchange for the return on the index (*total return swap*). By contrast, an *excess return index* requires no cash investment or funding payments. A customer may simply decide to periodically receive the return on the index

[1] The solution of this SDE is $\pi_t = B_t(\pi_0/B_0 + \int_0^t \theta_s d(X/B)_s)$. See [2] Append. A.1.

[2] $\pi_t = B_t(\pi_0/B_0 + \int_0^t \hat{\theta}_s d\hat{F}_s)$, $\hat{\theta} = \theta e^{[B,F]}/B$ and $\hat{F} = Fe^{-[B,F]}$. See [2] Append. A.2.

(*excess return swap*), more precisely to receive the return when positive, and pay when negative.

Algorithmic trading brings up a new requirement: since we are offering standard OTC transactions which are based on indices, it is important to make sure these transactions are correctly priced. This may depend on the nature of the index: for example, it would be a mistake to offer an excess return swap on an index which is not an excess return index, as one of the counterparties (most likely the bank marketing the index) would incur funding and convexity costs when replicating the index. Therefore, when planning to offer an excess return swap on a new index, it is important to make sure the index in question is indeed an excess return index. But how can we do this? How can we formally prove that a newly created index is an excess return index? There cannot be a formal proof without a formal definition.

In many cases, formal proofs are not required: things are sufficiently straightforward and clear for intuition and common sense to support us. However there have been situations when experienced and esteemed colleagues of mine (not to mention myself of course) would get confused over seemingly simple points. For example, if we are to trade a USD futures contract for the purpose of creating an EUR index, how and when should the profits be converted into EUR? Should the funding gains arising from past profits be included in the calculation of an excess return index? I personally do not think these questions are obvious, and I certainly cannot answer them without a formal analysis.

With a little bit of practice, total return indices are easily seen to behave like *synthetically tradable* processes. They are not directly tradable, but everything works as if they were. In this note, we shall formally define a *tradable process* with the belief that the notion equally represents the price process of a tradable asset, an asset which is synthetically tradable or a total return index. This view is not likely to be controversial. What is less obvious is the idea that an excess return index should behave in exactly the same way as the price process of a futures contract.[3] In this note, we shall formally define a *futures-tradable process* being understood that the notion also refers to an excess return index.

As we shall see, a futures-tradable process F is defined as a process for which there exists a tradable process B such that $BFe^{-[B,F]}$ is also tradable.[4] What is interesting about this definition (of course it has the right property of defining a process which *behaves* like a futures contract), is the fact that the tradable process B actually plays no particular role: if $BFe^{-[B,F]}$ is tradable for some B, then it is tradable for *any* tradable process B.[5] This remarkable and aesthetically pleasing fact has convinced me (at least) that definition (2) or anything equivalent, is the right formal definition of a futures-tradable process.

[3] See lemma (1).

[4] See definition (2).

[5] See proposition (4).

2 Excess Return Index vs Futures Contract

The belief that a futures-tradable process and an excess return index actually refer to the same notion is motivated by the following lemma:

Lemma 1 *Consider a financial model in which F denotes the price process of a futures contract, V denotes the price process of the discount factor maturing at some time t_2. We assume that the financial model is such that the bracket $[V, F]$ is a deterministic process.*[6] *If $0 < t_1 < t_2$ then the present value of the contingent claim paying the return $F_{t_2}/F_{t_1} - 1$ at time t_2 is:*

$$\pi_0 = V_0(e^{[V,F]_{t_2} - [V,F]_{t_1}} - 1)$$

Proof

Before we start, we see from this lemma that if the time interval between t_1 and t_2 is sufficiently small (so we can ignore the convexity correction), the present value of an excess return swap based on a futures contract is effectively null. So a futures contract *behaves* like an excess return index. This cannot be a coincidence: an excess return index must be *synthetically futures-tradable*, just like a total return index is *synthetically tradable*.

We now proceed with the proof: let π_0 be the present value of the claim and consider a strategy θ in F starting from π_0, taking V as numeraire. The associated wealth process π satisfies the SDE:

$$d\pi_t = \theta_t dF_t + \frac{\pi_t}{V_t} dV_t$$

From which we obtain the terminal wealth at time t_2 (since $V_{t_2} = 1$):

$$\pi_{t_2} = \frac{\pi_0}{V_0} + \int_0^{t_2} \hat{\theta}_s d\hat{F}_s$$

where $\hat{\theta} = \theta e^{[V,F]}/V$ and $\hat{F} = Fe^{-[V,F]}$. So the replicating condition is:

$$\frac{\pi_0}{V_0} + \int_0^{t_2} \hat{\theta}_s d\hat{F}_s = \frac{F_{t_2}}{F_{t_1}} - 1 \tag{1}$$

Since $[V, F]$ is a deterministic process, the payoff $F_{t_2}/F_{t_1} - 1$ of the claim is easily seen to be a *function of the history* of $\hat{F}$[7] and from the martingale representation theorem[8], the contingent claim is effectively replicable (i.e. there exist π_0 and

[6] [V,F] is the cross variation $\langle \log(V), \log(F) \rangle$ between $\log(V)$ and $\log(F)$. In fact other assumptions are needed as we need to apply the martingale representation theorem in the context of a Brownian setting.

[7] Less informally, the measurability condition required to apply the martingale representation theorem stands a good chance to being met.

[8] See [1] th. 4.15 p. 182 for a precise statement. The fact that this theorem can be applied would need to be justified based on the characteristics of the financial model. This lemma is rather informal and should not be believed too literally.

θ such that the replicating equation (1) holds). If Q is a probability measure under which $\hat{F}$ is a martingale[9], taking Q-expectations on both sides of (1):

$$\pi_0 = V_0 \left(E_Q \left[\frac{F_{t_2}}{F_{t_1}} \right] - 1 \right)$$

Now since $[V, F]$ is deterministic and $\hat{F} = Fe^{-[V,F]}$ is a Q-martingale, we have:[10]

$$\begin{aligned}
E_Q \left[\frac{F_{t_2}}{F_{t_1}} \right] &= E_Q \left[E_Q \left[\frac{F_{t_2}}{F_{t_1}} \middle| \mathcal{F}_{t_1} \right] \right] \\
&= E_Q \left[(1/F_{t_1}) E_Q[F_{t_2} | \mathcal{F}_{t_1}] \right] \\
&= E_Q \left[(1/F_{t_1}) e^{[V,F]_{t_2}} E_Q[\hat{F}_{t_2} | \mathcal{F}_{t_1}] \right] \\
&= E_Q \left[(1/F_{t_1}) e^{[V,F]_{t_2}} \hat{F}_{t_1} \right] \\
&= E_Q \left[e^{[V,F]_{t_2} - [V,F]_{t_1}} \right] \\
&= e^{[V,F]_{t_2} - [V,F]_{t_1}}
\end{aligned}$$

We conclude that $\pi_0 = V_0(e^{[V,F]_{t_2} - [V,F]_{t_1}} - 1)$. **QED**

3 Tradable Process

In the context of a financial model, we assume given some probability setting together with a set $\mathcal{S}$ of positive semi-martingales.[11] Each element of $\mathcal{S}$ represents the price process of some asset which is postulated as being tradable. The following definition formally describes the notion of a *synthetically tradable process*. Loosely speaking, a synthetically tradable process is simply the wealth process (provided it is positive) of a strategy involving tradable assets. The notion of *synthetically tradable* is relative to the set $\mathcal{S}$. Rather than use the phrase *$\mathcal{S}$-synthetically tradable process* we shall keep the simpler *tradable process*, while remembering that the notion is meaningful only in relation to a set $\mathcal{S}$.

Definition 1 *A positive semi-martingale X is said to be a* tradable process *if and only if it is of the form:*

$$X_t = B_t \left(\frac{X_0}{B_0} + \sum_{i=1}^{n} \int_0^t \theta_s^i d(X^i/B)_s \right) \tag{2}$$

for some $B \in \mathcal{S}$, $X^1, \ldots, X^n \in \mathcal{S}$ and integer $n \geq 0$, and where $\theta^1, \ldots, \theta^n$ are some admissible processes.[12]

[9] The fact that such measure exists would also need to be justified using Girsanov theorem.
[10] We denote $(\mathcal{F}_t)$ the filtration underlying the model. $\mathcal{F}_{t_1}$ is the σ-algebra for time t_1.
[11] All semi-martingales are assumed to be continuous in this note.
[12] i.e. processes for which the stochastic integrals involved are meaningful.

Whenever $n = 0$ in (2) it is understood that X reduces to $X_t = (X_0/B_0)B_t$. In particular, any process proportional to any element of $\mathcal{S}$ is a tradable process.

Proposition 1 *A positive semi-martingale X is a tradable process, if and only if it satisfies the stochastic differential equation:*

$$dX_t = \sum_{i=1}^{n} \theta_t^i dX_t^i \tag{3}$$

together with $X = \theta^1 X^1 + \ldots + \theta^n X^n$, for some $X^1, \ldots, X^n$ in $\mathcal{S}$, $n \geq 1$, and admissible processes $\theta^1, \ldots, \theta^n$.

Proof
Suppose X is a tradable process. There exist $B \in \mathcal{S}$, $X^1, \ldots, X^n \in \mathcal{S}$ and admissible processes $\theta^1, \ldots, \theta^n$ (where $n \geq 0$) such that equation (2) holds. Using Ito's formula, X is easily seen to satisfy the stochastic differential equation:

$$dX_t = \sum_{i=1}^{n} \theta_t^i dX_t^i + \frac{1}{B_t}(X_t - \theta_t^1 X_t^1 \ldots - \theta_t^n X_t^n) dB_t \tag{4}$$

Setting $X^0 = B$ and $\theta^0 = (X - \theta^1 X^1 \ldots - \theta^n X^n)/X^0$ we obtain the formula $X = \theta^0 X^0 + \ldots + \theta^n X^n$ together with the SDE:

$$dX_t = \sum_{i=0}^{n} \theta_t^i dX_t^i$$

Renumbering our processes from 1 to $n+1$ (rather than 0 to n), we see that X satisfies the conditions of proposition (1) with $n \geq 1$. Conversely, if X is of the form $X = \theta^1 X^1 + \ldots + \theta^n X^n$ for some $n \geq 1$ and satisfies the SDE (3), then setting $B = X^1$ we see that X also satisfies the SDE:

$$dX_t = \sum_{i=2}^{n} \theta_t^i dX_t^i + \frac{1}{B_t}(X_t - \theta_t^2 X_t^2 \ldots - \theta_t^n X_t^n) dB_t$$

the solution of which is:

$$X_t = B_t \left(\frac{X_0}{B_0} + \sum_{i=2}^{n} \int_0^t \theta_s^i d(X^i/B)_s \right)$$

Hence we see that X is a tradable process. **QED**

Proposition 2 *A positive semi-martingale X is a tradable process if and only if it is of the form:*

$$X_t = B_t \left(\frac{X_0}{B_0} + \sum_{i=1}^{n} \int_0^t \theta_s^i d(X^i/B)_s \right) \tag{5}$$

for some tradable processes B, $X^1, \ldots, X^n$, $n \geq 0$, and admissible $\theta^1, \ldots, \theta^n$.

Proof
The statement of proposition (2) looks almost identical to that of definition (1). Note however that we no longer require B or $X^1, \ldots, X^n$ to be elements of $\mathcal{S}$, but simply that they should be tradable processes. Now if X is a tradable process, then it clearly satisfies the property of proposition (2), since every element of $\mathcal{S}$ is a tradable process. So we assume conversely that X is a positive semi-martingale satisfying equation (5) for some tradable processes B and $X^1, \ldots, X^n$. We want to show that X is itself a tradable process. Setting $X^0 = B$, applying Ito's formula and defining θ^0 as in the proof of proposition (1), we see that $X = \theta^0 X^0 + \ldots + \theta^n X^n$ and satisfies the SDE:

$$dX_t = \sum_{i=0}^{n} \theta_t^i dX_t^i$$

Since each X^i is a tradable process, using proposition (1) it can be written as:

$$X^i = \sum_{j=1}^{n_i} \theta^{i,j} X^{i,j}$$

and it satisfies the SDE:

$$dX_t^i = \sum_{j=1}^{n_i} \theta_t^{i,j} dX_t^{i,j}$$

where the $X^{i,j}$ are elements of $\mathcal{S}$ and $n_i \geq 1$. It follows that:

$$X = \sum_{i=0}^{n} \sum_{j=1}^{n_i} \theta^i \theta^{i,j} X^{i,j}$$

and furthermore X satisfies the SDE:

$$dX_t = \sum_{i=0}^{n} \sum_{j=1}^{n_i} \theta_t^i \theta_t^{i,j} dX_t^{i,j}$$

Renumbering all our processes, since all $X^{i,j}$'s are elements of $\mathcal{S}$, we conclude by virtue of proposition (1) that X is a tradable process.[13] **QED**

Before we proceed, we shall need the following:

Lemma 2 *Let U and V be two positive semi-martingales and let $[U, V]$ denote the cross-variation between the local-martingale parts of $log(U)$ and $log(V)$[14]. Then:*

$$U_t V_t e^{-[U,V]_t} = U_0 V_0 + \int_0^t \phi_s dU_s + \int_0^t \psi_s dV_s$$

for some admissible processes ϕ and ψ.

[13] A more formal proof would need to show that all processes involved are admissible.

[14] Equivalently, $[U, V]_t = \int_0^t 1/(U_s V_s) d\langle U, V\rangle_s$ where $\langle U, V\rangle$ is the cross-variation.

Proof
Denoting $A = e^{-[U,V]}$ and applying Ito's formula:

$$\begin{aligned} d(UVe^{-[U,V]}) &= Ad(UV) - AUVd[U,V] \\ &= AUdV + AVdU + Ad\langle U,V\rangle - AUVd[U,V] \\ &= AUdV + AVdU \end{aligned}$$

This concludes the proof, taking $\phi = AV$ and $\psi = AU$. **QED**

Proposition 3 *Let X, Y, Z be tradable processes. Then G defined by:*

$$G_t = Z_t \frac{X_t}{Z_t}\frac{Y_t}{Z_t} \exp\left(-\left[\frac{X}{Z}, \frac{Y}{Z}\right]_t\right)$$

is itself a tradable process.

Proof
This is an immediate consequence of lemma (2). Note first that if X, Y and Z are positive semi-martingales, then so is G. Taking $U = X/Z$ and $V = Y/Z$, we see immediately from lemma (2) that G can be expressed as:

$$\begin{aligned} G_t &= Z_t\left(U_0V_0 + \int_0^t \phi_s dU_s + \int_0^t \psi_s dV_s\right) \\ &= Z_t\left(\frac{G_0}{Z_0} + \int_0^t \phi_s d(X/Z)_s + \int_0^t \psi_s d(Y/Z)_s\right) \end{aligned}$$

Since X, Y and Z are tradable processes, we conclude from proposition (2) that G is itself a tradable process. **QED**

4 Futures-Tradable Process

Having formally defined tradable processes, we are now in a position to deal with futures-tradable processes. The following definition may seem unnatural at first. However, when applying a strategy θ in a futures contract F using a tradable asset B as numeraire, the wealth process is of the form:

$$\pi_t = B_t\left(\frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d(Fe^{-[B,F]})_s\right)$$

And if we were to apply the strategy $\hat{\theta}$ to another tradable asset X while keeping B as numeraire, the wealth process would become:

$$\pi_t = B_t\left(\frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d(X/B)_s\right)$$

Comparing these two equations, it is very tempting to conjecture that $Fe^{-[B,F]}$ should in fact be some form of X/B. In other words, when formally defining a futures-tradable process, it is tempting to require that $X = BFe^{-[B,F]}$ be a tradable process. As we shall see, this definition works like a dream:

Definition 2 *A positive semi-martingale F is said to be a* futures-tradable *process if and only if there exists a tradable process B such that:*

$$BFe^{-[B,F]}$$

is itself a tradable process.

There is however something not quite satisfactory about definition (2), namely the role played by a particular tradable process B. After all, there are many possible choices of numeraire, and if G is another tradable process, we should expect $GFe^{-[G,F]}$ to be equally tradable. Fortunately this is the case, as the following proposition shows:

Proposition 4 *If a positive semi-martingale F is a futures-tradable process, then for any tradable process B, the positive semi-martingale:*

$$BFe^{-[B,F]}$$

is itself a tradable process.

Proof
Let F be a futures-tradable process. From definition (2) there exists a tradable process B such that $BFe^{-[B,F]}$ is itself a tradable process. Let G be an arbitrary tradable process. We need to show that $GFe^{-[G,F]}$ is also tradable. To do so, we apply proposition (3), taking $X = G$, $Y = BFe^{-[B,F]}$, and $Z = B$. Note first that X, Y and Z are tradable processes and it follows from proposition (3) that $Z.(X/Z).(Y/Z).\exp(-[X/Z, Y/Z])$ is a tradable process. However:

$$\begin{aligned}
Z\frac{X}{Z}\frac{Y}{Z}\exp\left(-\left[\frac{X}{Z},\frac{Y}{Z}\right]\right) &= B\left(\frac{G}{B}\right)\left(\frac{BFe^{-[B,F]}}{B}\right)\exp\left(-\left[\frac{G}{B}, Fe^{-[B,F]}\right]\right) \\
&= GFe^{-[B,F]}\exp\left(-\left[\frac{G}{B}, Fe^{-[B,F]}\right]\right) \\
&= GFe^{-[B,F]}\exp\left(-\left[\frac{G}{B}, F\right]\right) \\
&= GFe^{-[B,F]}\exp\left(-[G,F]+[B,F]\right) \\
&= GFe^{-[G,F]}
\end{aligned}$$

So we conclude that $GFe^{-[G,F]}$ is a tradable process.[15] **QED**

[15] Recall that since $[U,V] = \langle \log(U), \log(V)\rangle$ it is a process of finite variations, and furthermore $[U,V] = 0$ whenever U or V is of finite variations, and we have the property $[U_1U_2, V] = [U_1, V] + [U_2, V]$ and $[1/U, V] = -[U, V]$.

So far so good. However, we have not yet justified the fact that definition (2) is the *right* definition. In other words, it remains to check that a futures-tradable process as defined by definition (2) actually behaves like the price process of a futures contract. This is done via the following:

Proposition 5 *Let B be a tradable process, $F^1, \ldots, F^n$, $n \geq 0$, be futures-tradable processes and $\theta^1, \ldots, \theta^n$ be admissible processes such that the semi-martingale π defined by (setting $\hat{\theta}^i = \theta^i e^{[B,F^i]}/B$):*

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \sum_{i=1}^{n} \int_0^t \hat{\theta}^i_s d(F^i e^{-[B,F_i]})_s \right) \tag{6}$$

is positive, given an arbitrary initial value π_0. Then π is a tradable process.

Proof
Define $X^i = BF^i e^{-[B,F^i]}$. Then each X^i is tradable, B is tradable and π is a positive semi-martingale such that:

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \sum_{i=1}^{n} \int_0^t \hat{\theta}^i_s d(X^i/B)_s \right)$$

We conclude from proposition (2) that π is tradable. **QED**

The relevance of proposition (5) in comforting our choice of definition (2) can be explained as follows: before we formally define a *futures-tradable process*, we have the notion of *tradable process* which precedes it. However, regardless of the definition we wish do adopt, if we were to engage in a trading strategy $\theta^1, \ldots, \theta^n$ in futures contracts $F^1, \ldots, F^n$ taking some B as numeraire, we believe the associated wealth process π should follow the SDE:

$$d\pi_t = \sum_{i=1}^{n} \theta^i_t dF^i_t + \frac{\pi_t}{B_t} dB_t \tag{7}$$

Now provided this wealth process is a positive semi-martingale, the least we should expect from it, is that it is a synthetically tradable process. Otherwise, our definitions of *tradable* and *futures-tradable* would not make sense. This is exactly what proposition (5) guarantees: since equation (6) is the solution of the SDE (7), proposition (5) ensures that if we are to trade futures, the associated wealth process is a tradable process, provided it is positive.

The following proposition has a very simple interpretation. When trading futures contracts, the exact nature of the algorithm being used is irrelevant: the cumulative sum of the daily profit/loss contributions will always give rise to a futures-tradable process (provided it is positive). Note however that these contributions are not funded. There is no mention of a numeraire in which past trading gains would be re-invested. The conclusion of proposition (6) is therefore that no funding calculation should be incorporated in the definition of an excess return index.

Proposition 6 *Let $F^1, \ldots, F^n$, be futures-tradable processes and $\theta^1, \ldots, \theta^n$, $n \geq 0$, be admissible processes such that the semi-martingale F defined by:*

$$F_t = F_0 + \sum_{i=1}^{n} \int_0^t \theta_s^i dF_s^i \tag{8}$$

is positive, given an arbitrary initial value F_0. Then F is also futures-tradable.

Proof
Let B be an arbitrary tradable process. We need to show that $G = BFe^{-[B,F]}$ is also tradable. By assumption, each F^i is futures-tradable. It follows that each $G^i = BF^i e^{-[B,F^i]}$ is a tradable process. Now using equation (8) we obtain:

$$d\langle B, F\rangle = \sum_{i=1}^{n} \theta^i d\langle B, F^i\rangle$$

and consequently:

$$d[B,F] = \sum_{i=1}^{n} \frac{\theta^i F^i}{F} d[B, F^i] \tag{9}$$

Denoting $A = e^{-[B,F]}$ and using Ito's formula we obtain:

$$\begin{aligned}
d(Fe^{-[B,F]}) &= AdF - AFd[B,F] \\
&= \sum_{i=1}^{n} A\theta^i dF^i - \sum_{i=1}^{n} AF\frac{\theta^i F^i}{F} d[B,F^i] \\
&= \sum_{i=1}^{n} A\theta^i e^{[B,F^i]} \left(e^{-[B,F^i]}dF^i - e^{-[B,F^i]}F^i d[B,F^i]\right) \\
&= \sum_{i=1}^{n} A\theta^i e^{[B,F^i]} d(F^i e^{-[B,F^i]}) \\
&= \sum_{i=1}^{n} \psi^i d(G^i/B)
\end{aligned}$$

where we have put $\psi^i = A\theta^i e^{[B,F^i]}$. It follows that:

$$F_t e^{-[B,F]_t} = F_0 + \sum_{i=1}^{n} \int_0^t \psi_s^i d(G^i/B)_s$$

and multiplying both sides by B_t we obtain:

$$G_t = B_t \left(\frac{G_0}{B_0} + \sum_{i=1}^{n} \int_0^t \psi_s^i d(G^i/B)_s \right)$$

Since B and every G^i is tradable, we conclude from proposition (2) that G is itself a tradable process. **QED**

5 Foreign-Tradable Process

In this section, we investigate the relationship between tradable processes and futures-tradable processes in one currency, and those in another. Further to the set $\mathcal{S}$ which represents a given set of processes which are *directly* tradable in the first currency, we assume given a positive semi-martingale X which represents the foreign exchange rate between the two currencies. We then define a new set $\mathcal{S}^* = \{XG : G \in \mathcal{S}\}$ of positive semi-martingales, representing the processes which are directly tradable in the second currency. This definition of $\mathcal{S}^*$ is fairly natural: suppose our first currency is EUR and G is an element of $\mathcal{S}$, i.e. the price process of a directly tradable asset in EUR; if our second currency is USD then X is the EUR/USD exchange rate (i.e. the price of one EUR expressed in USD), and it is possible to buy the asset G with an amount in USD of XG. Likewise, it is possible to sell back this asset in USD at a price of XG. So XG is arguably the price process of a tradable asset in USD.

Given a positive semi-martingale G, we shall say that G is a tradable process if it is tradable in relation to $\mathcal{S}$. If it is tradable in relation to $\mathcal{S}^*$ we shall say that G is $*$-tradable. Likewise, we shall use the phrase $*$-*futures-tradable* when a positive semi-martingale is futures-tradable in relation to $\mathcal{S}^*$. The following proposition confirms that what is true for *directly* tradable processes by definition of $\mathcal{S}^*$, is in fact true for all (*synthetically*) tradable processes:

Proposition 7 *Given a positive semi-martingale G, we have the equivalence:*

$$G \text{ is tradable } \Leftrightarrow \ XG \text{ is } *\text{-tradable}$$

Proof
Since $\mathcal{S} = \{(1/X)G^* : G^* \in \mathcal{S}^*\}$, it is sufficient to prove the implication $\Rightarrow$. So let G be an arbitrary tradable process. We need to show that XG is a $*$-tradable process. Using proposition (1), there exist $X^1, \ldots, X^n \in \mathcal{S}$ and admissible processes $\theta^1, \ldots, \theta^n$, $n \geq 1$, such that $G = \theta^1 X^1 + \ldots + \theta^n X^n$, and furthermore G satisfies the SDE:

$$dG_t = \sum_{i=1}^{n} \theta_t^i dX_t^i \tag{10}$$

It follows in particular that $XG = \theta^1(XX^1) + \ldots + \theta^n(XX^n)$ and:

$$d\langle X, G\rangle = \sum_{i=1}^{n} \theta^i d\langle X, X^i\rangle$$

Hence, using Ito's formula, we obtain:

$$\begin{aligned} d(XG) &= XdG + GdX + d\langle X, G\rangle \\ &= \sum_{i=1}^{n} \theta^i X dX^i + \sum_{i=1}^{n} \theta^i X^i dX + \sum_{i=1}^{n} \theta^i d\langle X, X^i\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \theta^i (X dX^i + X^i dX + d\langle X, X^i \rangle) \\
&= \sum_{i=1}^{n} \theta^i d(X X^i)
\end{aligned}
$$

Since all the XX^i are elements of $\mathcal{S}^*$ we conclude from proposition (1) that XG is a $*$-tradable process. **QED**

Proposition 8 *Given a positive semi-martingale F, we have the equivalence:*

$$F \text{ is futures-tradable } \Leftrightarrow \; Fe^{[X,F]} \text{ is } *\text{-futures-tradable}$$

Proof
As in the tradable case, since furthermore we have $[1/X, Fe^{[X,F]}] = -[X,F]$, it is sufficient to prove the implication $\Rightarrow$. So we assume that F is a futures-tradable process and we shall prove that $F^* = Fe^{[X,F]}$ is $*$-futures-tradable. Now there exists a tradable process B such that $BFe^{-[B,F]}$ is also tradable. From proposition (7) it follows that $XBFe^{-[B,F]}$ is $*$-tradable, that is to say $XBF^*e^{-[XB,F^*]}$ is $*$-tradable. Since XB is itself a $*$-tradable process, we conclude that F^* is a $*$-futures-tradable process. **QED**

This last proposition has a surprising interpretation: suppose F is an EUR futures contract and X is EUR/USD. Then $Fe^{[X,F]}$ is a USD futures-tradable process and if we define $A = e^{[X,F]}$, then from proposition (6) the index:

$$I_t = I_0 + \int_0^t A_s^{-1} d(Fe^{[X,F]})_s$$

is an excess return index (provided it is positive). If we heuristically approximate the cross-variation increment $d\langle X, F\rangle$ by the product $dXdF$, we obtain:

$$
\begin{aligned}
I_t &= I_0 + \int_0^t A_s^{-1}(A_s dF_s + A_s F_s d[X,F]_s) \\
&= I_0 + \int_0^t dF_s \left(\frac{X_{s+ds}}{X_s} \right)
\end{aligned}
$$

So every EUR gain dF_s needs to be adjusted by the ratio X_{s+ds}/X_s, if one is to create an excess return index in USD. To those who (like me) were ever baffled by this formula, I hope this note will be helpful.

References

[1] Karatzas, I. and Shreve, S.E (1991). Brownian motion and stochastic calculus. 2nd Ed. Springer-Verlag

[2] N. Vaillant (2001). A Beginner's Guide to Credit Derivatives. Nomura, Working Paper.