A Beginner's Guide to Credit Derivatives*

Noel Vaillant
Debt Market Exotics
Nomura International

November 17, 2001

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*I am greatly indebted to my colleagues Evan Jones and Kevin Sinclair for their valuable comments and recommendations.

1 Introduction

This document will attempt to describe how simple credit derivatives can be formally represented, shown to be replicable and ultimately priced, using reasonable assumptions. It is a *beginner's guide* on more than one count: its subject matter is limited to the most simple types of claims (those involved in credit default swaps, plus a few more) and its treatment so detailed that most beginners should be able to follow it. Basic definitions of general option pricing are also included to establish a common and consistent terminology, and to avoid any possible misunderstanding. It is also a *beginner's guide* in the sense that I am myself a complete beginner on the subject of credit. I have no trading experience of credit default swaps, and my modeling background is limited to that of the default-free world.

When I became acquainted with the concept of credit default swap (**CDS's**), and was told about their rising importance and liquidity, I was struck by the obvious parallel that could be drawn between interest rate swaps (**IRS's**) with their building blocks (the **default-free zeros**), and CDS's with their own fundamental components (the **risky zeros**). In the early 1980's, the emergence of IRS's and the realization that these could be **replicated** with almost static[1] **trading strategies** in terms of default-free zeros, rendered the whole exercise of **bootstrapping** meaningful. The ultimate simplicity of default-free zeros, added to the fact that their **prices** could now be inferred from the market place, made them the obvious choice as basic **tradable instruments** in the modeling of many interest rate derivatives. Having assumed default-free zeros to be tradable, the whole question of **contingent claim pricing** was reduced to the mathematical problem of establishing the existence of a **replicating strategy**: a dynamic trading strategy involving those default-free zeros with an associated **wealth process** having a **terminal value** at **maturity**, matching the **payoff** of the given claim.

In a similar manner, the emergence of CDS's offers the very promising prospect of promoting risky zeros to the high status enjoyed by their counterparts, the default-free zeros. Although the relationship between CDS's and risky zeros will be shown to be far more complex than generally assumed[2], by ignoring the risk on the **recovery rate** and discretising the **default leg** into a finite set of possible payment dates, it is possible to show that a CDS can indeed be replicated in terms of risky zeros[3]. This makes the whole process of bootstrapping the default swap curve a legitimate one, which appears to be taken for granted by most practitioners. My assertion that this process is non-trivial and requires rigor may seem surprising, but in fact the process can only be made trivial by assuming no correlation between **survival probabilities** and interest rates, or indulging in the sort of naive pricing which ignores **convexity adjustments** similar to those encountered in the pricing of Libor-in-Arrears swaps.

[1] The replication of a standard Libor payment involves a borrowing/deposit trade at some time in the future, and is arguably non-static.

[2] The default leg paying $(1 - R)$ at time of default does not seem to be replicable.

[3] Provided survival probabilities have deterministic volatility and correlation with rates.

Although the assumption of zero correlation between survival probabilities and interest rates may have little practical significance, I would personally prefer to avoid such assumption, as the added generality incurs very little cost in terms of tractability, and the ability to measure exposures to correlation inputs is a valuable benefit. As for convexity adjustments, it is well-known that **forward default-free zeros**, **forward Libor rates** or **forward swap rates** should have no drift under the measure associated with their **natural numeraire**. When considered under a different measure, everyone expects these quantities to have drifts, and it should therefore not be a surprise to find similar drifts when dealing with the highly unusual **numeraire** of a risky zero. In some cases, this can be expressed as the following idea: a survival probability with maturity T is a probability for a fixed payment occurring at time T, and should the payment be delayed or the amount being paid be random, the survival probability needs to be convexity adjusted.

Assuming risky zeros to be tradable can always be viewed as a legitimate assumption. However, such assumption is rarely fruitful, unless one has the ability to infer the prices of these tradable instruments from the market. The fact that CDS's can be linked to risky zeros is therefore very significant, and reveals similar opportunities to those encountered in the default-free world. Several **credit contingent claim** can now be assessed from the point of view of **non-arbitrage pricing** and replication. The question of pricing these credit contingent claims is now reduced to that of the existence of replicating trading strategies in terms of risky and default-free zeros.

Although most of the techniques used in a default-free environment can be applied in the context of credit, some new difficulties do appear. The existence of replicating trading strategies fundamentally relies on the so-called **martingale representation theorem**[4] in the context of brownian motions. As soon as new factors of risk which are not explicable in terms of brownian motions (like a random **time of default**), are introduced into one's model, the question of replication may no longer be solved[5]. One way round the problem is to use risky zeros solely as numeraire. However, this raises a new difficulty. A risky zero is a **collapsing numeraire**, in the sense that its price can suddenly collapse to zero, at the random time of default. This document will show how to deal with such difficulties.

[4] See [1], Theorem 4.15 page 182.

[5] Assuming your time of default to be a stopping w.r. to a brownian filtration does not seem to help: there is no measure under which a non-continuous process will ever be a martingale, w.r. to a brownian filtration.

2 Trading Strategies and Replication

2.1 Contingent Claims

A **single claim** or **single contingent claim** is defined as a single arbitrary payment occurring at some date in the future. The date of such payment is called the **maturity** of the single claim, whereas the payment itself is called the **payoff**. By extension, a set of several random payments occurring at several dates in the future , is called a **claim** or **contingent claim**. A contingent claim can therefore be viewed as a portfolio of single contingent claims. The maturity of such claim is sometimes defined as the longest maturity among those of the underlying single claims. In some cases, the payoff of a single claim may depend upon whether a certain reference entity has defaulted prior to the maturity of the single claim. The time when such entity defaults is called the **time of default**. A **single credit contingent claim** is defined as a single claim whose payoff is linked to the time of default. A **credit contingent claim** is nothing but a portfolio of single credit contingent claims. As very often a claim under investigation is in fact a *single* claim, and/or clearly a *credit* claim, it is not unusual to drop the words *single* and/or *credit* and refer to it simply as the *claim*.

Examples of claims are numerous. The **default-free zero** with maturity T is defined as the single claim paying one unit of currency at time T. Its payoff is 1, and maturity T. The **risky zero** with maturity T is defined as the single credit claim paying one unit of currency at time T, provided the time of default is greater than T[6], and zero otherwise. Its payoff is $1_{\{D>T\}}$ and maturity T, where D is the time of default.

Two contingent claims are said to be **equivalent**, if one can be replicated from the other, at no cost. This notion cannot be made precise at this stage, but a few examples will suffice to illustrate the idea. If $T < T'$ are two dates in the future, and V_t denotes the price at time t of the default-free zero with maturity T', then this default-free zero is in fact equivalent to the single claim with maturity T and payoff V_T. This is because receiving V_T at time T allows you to buy the default-free zero with maturity T', and therefore replicate such default-free zero at no cost. More generally, a contingent claim is always equivalent to the single claim with maturity T and payoff equal to the *price* at time T of this claim, provided this claim is replicable (i.e. it is meaningful to speak of its *price*) and no payment has occurred prior to time T. A well-known but less trivial example is that of a standard (default-free) Libor payment between T and T'[7]. This payment is equivalent to a claim, consisting of a long position of the default-free zero with maturity T, and a short position in the default-free zero with maturity T'[8].

[6] Saying that the time of default is greater than T is equivalent to saying that default still hasn't occurred by time T.

[7] Fixing at T and payment at T' of the Libor rate between T and T'.

[8] This is assuming a zero spread between Libor fixings and cash. Relaxing this assumption offers a consistent and elegant way of pricing cross-currency basis swaps.

2.2 Stochastic Processes

A **stochastic process** is defined as a quantity moving with time, in a potentially random way. If X is a stochastic process, and ω is a particular *history of the world*, the **realization** of X in ω at time t is denoted $X_t(\omega)$. It is very common to omit the 'ω' and refer to such realization simply as X_t. A stochastic process X is very often denoted (X_t) or X_t.

When a stochastic process is non-random, i.e. its realizations are the same in all histories of the world, it is said to be **deterministic**. A deterministic process is only a function of time, there is no surprise about it. When a deterministic process has the same realization at all times, it is called a **constant**. A constant is the simplest case of stochastic process.

When a stochastic process is not a function of time, i.e. its realizations are constant with time in all histories of the world, it is called a **random variable** (rather than a process). A random variable is only a function of the history of the world, and doesn't change with time. The payoff of a single claim is a good example of a random variable. If X is a stochastic process, and t a particular point in time, the various realizations that X can have at time t is also a random variable, denoted X_t. Needless to say that the notation X_t can be very confusing, as it potentially refers to three different things: the random variable X_t, the process X itself and the realization $X_t(\omega)$ of X at time t, in a particular history of the world ω.

A stochastic process is said to be **continuous**, when its **trajectories** or **paths** in all histories of the world are continuous functions of time. A continuous stochastic process has no jump.

Among stochastic processes, some play a very important role in financial modeling. These are called **semi-martingales**. The general definition of a semi-martingale is unimportant to us. In practice, most semi-martingales can be expressed like this:

$$dX_t = \mu_t dt + \sigma_t dW_t \tag{1}$$

where W is a **Brownian motion**. The stochastic process μ is called the **absolute drift** of the semi-martingale X. The stochastic process σ is called the **absolute volatility** (or **normal volatility**) of the semi-martingale X. Note that μ and σ need not be deterministic processes. A semi-martingale of type (1) is a continuous semi-martingale. This is the most common case, the only exception being the price process of a risky zero, and the wealth process associated with a trading strategy involving risky zeros.

When X is a continuous semi-martingale, and θ is an arbitrary process[9], the **stochastic integral** of θ with respect to X is also a continuous semi-martingales, and is denoted $\int_0^t \theta_s dX_s$. The stochastic integral is a very important concept. It allows us to *construct* a lot of new semi-martingales, from a simpler semi-martingale X, and arbitrary processes θ. In fact, the proper way

[9] There are normally restrictions on θ which are ignored here.

to write equation (1) should be:

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \tag{2}$$

and X is therefore *constructed* as the sum of its initial value X_0 with two other semi-martingales, themselves *constructed* as stochastic integrals.

To obtain an intuitive understanding of the stochastic integral $\int_0^t \theta_s dX_s$, one may think of the following: suppose X represents the price process of some tradable asset, and θ_s represents some quantity of tradable asset held at time s[10]. Each $\theta_s dX_s$ can be viewed as the P/L arising from the change in price dX_s of the tradable asset over a small period of time. It is helpful to think of the stochastic integral $\int_0^t \theta_s dX_s$ as the sum of all these P/L contributions, between 0 and time t. Of course, the reality is such that various cashflows incurred at various point in time, are normally re-invested as they come along, possibly in other tradable assets. The total P/L arising from trading X between 0 and t may therefore be more complicated than a simple stochastic integral $\int_0^t \theta_s dX_s$.

A semi-martingale of type (1) is called a **martingale** if it has no drift[11], i.e. $\mu = 0$. A well-known example of martingale is that of a brownian motion. Martingales are important for two specific reasons. If X is a martingale, then for all future time t, the expectation of the random variable X_t is nothing but the current value X_0 of X, i.e.

$$E[X_t] = X_0 \tag{3}$$

Another reason for the importance of martingales, is that the stochastic integral $\int_0^t \theta_s dX_s$ is also a continuous martingale, whenever X is a continuous martingale[12]. The stochastic integral is therefore a very good way to *construct* new continuous martingales, from a simpler martingale X, and arbitrary processes θ. Furthermore, applying equation (3) to the stochastic integral $\int_0^t \theta_s dX_s$ (which is a martingale since X is a martingale), we obtain immediately:

$$E\left[\int_0^t \theta_s dX_s\right] = 0 \tag{4}$$

Equations (3) and (4) are pretty much all we need to know about martingales. These equations are very powerful: expectations and/or stochastic integrals can be very tedious to compute. Knowing that a process X is a martingale can make your life a whole lot easier.

[10]A short position at time s corresponds to $\theta_s < 0$.
[11]Not quite true. It may be a local-martingale. The distinction is ignored here.
[12]True if we ignore the distinction between local-martingales and martingales.

2.3 Tradable Instruments and Trading Strategies

A **tradable instrument** is defined as something you can *buy* or *sell*. The price process of a tradable instrument is normally represented by a positive continuous semi-martingale. When X is such semi-martingale, it is customary to say that X is a **tradable process**. A tradable process is not tradable by virtue of some mathematical property: it is postulated as so, within the context of a financial model. If X is a tradable process, it is understood that over a small period of time, an investor holding an amount θ_t of X at time t, will incur a P/L contribution of $\theta_t dX_t$ over that period. It is also understood that an amount of cash equal to $\theta_t X_t$ was necessary for the purchase of the amount θ_t of X at time t[13]. When no cash is required for the purchase of X, we say that X is a **futures-tradable process**. The phrase **cash-tradable process** may be used to emphasize the distinction from futures-tradable process. A futures-tradable process normally represents the price process of a futures contract. In some cases, the purchase of X provides the investor with some dividend yield, or other re-investment benefit. When that happens, the P/L incurred by the investor over a small period of time needs to be adjusted by an additional term, reflecting this benefit. This is the case when X is the price process of a dividend-paying stock, or that of a spot-FX rate. The phrase **dividend-tradable process** may be used to emphasize the distinction from a mere cash-tradable process.

If X is a tradable process, we define a **trading strategy** in X, as any stochastic process θ. In essence, a trading strategy is just a stochastic process with a specific meaning attached to it. When θ is said to be a trading strategy in X, it is understood that θ_t represents an amount of X held at time t[14]. In general, an investor will want to use available market information (like the price X_t of X at time t), before deciding which quantity θ_t of X to buy. The strategy θ is therefore rarely deterministic, as it is randomly influenced by the random moves of the tradable process X. If a trading strategy θ is constant, it is said to be **static**. Otherwise, it is said to be **dynamic**. When several tradable processes X, Y and Z are involved, the term *trading strategy* normally refers to the full collection of individual trading strategies θ, ψ and ϕ in X, Y and Z respectively.

A **numeraire** is just another term for *tradable instrument*. If X and B are two tradable processes, both are equally numeraires. A numeraire is a tradable asset used by an investor to meet his funding requirement: if an investor engages in a trading strategy θ with respect to X, his cash requirement at time t is $\theta_t X_t$. If θ_t is positive, the investor needs to borrow some cash, which cannot be done for free. One way for the investor to meet his **funding requirement** is to contract a short position in another tradable asset B. Such tradable asset is then called a numeraire. If θ_t is negative, the investor has a short position in X, and does not need to borrow any cash. He can use his numeraire to re-invest the proceeds of the short-sale of X.

If r is a stochastic process representing the overnight money-market rate,

[13] If $\theta_t < 0$, this indicates a positive cashflow to the investor of $-\theta_t X_t$ at time t.

[14] $\theta_t > 0$ is a long position. $\theta_t < 0$ is a short position.

the numeraire defined by:

$$B_t = \exp\left(\int_0^t r_s ds\right) \tag{5}$$

is called the **money-market numeraire**. Because $dB_t = r_t B_t dt$ and r_t, B_t are known at time t, the changes in the money-market numeraire over a small period of time, are known. Hence, the money-market numeraire is said to be **risk-free**. It is not a very useful numeraire, when an investor wishes to protect himself against future re-investment risks, as the overnight rate r_t is generally not deterministic. From that point of view, the money-market numeraire is far from being risk-free.

If F is a stochastic process representing a forward rate (or forward price), there normally exists a numeraire B, for which BF is a tradable process. Such numeraire B is called the **natural numeraire** of the forward rate F. For example, the natural numeraire of a forward Libor rate is the default-free zero with maturity equal to the end date of the forward Libor rate. It is indeed a tradable process for which BF is itself tradable[15].

2.4 The Wealth Process

In the previous section, we saw that an investor engaging in a trading strategy θ relative to a tradable process X, had a funding requirement of $\theta_t X_t$ at time t. This is not quite true. In fact, at any point in time, the true funding requirement needs to account for the **total wealth** π_t an investor may have. Such total wealth is defined as the total amount of cash (possibly negative) an investor would own, after liquidating all his positions in tradable instruments. A total wealth π_t at time t, is to a large extent dependent upon the **initial wealth** π_0 (possibly negative) the investor has, prior to trading. Each π_t is also the product of the trading performance up to time t. The evolution of π_t with time, is therefore a stochastic process denoted π. It is called the **wealth process** of the investor. Assuming X is the only tradable instrument used by the investor (excluding some numeraire), his total cash position after the purchase of θ_t of X at time t, is $\pi_t - \theta_t X_t$. If this is negative, the investor will need to take a short position in some numeraire B, to meet his funding requirement. The price of one unit of numeraire at time t being B_t, the total amount of numeraire which needs to be shorted is $-(\pi_t - \theta_t X_t)/B_t$. If the cash position of the investor is positive, the investor is not obligated to invest in the numeraire B. However, it is generally agreed that it is highly sub-optimal not to invest a positive cash position. An investor may not like the risk profile of a given numeraire. He may choose another numeraire, but will not choose not to invest at all. Hence, whatever the sign of the cash position $\pi_t - \theta_t X_t$, the investor will enter into a position $\psi_t = (\pi_t - \theta_t X_t)/B_t$ of numeraire B at time t.

[15] $BF = (V - B)/\alpha$, where V is the default-free zero with maturity equal to the start date of the forward Libor rate, and α the money-market day count fraction. As a portfolio of two tradable assets, BF is tradable.

In this example, the investor having engaged in a strategy θ relative to X and ψ relative to B, will experience a change in wealth $d\pi_t$ over a small period of time, equal to $d\pi_t = \theta_t dX_t + \psi_t dB_t$, or more specifically:

$$d\pi_t = \theta_t dX_t + \frac{1}{B_t}(\pi_t - \theta_t X_t)dB_t \tag{6}$$

An equation such as (6) is called a **stochastic differential equation**. It is the stochastic differential equation (**SDE**) *governing the wealth process of an investor, following a strategy θ in a cash-tradable process X, having chosen a cash-tradable process B as numeraire*. More generally, an SDE is an equation linking small changes in a stochastic process, for example '$d\pi_t$' on the left-hand side of (6), to the process itself, for example 'π_t' on the right-hand side of (6)[16].

The **unknown** to the SDE (6) is the wealth process π, which is only determined implicitly, through the relationship between $d\pi_t$ and π_t. The **inputs** to the SDE (6) are the two tradable processes X and B, the strategy θ and initial wealth π_0. A **solution** to the SDE (6) is an expression linking the wealth process π explicitly in terms of the inputs X, B, θ and π_0. In fact, using Ito's lemma as shown in appendix A.1, the solution to the SDE (6) is given by:

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \int_0^t \theta_s d\hat{X}_s \right) \tag{7}$$

where the semi-martingale $\hat{X}$ is the **discounted tradable process** $\hat{X} = X/B$, i.e. the tradable process X divided by the price process of the numeraire B[17]. In equation (7), B_0 is the initial value of the numeraire B, and π_0 is the initial wealth of the investor. So π_0/B_0 is just a constant. The stochastic integral $\int_0^t \theta_s d\hat{X}_s$ of the process θ with respect to the continuous semi-martingale $\hat{X}$, defines a new continuous semi-martingale. The wealth process π as given by equation (7), is the product of the continuous semi-martingale B, with the continuous semi-martingale $\pi_0/B_0 + \int_0^t \theta_s d\hat{X}_s$. The wealth process π is therefore itself[18] a continuous semi-martingale.

The SDE (6) and its solution (7) are just a particular example. Other SDE's can play an important role, when modeling a financial problem. For instance:

$$d\pi_t = \theta_t dX_t + \psi_t dY_t + \frac{1}{B_t}(\pi_t - \theta_t X_t - \psi_t Y_t)dB_t \tag{8}$$

This is the *SDE governing the wealth process of an investor, following the strategies θ and ψ in two tradable processes X and Y respectively, having chosen a tradable process B as numeraire*. It is very similar to the SDE (6), the only difference being the presence of an additional tradable process Y. As a consequence,

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[16] In fact, the proper way to write (6) is $\pi_t = \pi_0 + \int_0^t \theta_s dX_s + \int_0^t B_s^{-1}(\pi_s - \theta_s X_s)dB_s$. So an SDE is an equation linking a process, to a stochastic integral involving that same process.

[17] As a ratio of a continuous semi-martingale, with a positive continuous semi-martingale, $\hat{X}$ is a well-defined continuous semi-martingale, as shown by Ito's lemma.

[18] Also a consequence of Ito's lemma.

the total cash position of the investor at any point in time, is $\pi_t - \theta_t X_t - \psi_t Y_t$ which explains the particular form of the SDE (8). Similarly to equation (6), the solution to the SDE (8) is given by:[19]

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \int_0^t \theta_s d\hat{X}_s + \int_0^t \psi_s d\hat{Y}_s \right) \tag{9}$$

where $\hat{X}$, $\hat{Y}$ are the discounted processes defined by $\hat{X} = X/B$ and $\hat{Y} = Y/B$.

Another interesting SDE is the following:

$$d\pi_t = \theta_t dX_t + \frac{\pi_t}{B_t} dB_t \tag{10}$$

This SDE looks even simpler than the SDE (6), the main difference being that the total cash position in (10), appears to be equal to the total wealth π_t at any point in time. In fact, equation (10) is the *SDE governing the wealth process of an investor, following a strategy θ in a futures-tradable process X, having chosen a cash-tradable process B as numeraire*. The fact that the tradable process X is futures-tradable and not cash-tradable, is not due to any particular mathematical property. It is just an assumption. This assumption in turn leads to a different SDE, modeling the wealth process of an investor.[20] The solution to the SDE (10) is given by:[21]

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d\hat{X}_s \right) \tag{11}$$

where the semi-martingale $\hat{X}$ is defined by $\hat{X} = Xe^{-[X,B]}$, the process $\hat{\theta}$ is defined by $\hat{\theta} = (\theta e^{[X,B]})/B$, and $[X, B]$ is the **bracket** between X and B[22]. Note that contrary to equation (7), $\hat{X}$ is not the discounted process X/B, and the stochastic integral does not involve θ itself, but the *adjusted* process $\hat{\theta}$.

Last but not least, the following SDE will prove to be the most important of this document:

$$d\pi_t = \theta_t dX_t - \frac{\theta_t X_t}{Y_t} dY_t + \frac{\pi_t}{B_t} dB_t \tag{12}$$

This SDE is in fact a particular case of the SDE (8), where the trading strategy ψ relative to the tradable asset Y, has been chosen to be $\psi = -\theta X/Y$. In particular, we have $\theta_t X_t + \psi_t Y_t = 0$ at all times, and the cash position associated with the strategies θ and ψ, is therefore equal to the total wealth π_t at all times.

[19] See appendix A.1.

[20] SDE (10) is important when modeling the effect of convexity between futures and FRA's.

[21] See appendix A.2.

[22] The bracket $[X, B]$ between two positive continuous semi-martingales, is the process defined by $[X, B]_t = \int_0^t \sigma_s^X \sigma_s^B \rho_s^{X,B} ds$, where σ^X and σ^B are the **volatility** processes of X and B respectively, and $\rho^{X,B}$ is the correlation process between X and B. Given a positive semi-martingale of type (1), the volatility process is defined as the absolute volatility divided by the process itself. If X or B are not of type (1), the bracket $[X, B]$ can be defined as the cross-variation process between $\log X$ and $\log B$, or equivalently $[X, B]_t = \int_0^t X_s^{-1} B_s^{-1} d\langle X, B\rangle_s$.

It is possible to describe equation (12), as the *SDE governing the wealth process of an investor, following a strategy θ in a tradable process X, funding the strategy θ in X with another tradable process Y, having chosen a tradable process B as numeraire*. Being a particular case of (8), this SDE has a valid solution in equation (9). However, in view of the particular choice of $\psi = -\theta X/Y$, this solution can be simplified as:[23]

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d\hat{X}_s \right) \tag{13}$$

where the semi-martingale $\hat{X}$ is defined as $\hat{X} = X' e^{-[X',B']}$, the process $\hat{\theta}$ is defined as $\hat{\theta} = (\theta e^{[X',B']})/B'$, the two positive continuous semi-martingale X' and B' are given by $X' = X/Y$ and $B' = B/Y$, and $[X', B']$ is the bracket process between X' and B'.

Anticipating on future events, it may be worth emphasizing now the crucial importance of equation (13), in the pricing of credit derivatives. Strictly speaking, equation (13) cannot be applied to the price process B of a risky zero, which can be discontinuous with a sudden jump to zero. However, we shall see that only minor adjustments are required, to account for such particular feature. The advantage of equation (13), is that all the jump risk is concentrated in the numeraire B. In particular, the stochastic integral in (13) only involves[24] the continuous semi-martingale $X' = X/Y$. This process can realistically be modeled with a brownian diffusion. This is a crucial point, as it will allow a smooth application of the martingale representation theorem, and ensure the existence of replicating strategies, for a wide range of credit contingent claims.

2.5 Replication and Non-Arbitrage Pricing

In this section, we consider the issue of **non-arbitrage pricing** of a single contingent claim, possibly a credit claim, with maturity T and payoff h_T. To an investor starting with initial wealth π_0 and engaging into a strategy θ relative to some tradable assets[25], (having singled out one of them as numeraire), we can associate a wealth process π. We call **terminal wealth** associated with π_0 and the strategy θ, the value of the wealth process π_T on the maturity date of the claim. We say that a contingent claim is **replicable**, if there exists an initial wealth π_0, together with a trading strategy θ, for which the associated terminal wealth π_T is equal to the payoff h_T of the claim. The condition $\pi_T = h_T$ is called the **replicating condition** of the claim. A strategy θ, for which the replication condition is met, is called a **replicating strategy**. The initial wealth π_0 for which[26] the replicating condition is met, is called the **non-arbitrage price** or **price** of the contingent claim. The question of **contingent claim pricing** is defined as the question of determining the non-arbitrage price of a contingent

[23]See appendix A.3.
[24]Provided we assume the bracket $[X', B']$ to be deterministic.
[25]A *strategy* refers to a full collection of individual strategies relative to various assets.
[26]It will be shown to be unique.

claim. This question is only meaningful in the context of a replicable contingent claim. When faced with a non-replicable contingent claim, one cannot speak of its price[27].

For example, a European payer swaption with maturity T is a single claim with payoff $h_T = B_T(F_T - K)^+$, where B and F are processes representing the annuity[28] and forward rate of the underlying swap, and K is the strike of the swaption. B being the natural numeraire of the forward rate F, the process BF is tradable[29]. Starting with an initial wealth π_0, engaging in a strategy θ with respect to BF and choosing B as numeraire, the associated terminal wealth π_T can be derived from equation (7)[30], and the replicating condition is:

$$\frac{\pi_0}{B_0} + \int_0^T \theta_s dF_s = (F_T - K)^+ \tag{14}$$

Hence, the question of whether a European payer swaption is replicable, is reduced to that of the existence of π_0 and θ, satisfying equation (14).

In general, the question of whether a contingent claim is replicable, can only be answered using the **martingale representation theorem**. Fundamentally,[31] this theorem states that *if a random variable H is a function of the history[32] of some continuous semi-martingale X, from time 0 to time T, and provided that X has a brownian diffusion involving no more than one brownian motion[33], then H can be represented in terms of a constant, plus a stochastic integral with respect to X. In other words, there exists a constant x_0 and a stochastic process θ, such that:*

$$x_0 + \int_0^T \theta_s dX_s = H \tag{15}$$

For example, in the case of the European swaption above, the random variable $H = (F_T - K)^+$ being a function of the terminal value F_T of F at time T, is *a fortiori* a function of the history of the semi-martingale F between 0 and T. It follows that if our model is such that the process F is assumed to have a brownian diffusion, there is a good chance that the martingale representation theorem can be applied, and in light of equation (14), the swaption appears to be replicable in the context of this model[34]. The only case when the martingale representation theorem may fail to apply, is if our model assumes a brownian diffusion for F involving more than one brownian motion. This would be the

[27] Unless *price* refers to a notion which is distinct from that of *non-arbitrage price*.

[28] Annuity, delta, pvbp, pv01 are all possible terms.

[29] This is in fact an assumption. Since both B and BF can be viewed as linear combinations of default-free zeros with positive values, assuming them tradable is very reasonable.

[30] Applying (7) to $X = BF$ gives a terminal wealth of $\pi_T = B_T \left(\frac{\pi_0}{B_0} + \int_0^T \theta_s dF_s \right)$.

[31] See [1] th. 4.15 p. 182 for a possible precise mathematical statement.

[32] A lot of care is being taken to avoid mentioning *filtrations* or *measurability* conditions.

[33] i.e. X is a semi-martingale of type (1), where μ and σ only depend on the history of W. A convoluted way of saying that our filtration is brownian and one-dimensional.

[34] Apply (15) to $X = F$ and $H = (F_T - K)^+$, and take $\pi_0 = x_0 B_0$.

case, for instance, if our model assumed stochastic volatility introduced as an additional brownian source of risk. In such a model, where only B and BF exist as tradable processes, a European swaption is arguably not replicable. Note however, that stochastic volatility is not a problem by itself, provided it is driven by the same brownian motion, as the one underlying the diffusion of F[35]. As we can see from this example, being replicable is not an inherent property of a contingent claim, but rather a consequence of our modeling assumptions.

Once a contingent claim is shown to be replicable, we are faced with the task of computing its price. In general, this can be done using the replicating condition, which is most likely to be of the form:

$$\frac{\pi_0}{B_0} + \int_0^T \theta_s d\hat{X}_s = B_T^{-1} h_T \tag{16}$$

where $\hat{X}$ is a certain continuous semi-martingale, representing the price process of some tradable instrument, and which has been adjusted in some way.[36] In order to calculate π_0, all we have to do is use equation (4), taking the expectation relative to a specific **probability measure** Q, under which the semi-martingale $\hat{X}$ is in fact a martingale.[37] We obtain:

$$E_Q \left[\int_0^T \theta_s d\hat{X}_s \right] = 0 \tag{17}$$

This particular *trick* of considering a very convenient new measure is usually referred to as a **change of measure**. The new measure Q is called the **pricing measure**, or sometimes the **risk-neutral measure**.[38] Taking Q-expectation on both side of (16), using (17) we finally see that:

$$\pi_0 = B_0 E_Q[B_T^{-1} h_T] \tag{18}$$

For example, provided the European swaption is replicable, we have:

$$\pi_0 = B_0 E_Q[(F_T - K)^+] \tag{19}$$

where the pricing measure Q is such that F is a martingale under Q.

[35] It is however a lot harder to compute an expectation in that case.

[36] The nature of this adjustment may vary, see e.g. (7), (11) or (13).

[37] The existence of Q is normally derived from Girsanov theorem. See e.g. [1] Th.5.1 p. 191. The uniqueness of Q is not necessary in the coming argument, but if the claim is replicable, such measure is very likely to be unique.

[38] Particularly if the numeraire B is the money-market numeraire.

3 Credit Contingent Claims

3.1 Collapsing Numeraire

Recall that a **risky zero** with maturity T is defined as a single credit claim with payoff $1_{\{D>T\}}$ and maturity T, where D is the **time of default**. We would like to assume risky zeros to be tradable, an assumption which will be vindicated by the fact that CDS's can be replicated in terms of risky zeros, allowing prices of risky zeros to be inferred from the market place. Suppose B is the price process of the risky zero with maturity T. If the time of default occurs prior to time T, the final payoff B_T of the claim is zero. It follows that the risky zero must be worthless between time D and time T. Its price process B must have a value of zero, between time D and time T. Hence, it is impossible to model the price process of a risky zero with a positive continuous semi-martingale, as this would be completely unrealistic. Such price process must be allowed to be discontinuous at time D with a sudden jump to zero, and it cannot be non-zero after time D.

We say that a process B is a **collapsing numeraire**, or a **collapsing tradable process**, if it is a tradable process of the form $B_t = B_t^* 1_{\{t<D\}}$, where B^* is a positive continuous semi-martingale, called the **continuous part** of B. A collapsing numeraire satisfies the requirements of having a jump to zero at time D, and remaining zero-valued thereafter. It is an ideal candidate to represent the price process of a risky zero. We shall therefore assume that all our risky zeros have price processes which are collapsing numeraires. In short, we shall say that a risky zero *is a collapsing numeraire*.

Suppose B is a collapsing numeraire, and X, Y are two tradable processes. We assume that an investor engages into a strategy θ (up to time D)[39] relative to X, using Y to fund his position in X, having chosen the collapsing process B as numeraire. It is very tempting to write down the SDE governing the wealth process π of the investor, as the exact copy of equation (12):

$$d\pi_t = \theta_t dX_t - \frac{\theta_t X_t}{Y_t} dY_t + \frac{\pi_t}{B_t} dB_t \qquad (20)$$

However, this SDE is not quite satisfactory: the process B having potentially a jump at time D, the same may apply to the wealth process π. The ratio π/B, which should represent the total amount of numeraire held at any point in time, should therefore itself be discontinuous at time D. If follows that when $t = D$, there is potentially a big difference between π_{t-}/B_{t-} (the amount of numeraire held just prior to the jump), and π_t/B_t (the amount of numeraire held after the jump). When it comes to assessing the P/L contribution which arises from a jump in the numeraire, one need to choose very carefully between $(\pi_{t-}/B_{t-})dB_t$ and $(\pi_t/B_t)dB_t$. This can be done using the following argument: at any point in time, the total wealth π_t of the investor is split between three different assets. In fact, because the position in X is always funded with the appropriate position

[39] *Up to time D* is a way of expressing the fact that the investor stops trading after time D.

in Y, the total wealth held in X and Y is always zero. The entire wealth of the investor is continuously invested in the collapsing numeraire B. It follows that in the event of default, the total wealth of the investor suddenly collapses to zero, and therefore $\pi_D = 0$.[40] We conclude that $(\pi_t/B_t)dB_t$, is wholly inappropriate to reflect the sudden jump in the wealth of the investor.[41] Since $dB_t = 0$ for $t > D$, and $B_{t-} = B^*_t$ for $t \leq D$, the P/L contribution arising from numeraire re-investment can equivalently be expressed as $(\pi_{t-}/B^*_t)dB_t$, where B^* is the continuous (and positive) part of the collapsing process B.

Having suitably adjusted equation (20), to account for the collapsing numeraire, one final touch needs to be made to formally express the fact that the investor will no longer trade after time D. One possible way, is to replace θ_t by the strategy $\theta_t 1_{\{t\leq D\}}$. Equivalently, X^D and Y^D being the stopped processes[42], We have $dX^D_t = dY^D_t = 0$ for $t > D$. Hence the same purpose may be achieved by replacing dX_t and dY_t, with dX^D_t and dY^D_t respectively. This would ensure that no P/L contribution would arise from θ, after time D. We are now in a position to write down the SDE *governing the wealth process of an investor, engaging in a strategy θ in X (up to time D), using Y to fund his position in X, having chosen the collapsing process B as numeraire*:

$$d\pi_t = \theta_t dX^D_t - \frac{\theta_t X_t}{Y_t} dY^D_t + \frac{\pi_{t-}}{B^*_t} dB_t \tag{21}$$

where B^* is the continuous part of the collapsing numeraire B. As shown in appendix A.4, the solution to this SDE is:

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d\hat{X}_s \right) \tag{22}$$

where the semi-martingale $\hat{X}$ is defined as $\hat{X} = X' e^{-[X',B']}$, the process $\hat{\theta}$ is defined as $\hat{\theta} = (\theta e^{[X',B']})/B'$, the two positive continuous semi-martingale X' and B' are given by $X' = X/Y$ and $B' = B^*/Y$, and $[X', B']$ is the bracket process between X' and B'.[43] It is remarkable that equation (22) is formally identical to equation (13). The only difference is that the positive continuous semi-martingale B' is defined in terms B^*, and not B itself. It is also remarkable that the time of default D, does not appear anywhere in equation (22). The only dependence in D, is contained via the collapsing numeraire B. In fact, the wealth process π is the product of the collapsing numeraire B with a continuous semi-martingale,[44] which can realistically be modeled with a brownian diffusion. This will allow us to apply the martingale representation theorem, and show that several credit contingent claims are replicable, and can therefore be submitted to non-arbitrage pricing.

[40] In fact, $\pi_t = 0$ for all $t > D$ as the investor stops trading altogether.

[41] $(\pi_D/B_D)dB_D$ would be zero.

[42] $X^D_t = X_{t\wedge D}$ is defined as X_t for $t < D$ and X_D for $t \geq D$.

[43] $[X', B']_t = \int_0^t X'^{-1}_s B'^{-1}_s d\langle X', B'\rangle_s$.

[44] The process $\frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d\hat{X}_s$ is a continuous semi-martingale.

3.2 Delayed Risky Zero

Given $T < T'$, we call **delayed risky zero** with maturity T' and observation date T, the single credit contingent claim with payoff $1_{\{D>T\}}$ and maturity T'. A delayed risky zero with observation date T, has the same payoff as that of a risky zero with maturity T. However, the payment date of a delayed risky zero, is *delayed*, relative to that of a risky zero. Delayed risky zeros will be seen to play an important role in the pricing of the default leg of a CDS.

Given a delayed risky zero with maturity T' and observation date T, we denote B the collapsing numeraire, representing the price process of the risky zero with maturity T. We denote W the price process of the default-free zero with maturity T'[45], and V the price process of the default-free zero with maturity T. All three processes B, W, V are assumed to be tradable. It is clear that the delayed risky zero is equivalent to the single claim with maturity T and payoff $B_T W_T$. An investor entering into a strategy θ relative to W (up to time D), using V to fund his position in W, having chosen the collapsing process B as numeraire, has a wealth process π following the SDE:

$$d\pi_t = \theta_t dW_t^D - \frac{\theta_t W_t}{V_t} dV_t^D + \frac{\pi_{t-}}{B_t^*} dB_t \tag{23}$$

where B^* is the continuous part of the collapsing numeraire B. The terminal wealth π_T of the investor is given by:

$$\pi_T = B_T \left(\frac{\pi_0}{B_0} + \int_0^T \hat{\theta}_s d\hat{W}_s \right) \tag{24}$$

where the semi-martingale $\hat{W}$ is defined as $\hat{W} = W' e^{-[W',B']}$, the process $\hat{\theta}$ is defined as $\hat{\theta} = (\theta e^{[W',B']})/B'$, the two positive continuous semi-martingale W' and B' are given by $W' = W/V$ and $B' = B^*/V$, and $[W', B']$ is the bracket process between W' and B'. Note that the process W' represents the forward price process (with expiry T) of the default-free zero with maturity T'. As for B', it is the continuous part of the collapsing process B/V.[46] The process B/V is called the **survival probability** process, denoted P, with maturity T. Alternatively, at any point in time t, the ratio B_t/V_t is called the survival probability at time t, denoted P_t, with maturity T. A survival probability is therefore the ratio between the price process of a risky zero, and the price process of the default-free zero with same maturity. Having defined the survival probability, B' appears as the continuous part of the survival probability process P. For a wide range of distributional assumptions, the bracket $[W', B']$ is given by:

$$[W', B']_t = \int_0^t \sigma_{W'}\, \sigma_P\, \rho\, ds \tag{25}$$

where $\sigma_{W'}$ is the volatility process of W', σ_P is the volatility process of B', and ρ is the correlation process between W' and B'. Contrary to what the

[45] W is not a brownian motion, it is a positive continuous semi-martingale.

[46] $(B/V)_t = (B^*/V)_t 1_{\{t<D\}}$. Hence it is a collapsing process (but not assumed tradable).

notation suggests, σ_P is not the volatility process of the survival probability P. We call σ_P the **no-default volatility** of the survival probability P. It is the volatility of the continuous part of P, i.e. the volatility of P prior to default, or equivalently the volatility of P, if **no default** were to occur. Likewise, we call ρ the **no-default correlation** process, between the forward default-free zero W', and survival probability P. The distinction between volatility and no-default volatility is essential. As the survival probability P is a collapsing process, its volatility beyond the time of default D is not a very well-defined quantity. Assuming we were to adopt the convention that a zero-valued process has zero-volatility, then the volatility process of the survival probability has a sudden jump to zero, on the time of default. Such volatility process cannot ever be modeled as a deterministic process.[47] In contrast, the no-default volatility process σ_P, can realistically be modeled as a deterministic process, as no jump is to occur on the time of default. Likewise, the no-default correlation process can freely be modeled as a deterministic process. In what follows, we shall therefore assume that the **bracket** $[W', B']$ **is a deterministic process**.

Having established the terminal wealth π_T in the form of equation (24), the replicating condition $\pi_T = B_T W_T$ will be satisfied, whenever the following sufficient condition holds:

$$\frac{\pi_0}{B_0} + \int_0^T \hat{\theta}_s d\hat{W}_s = W_T \tag{26}$$

The question of whether a delayed risky zero is replicable, can therefore be positively answered, provided an initial wealth π_0 and trading strategy θ satisfying (26), can be shown to exist. Since $V_T = 1$, it is possible to write W_T as $W_T = \hat{W}_T e^{[W',B']_T}$. Having assumed the bracket process $[W', B']$ to be deterministic, its terminal value $[W', B']_T$ is therefore non-random. It follows that W_T is just $\hat{W}_T$, multiplied by the constant $e^{[W',B']_T}$. In particular, W_T is *a function of the history of* of the process $\hat{W}$. This shows that provided reasonable distributional assumptions are made,[48] the martingale representation theorem will be successfully applied, and the delayed risky zero will be shown to be replicable.[49]

When this is the case, denoting Q a probability measure relative to which the continuous semi-martingale $\hat{W}$ is in fact a martingale, taking Q-expectation on both side of (26), we see that the non-arbitrage price π_0 of the delayed risky zero is given by:

$$\pi_0 = B_0 E_Q[W_T] = B_0 E_Q[\hat{W}_T] e^{[W',B']_T} = B_0 \frac{W_0}{V_0} e^{[W',B']_T} \tag{27}$$

where we have used the fact[50] that $E_Q[\hat{W}_T] = \hat{W}_0 = W_0/V_0$. Re-expressing (27)

[47] It would require the time of default D to be assumed non-random ...

[48] W' should have a simple one-dimensional brownian diffusion.

[49] Having x_0 and ψ with $x_0 + \int_0^T \psi_s d\hat{W}_s = W_T$, take $\pi_0 = B_0 x_0$ and $\theta = \psi e^{-[W',B']} B'$.

[50] $\hat{W}$ being a Q-martingale. See equation (3).

in terms of the survival probability $P_0 = B_0/V_0$, we conclude that:

$$\pi_0 = P_0 W_0 \exp\left(\int_0^T \sigma_{W'}\, \sigma_P\, \rho\, dt\right) \tag{28}$$

A naive valuation would have yielded $\pi_0 = P_0 W_0$. Assuming a positive correlation ρ between survival probabilities and bonds,[51] equation (28) indicates that a delayed risky zero, should be more valuable than what the naive valuation suggests, i.e. $\pi_0 > P_0 W_0$. This can be explained by the following argument: when dynamically replicating a delayed risky zero, an investor is essentially long an amount W/V of risky zero B. As soon as the bond market rallies, W/V goes up and the investor finds himself *under-invested* in B. With positive correlation, the risky zero will be more expensive to buy. It follows that the investor will have to *buy at the high*, (and similarly *sell at the low*), finding himself is a *short gamma* position. This short gamma position being a cost to the investor, a higher amount of cash is required to achieve the replication of the delayed risky zero. In other words, the non-arbitrage price of a delayed risky zero should be higher. The opposite conclusion would obviously hold, in the context of negative correlation between survival probabilities and bond prices.

3.3 Credit Default Swap

Let $t_0 < t_1 < \ldots < t_n$, be a date schedule. We call **CDS fixed leg** (associated with the schedule $t_0, \ldots, t_n$), the contingent claim paying $\alpha_i K 1_{\{D>t_i\}}$ at time t_i for all $i = 1, \ldots, n$,[52] where K is a constant and each α_i is the day-count fraction between t_{i-1} and t_i.[53] The constant K is called the **fixed rate** of the CDS fixed leg. A CDS fixed leg is therefore a portfolio of $n \geq 1$ risky zeros with maturity $t_1, \ldots, t_n$, held in amounts $\alpha_1 K, \ldots, \alpha_n K$ respectively.[54] Assuming risky zeros are tradable, a CDS fixed leg is replicable, and its non-arbitrage price is given by:

$$\pi_0 = \sum_{i=1}^{n} \alpha_i K P_0^i V_0^i \tag{29}$$

where each P_0^i is the current survival probability with maturity t_i, and V_0^i is the current default-free zero with maturity t_i.

We call **CDS default leg** (associated with the schedule $t_0, \ldots, t_n$), the contingent claim comprised of $n \geq 1$ single claims C_i, $i = 1, \ldots, n$, where each single claim C_i has a maturity t_i and payoff $(1-R)1_{\{t_{i-1}<D\leq t_i\}}$, where R is

[51] It is not obvious this should be the case. One one hand, a bullish bond market may be viewed as cheaper funding cost for companies, and therefore higher survival probabilities. On the other hand, a bullish bond market can be the sign of an economic contraction, higher rate of bankruptcies, flight to quality and credit collapse.

[52] There is no payment on date t_0.

[53] Relative to a given accruing basis.

[54] In real life, if the time of default D occurs prior to t_n, a CDS fixed leg would normally pay a last coupon, accruing from the last payment date to the time of default. The present definition ignores this potential last fractional coupon.

a constant. The constant R is called the **recovery rate** of the CDS default leg. Essentially, a CDS default leg pays $(1 - R)$ at time t_i, provided default occurs in the interval $]t_{i-1}, t_i]$.[55] Each single claim C_i is clearly equivalent to a long position of $(1 - R)$ in the delayed risky zero with maturity t_i and observation date t_{i-1}, and a short position of $(1 - R)$ in the risky zero with maturity t_i. Provided similar assumptions to those of section 3.2 hold, delayed risky zeros are replicable and a CDS default leg is therefore itself replicable. Using equation (28), the non-arbitrage price of the CDS default leg is:

$$\pi_0' = (1 - R) \sum_{i=1}^{n} (\hat{P}_0^{i-1} - P_0^i) V_0^i \tag{30}$$

where $V_0^1, \dots, V_0^n$ are the current values of the default-free zeros with maturity $t_1, \dots, t_n$, $P_0^1, \dots, P_0^n$ are the current survival probabilities with maturity $t_1, \dots, t_n$, and $\hat{P}_0^0, \dots, \hat{P}_0^{n-1}$ are the current *convexity adjusted* survival probabilities with maturity $t_0, \dots, t_{n-1}$. Specifically, for all $i = 1, \dots, n$, we have:

$$\hat{P}_0^{i-1} = P_0^{i-1} \exp\left(\int_0^{t_{i-1}} u_i(s) v_{i-1}(s) \rho(s) ds \right) \tag{31}$$

where P_0^{i-1} is the current survival probability with maturity t_{i-1}, u_i is the local volatility structure of the forward default-free zero with expiry t_{i-1} and maturity t_i, v_{i-1} is the no-default local volatility of the survival probability with maturity t_{i-1}, and ρ some sort of (no-default) correlation structure between survival probabilities and bonds.

We call a **credit default swap** or **CDS**, any claim comprised of a long position in a CDS default leg, and a short position in a CDS fixed leg,[56] (not necessarily relative to the same date schedule).

3.4 Risky Floating Payment and Related Claim

Given $T < T'$, we call **risky floating payment** with maturity T' and expiry T, the single credit contingent claim with maturity T' and payoff $F_T 1_{\{D>T'\}}$, where D is the time of default, and F is the forward Libor process between T and T'. More generally, we call **floating related claim** (with maturity T' and expiry T), any single credit contingent claim with maturity T' and payoff of the form $g(F_T)1_{\{D>T'\}}$, for some payoff function g.

Given a floating related claim with maturity T' and expiry date T, we denote B the collapsing numeraire, representing the price process of the risky zero

[55] In real life, a CDS default leg would not pay on a discrete schedule of payment dates, but rather on the time of default itself (or a few days later). furthermore the payoff would not be $(1 - R)$: the long of the CDS default leg (the buyer of protection) would receive 1, and deliver a bond (deliverable obligation) to the short. It follows that the net payoff to the long can indeed be viewed as $(1 - R)$ (where R is the market price of the delivered bond), but R is not a constant specified by the CDS transaction. This makes our definition highly simplistic, but in line with current practice.

[56] A long CDS position correspond to being long protection and short credit.

with maturity T'. We denote W the price process of the default-free zero with maturity T', and V the price process of the default-free zero with maturity T. All three processes B, W, V are assumed to be tradable. In fact, it shall be convenient to define $F = (V/W - 1)/\alpha$ (where α is the money market day-count fraction between T and T') and assume that B, W and FW are tradable. It is clear that the floating related claim is equivalent to the single claim with maturity T and payoff $g(F_T)B_T$. An investor entering into a strategy θ relative to FW (up to time D), using W to fund his position in FW, having chosen the collapsing process B as numeraire, has a wealth process π following the SDE:

$$d\pi_t = \theta_t d(FW)_t^D - \theta_t F_t dW_t^D + \frac{\pi_{t-}}{B_t^*} dB_t \tag{32}$$

where B^* is the continuous part of the collapsing numeraire B. The terminal wealth of the investor is given by:[57]

$$\pi_T = B_T \left(\frac{\pi_0}{B_0} + \int_0^T \hat{\theta}_s d\hat{F}_s \right) \tag{33}$$

where the semi-martingale $\hat{F}$ is defined as $\hat{F} = F e^{-[F,P]}$, the process $\hat{\theta}$ is defined as $\hat{\theta} = (\theta e^{[F,P]})/P$, P is the continuous part of the survival probability process B/W, with maturity T', and $[F, P]$ is the bracket process between F and P. A sufficient condition for replication is:

$$\frac{\pi_0}{B_0} + \int_0^T \hat{\theta}_s d\hat{F}_s = g(F_T) \tag{34}$$

Since $F_T = \hat{F}_T e^{[F,P]_T}$, the martingale representation theorem can successfully be applied for a wide range of distributional assumptions on F, provided the bracket $[F, P]$ is deterministic. When this is the case, the floating related claim is replicable, and its non-arbitrage price is given by:

$$\pi_0 = B_0 E_Q[g(F_T)] \tag{35}$$

where the pricing measure Q is such that the semi-martingale $\hat{F}$ is in fact a martingale under Q. In particular, when $g(x) = x$, we obtain the non-arbitrage price of a risky floating payment, as:[58]

$$\pi_0 = P_0 W_0 F_0 e^{[F,P]_T} \tag{36}$$

where P_0 is the current survival probability with maturity T', W_0 is the current default-free zero with maturity T', and F_0 the current forward Libor rate between T and T'. Note that the convexity adjustment $e^{[F,P]_T}$ indicates that a positive correlation between rates and survival probabilities would make the risky floating payment more valuable than suggested by a naive valuation.

[57] See equation (22).

[58] $[F, P]_T = \int_0^T \sigma_F \sigma_P \rho \, ds$. Note that σ_P and ρ are *no-default* volatility and correlation.

3.5 Foreign Credit Default Swap

Suppose we are given two currencies, one being called **foreign** and the other **domestic**. We call **foreign credit default swap** or **foreign CDS** any credit default swap denominated in foreign currency. A foreign CDS is therefore nothing but a normal CDS. Similarly, a **domestic CDS** is nothing but a normal CDS, denominated in domestic currency. The purpose of this section is to investigate whether a non-arbitrage relationship exists between domestic and foreign CDS's. Specifically, having assumed that domestic risky zeros are tradable, we shall see that foreign CDS's can be replicated through dynamic strategies involving domestic risky zeros. The conclusion is quite interesting: given the two yield curves in domestic and foreign currencies, given the default swap curve in domestic currency, foreign CDS's are fully determined through some sort of *quanto adjustment*, and cannot be specified independently.[59]

A CDS being a linear combination of risky zeros and delayed risky zeros,[60] It is sufficient for us to show that foreign (delayed) risky zeros can be replicated in terms of domestic risky zeros. Given a foreign risky zero with maturity T, we denote B the collapsing numeraire representing the price process of the domestic risky zero with maturity T. We denote V the price process of the domestic default-free zero with maturity T, and W the price process of the foreign default-free zero with maturity T. We also denote X the spot FX rate process, quoted with the foreign currency as the base currency.[61] We assume that W is tradable in foreign currency, whereas B and V are tradable in domestic currency. In fact, we assume that all three processes W, V/X and B/X are tradable in foreign currency. An investor entering into a strategy θ relative to W (up to time D), using V/X to fund his position in W, having chosen the collapsing process B/X[62] as numeraire, has a wealth process π (in foreign currency) following the SDE:

$$d\pi_t = \theta_t dW_t^D - \frac{\theta_t W_t X_t}{V_t} d(V/X)_t^D + \frac{X_t \pi_{t-}}{B_t^*} d(B/X)_t \tag{37}$$

where B^* is the continuous part of the collapsing process B. The terminal wealth of the investor (in foreign currency) is given by:[63]

$$\pi_T = \frac{B_T}{X_T} \left(\frac{X_0 \pi_0}{B_0} + \int_0^T \hat{\theta}_s d\hat{Y}_s \right) \tag{38}$$

where the semi-martingale $\hat{Y}$ is defined as $\hat{Y} = Ye^{-[Y,P]}$, the process $\hat{\theta}$ is defined as $\hat{\theta} = (\theta e^{[Y,P]})/P$, $P = B^*/V$ is the continuous part of the survival probability (in domestic currency) with maturity T, $Y = WX/V$ is the forward FX rate with maturity T,[64] and $[Y,P]$ is the bracket process between Y and P. The

[59]Note however that this section only applies to the case where both domestic and foreign currencies are G7+ currencies. The reason for this restriction will become clear below.

[60]See section 3.3.

[61]X_t is the price in domestic currency at time t, of one unit of foreign currency.

[62]$(B/X)_t = (B^*/X)_t 1_{\{t<D\}}$ is indeed a collapsing process, also assumed to be tradable.

[63]See equation (22).

[64]Y is also quoted with the foreign currency as the base currency.

payoff (in foreign currency) of the foreign risky zero with maturity T being $1_{\{D>T\}} = B_T$, a sufficient condition for replication is:

$$\frac{X_0 \pi_0}{B_0} + \int_0^T \hat{\theta}_s d\hat{Y}_s = X_T \tag{39}$$

Since $W_T = V_T = 1$, it is possible to write $X_T = \hat{Y}_T e^{[Y,P]_T}$, and provided the bracket $[Y, P]$ can be assumed to be deterministic, the martingale representation theorem will be successfully applied for a wide range of distributional assumptions on Y. However, assuming the bracket $[Y, P]$ to be deterministic may not be possible in cases where the reference entity underlying the time of default D, is a sovereign entity controlling either the foreign or domestic currency.[65] To avoid dealing with such problem, we shall restrict this analysis to the case when both domestic and foreign currency are G7+ currencies.

Q being a measure under which the semi-martingale $\hat{Y}$ is in fact a martingale, taking Q-expectation on both side of (39), we obtain the non-arbitrage price of the foreign risky zero as:

$$\pi_0 = \frac{B_0}{X_0} E_Q[\hat{Y}_T] e^{[Y,P]_T} = \frac{B_0}{X_0} \hat{Y}_0 e^{[Y,P]_T} = P_0 W_0 e^{[Y,P]_T} \tag{40}$$

where P_0 is the current (domestic) survival probability with maturity T and W_0 is the current foreign default-free zero with maturity T. Recall that the forward FX rate Y, appearing in the *quanto adjustment* $e^{[Y,P]_T}$, must be quoted with the foreign currency as the base currency. A positive correlation between Y and P, would therefore indicate a strengthening foreign currency, in line with higher (domestic) survival probabilities. When this is the case, equation (40) indicates a higher price than what a naive valuation would suggest. This can be explained by the following heuristic argument:[66] an investor dynamically replicating a foreign risky zero, is essentially long a certain amount of domestic risky zero. If the foreign currency strengthens, the investor will find himself *under-invested* in the domestic risky zeros. However, a positive correlation implies that domestic risky zeros will be more expensive to buy. The investor will therefore *buy at the high* and *sell at the low*, facing the equivalent of a *short gamma* position. This short gamma position being a cost to the investor, a higher initial wealth is required to achieve the replication of a foreign risky zero. In other words, the non-arbitrage price of a foreign risky zero should be higher.

When $T < T'$, the case of a foreign delayed risky zero with observation date T and maturity T', is handled in a similar manner, trading W' (the foreign default-free zero with maturity T') instead of W. We obtain:

$$\pi_0 = P_0 W'_0 e^{[Y',P]_T} \tag{41}$$

[65] Default may be accompanied by a substantial devaluation, which would amount to a sudden jump in FX volatility and breakdown in correlations. In fact, if the domestic or foreign currency is not an G7+ currency, the collapse of any major corporation in the country of that currency, may be accompanied by sharp FX moves.

[66] This sort of casual explanation is useful to check that we got the *sign* right.

where P_0 is the current (domestic) survival probability with maturity T and W'_0 is the current foreign default-free zero with maturity T'. However, contrary to equation (40), the convexity adjustment $e^{[Y',P]_T}$ does not involve the forward FX rate Y, but $Y' = XW'/V$. Writing $Y' = YW'/W$, we have:[67]

$$[Y', P]_T = [Y, P]_T + [W'/W, P]_T \tag{42}$$

and we conclude that the convexity adjustment in (41) is in fact the same *quanto adjustment* as in (40), compounded by a *delay adjustment* $e^{[W'/W,P]_T}$ formally identical to that encountered in the pricing of a domestic delayed risky zero.[68]

3.6 Equity Option with Possible Bankruptcy

In this section, we assume that the reference entity which underlies the time of default D, is a corporation with a non-dividend paying stock X. Furthermore, contrary to market practice, we would like to assume that X is no longer a positive continuous semi-martingale, but rather a collapsing tradable process,[69] i.e. a process of the form $X_t = X_t^* 1_{\{t<D\}}$ where X^* is a positive continuous semi-martingale (the continuous part of X). Such assumption allows the price process X to display a sudden *jump to zero* in the event of default, and can therefore legitimately be viewed as more realistic than the standard log-normal assumption. The purpose of this section is to investigate the impact of such assumption, on the pricing of various equity claims, which are contingent on the terminal value of X.

Specifically, given a date T, we consider the claim with maturity T and payoff $f(X_T)$, where f is an arbitrary payoff function. We denote B the collapsing process representing the price process of the risky zero with maturity T. We assume that both X and B are tradable processes. Since the payoff $f(X_T)$ can be expressed as:

$$f(X_T) = B_T[f(X_T^*) - f(0)] + f(0) \tag{43}$$

by considering $g(x) = f(x) - f(0)$, we can reduce our attention to the claim with maturity T, and payoff $B_T g(X_T^*)$.[70]

An investor entering into a strategy θ relative to X (up to time D), having chosen the collapsing process B as numeraire, has a wealth process π following the SDE:

$$d\pi_t = \theta_t dX_t + \frac{1}{B_t^*}(\pi_{t-} - \theta_t X_{t-})dB_t \tag{44}$$

This is the first time in this document, that an attempt is made to model the wealth process associated with tradable assets which are both collapsing processes. Up till now, the use of collapsing processes was limited to the numeraire. The SDE (44) is therefore unknown to us, and some explanation is probably welcome: it should be noted that (44) looks pretty natural in light of similar

[67]Recall that the bracket $[Y', P]$ is the cross-variation process $\langle \log Y', \log P\rangle$.
[68]See equation (28), where $W' = W/V$ is also a forward default-free zero.
[69]See section 3.1.
[70]Since $g(0) = 0$, we have $B_T g(X_T^*) = g(X_T)$.

SDE's, and the SDE (6) in particular. However, since both X and B are potentially discontinuous, and trading is assumed to be interrupted after the time of default, one has to be very careful that the P/L contributions expected from *collapsing* prices, are properly reflected in (44), and furthermore that no P/L contribution arises after time D.[71] This last point is actually guaranteed by the fact that $dX_t = dB_t = 0$ for $t > D$.[72] As for a proper accounting of P/L jumps, the following argument will probably convince us that (44) is *doing the right thing*: since at any point in time the total wealth π_t of the investor, is *split* between the two collapsing processes X and B, the investor would lose everything in the event of default. It follows that the total wealth after default is $\pi_D = 0$, and the jump $d\pi_D$ on the time of default is $d\pi_D = -\pi_{D-}$.[73] This jump is properly reflected by the SDE (44), as shown by the following derivation:

$$\begin{aligned} d\pi_D &= \theta_D dX_D + \frac{1}{B^*_D}(\pi_{D-} - \theta_D X_{D-})dB_D \\ &= -\theta_D X_{D-} - \frac{1}{B_{D-}}(\pi_{D-} - \theta_D X_{D-})B_{D-} \\ &= -\pi_{D-} \end{aligned} \tag{45}$$

As shown in appendix A.5, the solution to the SDE (44) is given by:

$$\pi_t = B_t\left(\frac{\pi_0}{B_0} + \int_0^t \theta_s d\hat{X}_s\right) \tag{46}$$

where the continuous semi-martingale $\hat{X}$ is defined as $\hat{X} = X^*/B^*$. This solution is formally identical to (7), except that $\hat{X}$ is defined in terms of the *continuous parts* X^*, B^*, and not X, B themselves.[74]

Since $B_T = 1$ implies $B^*_T = 1$, the replication condition $\pi_T = B_T g(X^*_T)$ is equivalent to $\pi_T = B_T g(\hat{X}_T)$, and a sufficient condition for replication is:

$$\frac{\pi_0}{B_0} + \int_0^T \theta_t d\hat{X}_t = g(\hat{X}_T) \tag{47}$$

and because $g(\hat{X}_T)$ is obviously a *function of the history* of $\hat{X}$ between 0 and T, (and $\hat{X}$ is a continuous semi-martingale), the martingale representation theorem will be successfully applied for a wide range of distributional assumptions on $\hat{X}$. When that is the case, the equity claim is replicable, and its non-arbitrage price is given by:

$$\pi_0 = B_0 E_Q[g(\hat{X}_T)] \tag{48}$$

where Q is a measure under which the semi-martingale $\hat{X}$ is in fact a martingale. Going back to (43), we obtain the price of the equity claim with payoff $f(X_T)$:

$$\pi'_0 = V_0\left[P_0 E_Q[f(\hat{X}_T)] + (1 - P_0)f(0)\right] \tag{49}$$

[71] See section 3.1 on the collapsing numeraire, for a similar discussion.
[72] It is therefore unnecessary to introduce dX^D_t as in the SDE (21).
[73] π_{D-} is the total wealth just prior to default.
[74] The process X/B would not be defined beyond time D.

where P_0 is the current survival probability with maturity T, and V_0 is the current default-free zero with maturity T. Note that contrary to standard equity option pricing, the pricing measure Q is such that, the process $\hat{X} = X^*/B^*$ (and not the **equity forward** process X/V) should be a martingale. We call this process $\hat{X}$ the **no-default credit equity forward** process. It is a *credit* forward, as the stock price X is effectively compounded up at the credit yield implied by B (as opposed to the Libor yield implied by V), and it is a *no-default* forward, as it is defined in terms of the continuous parts X^* and B^*, which coincide with X and B, in the event of no default.

The term volatility $[\hat{X}, \hat{X}]_T$ of the no-default credit equity forward, which is crucial for any implementation of (49), can be derived from the term volatility of the equity forward[75] $[Y, Y]_T$ as follows: from $Y = X^*/V$, we have $\hat{X} = Y/P$ where $P = B^*/V$ is the continuous part of the survival probability with maturity T, and therefore:

$$[\hat{X}, \hat{X}]_T = [Y, Y]_T - 2[Y, P]_T + [P, P]_T \tag{50}$$

As we can see from equation (50), the no-default volatility and correlation (with equity) of the survival probability, will also be required.

3.7 Risky Swaption and Delayed Risky Swaption

Given a date T, we define the **risky payer swaption** with expiry T as the single claim with maturity T and payoff $1_{\{D>T\}} C_T (F_T - K)^+$, where F is a forward swap rate and C its natural numeraire[76], K is a constant (called the strike) and D is the time of default. Note that the effective date of the underlying swap (F, C) must be greater than the expiry date T, but need not be equal to it. A risky payer swaption is equivalent to the right to enter into a forward payer swap, provided no default has occurred by the time of the expiry. Given $T < T'$, we call **delayed risky payer swaption** with observation date T and expiry T', the single claim with maturity T' and payoff $1_{\{D>T\}} C_{T'} (F_{T'} - K)^+$. A delayed risky swaption is equivalent to the right to enter into a forward payer swap on the expiry date T', provided no default has occurred by the time of the observation date T. Note that a long position in a delayed risky swaption with observation date T and expiry T', together with a short position in a risky swaption with expiry T', is equivalent to the right to enter into a forward payer swap on the expiry date T' provided default *has* occurred, in the time interval $]T, T']$. Risky swaptions and delayed risky swaptions will be seen to play an important role in the next section, where we study the impact of possible default, on the pricing of an interest rate swap transaction.

In this section, we concentrate on the question of non-arbitrage pricing of risky swaptions and delayed risky swaptions. More generally, we consider the single claim with maturity T and payoff $B_T C_T g(F_T)$, where g is an arbitrary payoff function, and B is the collapsing process representing the price process of

[75]Strictly speaking, its continuous part $Y = X^*/V$.

[76]i.e. the underlying annuity, delta, pv01 , pvbp. . .

the risky zero with maturity T. The case of a risky payer swaption corresponds to $g(x) = (x-K)^+$, whereas a delayed risky payer swaption is clearly equivalent to $g(x)$ being *the undiscounted price at time* T *of a payer swaption with strike* K *and expiry* T', *given an underlying forward swap rate of* x.[77]

We denote V the price process of the default-free zero with maturity T. The four processes C, CF, V and B are assumed to be tradable. An investor entering into a strategy θ and ψ (up to time D) relative to CF and C respectively, funding his position in CF and C with V, having chosen the collapsing process B as numeraire, has a wealth process π satisfying the SDE:

$$d\pi_t = \theta_t d(CF)_t^D + \psi_t dC_t^D - \frac{\theta_t C_t F_t + \psi_t C_t}{V_t} dV_t^D + \frac{\pi_{t-}}{B_t^*} dB_t \qquad (51)$$

where B^* is the continuous part of B. The associated terminal wealth is:[78]

$$\pi_T = B_T \left(\frac{\pi_0}{B_0} + \int_0^T \hat{\theta}_t dX_t + \int_0^T \hat{\psi}_t dY_t \right) \qquad (52)$$

where the continuous semi-martingales X, Y are defined as $X = C'Fe^{-[C'F,P]}$ and $Y = C'e^{-[C',P]}$, the processes $\hat{\theta}$ and $\hat{\psi}$ are defined as $\hat{\theta} = (\theta e^{[C'F,P]})/P$ and $\hat{\psi} = (\psi e^{[C',P]})/P$, the process $C' = C/V$ is the forward annuity of the underlying swap, and $P = B^*/V$ is the continuous part of the survival probability process B/V. A sufficient condition for replication is:

$$\frac{\pi_0}{B_0} + \int_0^T \hat{\theta}_t dX_t + \int_0^T \hat{\psi}_t dY_t = C_T g(F_T) \qquad (53)$$

Since $V_T = 1$ we have $C_T = Y_T e^{[C',P]_T}$, and from $[C'F,P]_T = [C',P]_T + [F,P]_T$ we see that $F_T = (X_T/Y_T)e^{[F,P]_T}$. Hence, provided both brackets $[C',P]$ and $[F,P]$ are assumed deterministic, the quantity $C_T g(F_T)$ can be viewed as a *function of the history of* X *and* Y *between time* 0 *and* T, and the martingale representation theorem[79] will be successfully applied, for a wide range of distributional assumptions on C' and F. When this is the case, our claim is replicable, and we have:

$$\pi_0 = B_0 E_{Q'}[C_T g(F_T)] \qquad (54)$$

[77] e.g. $g(x) = xN(d) - KN(d-u)$ where $d = (\ln(x/K) + u^2/2)/u$, and u is the non-annualized total volatility of F, between T and T'. More generally, $g(x) = E_Q[(F_{T'} - K)^+ | F_T = x]$ where F is a martingale under Q.

[78] This is yet another SDE! However, the fact that equation (52) is indeed the terminal wealth associated with (51) can be seen by applying (22) separately to:

$$d\pi_t^1 = \theta_t d(CF)_t^D - \frac{\theta_t C_t F_t}{V_t} dV_t^D + \frac{\pi_{t-}^1}{B_t^*} dB_t$$

and:

$$d\pi_t^2 = \psi_t dC_t^D - \frac{\psi_t C_t}{V_t} dV_t^D + \frac{\pi_{t-}^2}{B_t^*} dB_t$$

where $\pi = \pi^1 + \pi^2$, and arbitrary π_0^1 and π_0^2 such that $\pi_0 = \pi_0^1 + \pi_0^2$.

[79] Strictly speaking, a two-dimensional version of it.

where Q' is a measure under which the semi-martingales $C'Fe^{-[C'F,P]}$ and $C'e^{-[C',P]}$ are in fact martingales.

It may appear from (54) that our objective of pricing the claim with payoff $B_T C_T g(F_T)$ has been achieved. However, although it is probably fair to say that a lot of work has been done (in particular, showing that the claim is replicable under reasonable assumptions), equation (54) is not very satisfactory: C_T being *inside* the expectation, the relationship between (54) and the standard price of a European swaption or related claim (of the form $C_0 E_Q[g(F_T)]$), is not very clear. Equation (54) is also misleading, as it indicates that the distributional assumption made on C (or C') could play an important role, when in fact, the following will show that the distribution of C' only matters in as much as the terminal bracket $[C', P]_T$ is concerned: defining $Z_T = (V_0 e^{-[C',P]_T}/C_0)C_T$, using $V_T = 1$, and the fact that $C'e^{-[C',P]}$ is a Q'-martingale, we have :

$$E_{Q'}[Z_T] = \frac{V_0}{C_0} E_{Q'}[C'_T e^{-[C',P]_T}] = \frac{V_0}{C_0} C'_0 = 1 \qquad (55)$$

So Z_T is a probability density under Q', and if $dQ = Z_T dQ'$, from (54):

$$\pi_0 = B_0 \frac{C_0}{V_0} e^{[C',P]_T} E_{Q'}[Z_T g(F_T)] = P_0 C_0 e^{[C',P]_T} E_Q[g(F_T)] \qquad (56)$$

where $P_0 = B_0/V_0$ is the current survival probability with maturity T, and C_0 the current annuity of the underlying forward swap rate. The attractiveness of (56) is obvious: the non-arbitrage price π_0 of a risky swaption (or related claim), appears to be the standard price $C_0 E_Q[g(F_T)]$ multiplied by a survival probability P_0 (not a big surprise, the payoff being conditional on no default), with an additional (and by now fairly common), convexity adjustment $e^{[C',P]_T}$. The problem with equation (56), is that despite its remarkable appeal to intuition it is pretty useless, unless the distribution of F under Q is known.[80] When we said that $C_0 E_Q[g(F_T)]$ was *the standard price*, we were being economical with the truth: it is indeed *the standard price*, provided F is a martingale under Q. As far equation (56) is concerned, there is no reason why this should be the case. In fact, as shown in appendix A.6, the process $Fe^{-[F,P]}$ (and not F itself) is a martingale under Q.

So it seems that equation (56), with the knowledge that the pricing measure Q is such that $Fe^{-[F,P]}$ is a martingale, is a far better answer to our pricing problem than equation (54). And so it is. However, the road to (56) was long and tedious, making the whole argument somewhat unconvincing, with the belief that a more elegant and direct route should exist. The reason we obtained (54) instead of (56), was our choice of numeraire B: if the process BC had been a tradable process, we could have chosen BC as collapsing numeraire instead of B, giving us a terminal wealth π_T with $B_T C_T$ as a common factor (instead of just B_T). The replicating condition would have involved $g(F_T)$ (instead of $C_T g(F_T)$) and it is believable that (56) would have been derived without much more effort... The problem is that BC is not a tradable process.[81]

[80] Very often, knowing the distribution of F under Q, amounts to knowing its *drift* under Q.

[81] We can always assume anything to be tradable, but it would not make sense to do so.

One solution to the problem is to consider the SDE:

$$d\pi_t = \theta_t d(CF)_t^D - \theta_t F_t dC_t^D + \left(\frac{dB_t}{B_t^*} + \frac{dC_t^D}{C_t} - \frac{dV_t^D}{V_t}\right)\pi_{t-} \tag{57}$$

The financial interpretation of (57) could be phrased as *the SDE governing the wealth process of an investor entering into a strategy θ (up to time D) relative to CF, using C to fund his position in CF, investing his total wealth once in the collapsing numeraire B, and once in the numeraire C, using V to fund his position in B and C.* In appendix A.6, we show that the solution to (57) is given by:

$$\pi_t = \frac{B_t C_t}{V_t}\left(\frac{\pi_0 V_0}{B_0 C_0} + \int_0^t \hat{\theta}_s d\hat{F}_s\right) e^{-[C',P]_t} \tag{58}$$

where the continuous semi-martingale $\hat{F}$ is defined as $\hat{F} = Fe^{-[F,P]}$, the process $\hat{\theta}$ is defined as $\hat{\theta} = (\theta e^{[F,P]+[C',P]})/P$, the process $C' = C/V$ is the forward annuity of the underlying swap, and $P = B^*/V$ is the continuous part of the survival probability process B/V. Since $V_T = 1$, a sufficient condition for replication is:

$$\frac{\pi_0 V_0}{B_0 C_0} + \int_0^T \hat{\theta}_t d\hat{F}_t = g(F_T)e^{[C',P]_T} \tag{59}$$

and we see that the non-arbitrage price π_0 is indeed given by (56), where Q is a measure under which, the semi-martingale $\hat{F}$ is indeed a martingale. In the case of a risky swaption, we finally have:

$$\pi_0 = P_0 C_0 e^{[C',P]_T} E_Q[(F_T - K)^+] \tag{60}$$

where P_0 is the current survival probability with maturity T, and C_0 the current annuity of the underlying swap. This price is exactly the naive price, except for the adjustments $e^{[C',P]_T}$ and $e^{[F,P]_T}$ required on C_0 and F_0 respectively. The case of the delayed risky swaption is handled by applying (56) to the function $g(x) = E_Q[(F_{T'} - K)^+ | F_T = x]$. We obtain:[82]

$$\pi_0 = P_0 C_0 e^{[C',P]_T} E_Q[(F_{T'} - K)^+] \tag{61}$$

where Q is such that F is a Q-martingale, with adjusted initial value $F_0 e^{[F,P]_T}$.[83] This is also very close to the naive valuation, i.e. the standard price of the European swaption with expiry T', multiplied by the survival probability with maturity T. The only difference is the presence of the convexity adjustments $e^{[C',P]_T}$ and $e^{[F,P]_T}$, required on C_0 and F_0 respectively.

[82] Having adjusted the initial value F_0 to $F_0 e^{[F,P]_T}$, we can assume that the pricing measure Q in (56) is such that F is a Q-martingale. We have:

$$E_Q[g(F_T)] = E_Q[E_Q[(F_{T'} - K)^+ | F_T]] = E_Q[(F_{T'} - K)^+]$$

[83] Strictly speaking, $F_t e^{-[F,P]_{t\wedge T}}$ is a Q-martingale.

3.8 OTC Transaction with Possible Default

Let (T, h_T) denote a single claim with maturity T and payoff h_T. If we assume this claim to be replicable, it is meaningful to speak of its price at any point in time. In fact, such a price coincides with the value π_t at time t of the wealth process associated with the replicating strategy of the claim.[84] In general, an investor holding a long position in the claim (T, h_T) will *mark his book* at its current value π_t. This seems highly reasonable.

However, a long position in the claim (T, h_T) is most likely to be associated with an **external counterparty**, by whom the payoff h_T is meant to be paid. The fact that the external counterparty is potentially subject to default means that the payoff h_T may not be paid at all: what was thought to be a *long position in the claim* (T, h_T), is in fact a *long position in a claim* (T, h'_T) where the payoff h'_T may differ greatly from what the trade confirmation suggests. An investor marking his book at the price π_t, is not so much using the wrong price: he is rather pricing the wrong claim.

In the event of default, the payment of h_T at time T will not occur. If D denotes the time of default associated with the counterparty, the *true* payment occurring at time T is therefore $h_T 1_{\{D>T\}}$, as opposed to h_T itself. Furthermore, if $\pi_D \leq 0$, the claim after default is in fact an asset to the counterparty. The investor will have to settle his liability with a payment $-\pi_D$ on the time of default.[85]

As we can see, a trade confirmation which indicates a long position in the claim (T, h_T) to the investor, is in fact economically equivalent to a long position in the claim $(T, h_T 1_{\{D>T\}})$, together with a short position in the claim paying $(\pi_D)^-$ on the time of default.[86] We call **OTC claim** associated with the claim (T, h_T), such *economically equivalent* claim. More generally, we call OTC claim associated with a portfolio of replicable claims (T_1, h_1), ..., (T_n, h_n), the claim constituted by long positions in the single claims $(T_1, h_1 1_{\{D>T_1\}})$, ..., $(T_n, h_n 1_{\{D>T_n\}})$ together with a short position in the claim paying the negative part of the portfolio's value, on the time of default.

Since $\pi_D = (\pi_D)^+ - (\pi_D)^-$, a short position in the claim paying $(\pi_D)^-$ at time D, is equivalent to a long position in the claim paying π_D, together with a short position in the claim paying $(\pi_D)^+$. However, receiving π_D at time D is equivalent to receiving h_T at time T, provided $D \leq T$. In other words, it is equivalent to the claim paying $h_T 1_{\{D\leq T\}}$ at time T. It follows that a short position in the claim paying $(\pi_D)^-$ at time D, is equivalent to a short position in the claim paying $(\pi_D)^+$ at time D, together with a long position in the claim

[84] If π_t is of the form $\pi_t = B_t(\pi_0/B_0 + \int_0^t \hat{\theta}_s d\hat{X}_s)$, then $\pi_T = B_T(\pi_t/B_t + \int_t^T \hat{\theta}_s d\hat{X}_s)$. If the replicating condition $\pi_T = h_T$ is met, an initial investment of π_t at time t together with the replicating strategy θ (from t to T), will also replicate the claim (T, h_T). So π_t is the non-arbitrage price of the claim at time t.

[85] We are implicitly assuming no other claim is being held with this counterparty. Furthermore, if $\pi_D \geq 0$ the investor should receive a payment of $R \times (\pi_D)^+$ where R is the recovery rate associated with the counterparty. The present discussion assumes $R = 0$ but can easily be extended to account for a non-zero recovery rate.

[86] i.e. a claim paying the negative part of π_D on default. Note that $\pi_t = 0$ for $t > T$.

paying $h_T 1_{\{D \leq T\}}$ at time T. Since $h_T = h_T 1_{\{D>T\}} + h_T 1_{\{D \leq T\}}$, we conclude that the OTC claim associated with the claim (T, h_T), is equivalent to the default-free claim itself, together with a short position in the claim paying $(\pi_D)^+$ at time D. We call the claim paying $(\pi_D)^+$ at time D the **insurance claim** associated with the claim h_T.[87] (More generally, we can define an insurance claim associated with a portfolio of single claims as the claim paying the positive part $(\pi_D)^+$ of the value of the portfolio on the time of default.) As shown in the preceding argument, *the OTC claim associated with a single claim is nothing but the claim itself, together with a short position in the associated insurance claim.* This can easily be shown to be true, in the general case of multiple claims. It follows that upon entering into a first OTC transaction, an investor is not only buying the claim specified by the confirmation agreement, but is also selling the insurance claim associated with it. When viewing the whole portfolio facing a counterparty as one (multiple) claim, the investor is not only *long* the claim legally agreed, he is also *short* the insurance claim associated with his portfolio.

It should now be clear that when *marking his book* at the price π_t of the claims legally specified, an investor is being overly optimistic by ignoring his potential liability stemming from a short position in the insurance claim associated with his position. The extent of his error is precisely the price of the insurance claim[88] which has been ignored. The question of non-arbitrage pricing of the insurance claim is therefore of crucial importance, for the purpose of properly assessing the credit cost associated with a given claim. Unfortunately, it is not clear that the insurance claim should be replicable,[89] (and hence have a *price*). Furthermore, the general question of option pricing on a whole portfolio,[90] can soon become intractable.

To alleviate this last problem, the following remarks can be made: if a portfolio with value π_t is split into two separate sub-portfolios with values π_t^1 and π_t^2, then $(\pi_D)^+ \leq (\pi_D^1)^+ + (\pi_D^2)^+$. It follows that the insurance claim associated with the original portfolio, should be worth less than the sum of the two insurance claims associated with the sub-portfolios. More generally, an insurance claim should be cheaper than the sum of the insurance claims associated with any partition of the original portfolio. Hence, although the insurance claim associated with a portfolio may be nearly impossible to price, by breaking down this portfolio into smaller parts it may be possible to arrive at a valuable *upper-bound* for the price of the insurance claim, leading to a conservative (and therefore acceptable) estimate of the value of the portfolio as a whole. The ability to price the insurance claims associated with the most simple claims, can therefore turn out to be very useful. In any case, this ability would be required in the case of *first time transaction* with a new counterparty.

[87] With a non-zero recovery rate, the *insurance claim* pays $(1-R)(\pi_D)^+$ at time D.

[88] We are implicitly assuming the insurance claim is replicable. Maybe not so true...

[89] The issue is similar to that of the floating leg of a CDS: a claim paying *at the time of default* D, does not seem to be replicable in terms of a finite number of tradable instruments. It would seem that only within the framework of a *term structure model* (with a continuum of tradable zeros, for a finite number of risk factors), could such a claim be replicable.

[90] Paying the positive part $(\pi_D)^+$ looks like an option.

Furthermore, it may be argued that a counterparty of *lesser credit*, (one for which the insurance claim should not be ignored), is more likely not only to have fewer trades, but also to have trades with cumulative (rather than netting) effects on the risk. When this is the case, approximating the insurance claim of a portfolio by those of its constituents, will lead to a lesser discrepancy.

In order to deal with the issue of a non-replicable insurance claim, we may resort to the same approximation as that of section (3.3): by discretising the time interval $]0, T]$ between now and the maturity of a claim, into smaller intervals $]t_{i-1}, t_i]$ for $0 = t_0 < \ldots < t_n = T$, it seems reasonable view the insurance claim, as paying $(\pi_{t_i})^+$ at time t_i, provided default occurs in the interval $]t_{i-1}, t_i]$. In other words, we may approximate the insurance claim, as a portfolio of single claims $C_1, \ldots, C_n$, where each C_i has maturity t_i, and payoff $(\pi_{t_i})^+ 1_{\{t_{i-1} < D \leq t_i\}}$. In fact, since $1_{\{t_{i-1} < D \leq t_i\}} = 1_{\{D > t_{i-1}\}} - 1_{\{D > t_i\}}$, each single claim C_i can be exactly expressed as a long position in the claim with payoff $(\pi_{t_i})^+ 1_{\{D > t_{i-1}\}}$, together with a short position in the claim with payoff $(\pi_{t_i})^+ 1_{\{D > t_i\}}$.

For example, when the underlying claim is just an ordinary interest rate swap, the two claims $(\pi_{t_i})^+ 1_{\{D > t_{i-1}\}}$ and $(\pi_{t_i})^+ 1_{\{D > t_i\}}$ are respectively a *delayed risky swaption* and *risky swaption*, as defined in section 3.7. The insurance claim associated with an interest rate swap, can therefore be reasonably approximated and priced. However, the problem is slightly more complicated than suggested here. Strictly speaking, the swap underlying each risky swaption is not a forward starting swap, but a swap with slightly more complex features: Assuming the dates $t_0, \ldots, t_n$ have been chosen to match the floating schedule of the original swap, each floating payment (maybe associated with a fixed payment) occurring at time t_i has to be incorporated as part of the underlying swap of the two risky swaptions with expiry t_i. Obviously, if default was to occur in the interval $]t_{i-1}, t_i]$, the coupon payments due at time t_i would have a big impact on the mark-to-market of the original swap and cannot be ignored. It follows that the two risky swaptions are strictly speaking *swaptions with penalty*,[91] rather than normal swaptions. Furthermore, in the very common case when the frequency of the floating leg is higher than that of the fixed leg, the swap underlying each risky swaption pays a full first coupon on the fixed leg. This is equivalent to a normal swap (i.e. with a short first coupon), with an additional *penalty* paid in the near future.[92] This new difficulty cannot be ignored: the mismatch in frequency between a floating and fixed leg of a swap, is one of the major factors on its market-to-market. One can easily believe that the insurance claim associated with receiving annual vs 3s, should be significantly more expensive than that associated with paying annual vs 3s...

Despite these difficulties, the risky swaptions can easily be approximated by translating the *penalties* into an adjustment to the strike. Formula (60) and (61) can then be used to derive the price of each single claim C_i, and finally obtain the price of the insurance claim associated with an interest rate swap.

[91] The penalty is strictly speaking path-dependent, linked to the last floating fixing.
[92] This penalty bears a small discounting risk.

A Appendix

A.1 SDE for Cash-Tradable Asset and one Numeraire

In this appendix, we show how Ito's lemma can be used, to check that equation (9) is indeed a solution of the SDE (8). Taking $\psi = 0$, this will also prove that equation (7) is a solution of the SDE (6). Equation (9) can be written as $\pi_t = B_t C_t$, where C is the semi-martingale defined by:

$$C_t = \frac{\pi_0}{B_0} + \int_0^t \theta_s d\hat{X}_s + \int_0^t \psi_s d\hat{Y}_s \tag{62}$$

¿From Ito's lemma, we have:

$$d\pi_t = C_t dB_t + B_t dC_t + d\langle B, C\rangle_t \tag{63}$$

where $\langle B, C\rangle$ is the cross-variation between B and C. From (62), we obtain:

$$d\langle B, C\rangle_t = \theta_t d\langle B, \hat{X}\rangle_t + \psi_t d\langle B, \hat{Y}\rangle_t \tag{64}$$

and furthermore:

$$B_t dC_t = \theta_t B_t d\hat{X}_t + \psi_t B_t d\hat{Y}_t \tag{65}$$

Applying Ito's lemma once more to $X = B\hat{X}$ and $Y = B\hat{Y}$, we have:

$$\theta_t B_t d\hat{X}_t + \theta_t d\langle B, \hat{X}\rangle_t = \theta_t dX_t - \theta_t \hat{X}_t dB_t \tag{66}$$

and:

$$\psi_t B_t d\hat{Y}_t + \psi_t d\langle B, \hat{Y}\rangle_t = \psi_t dY_t - \psi_t \hat{Y}_t dB_t \tag{67}$$

Adding (64) together with (65), using (66) and (67), we obtain:

$$B_t dC_t + d\langle B, C\rangle_t = \theta_t dX_t - \theta_t \hat{X}_t dB_t + \psi_t dY_t - \psi_t \hat{Y}_t dB_t \tag{68}$$

and finally from (63):

$$d\pi_t = \frac{\pi_t}{B_t} dB_t + \theta_t dX_t + \psi_t dY_t - \theta_t \hat{X}_t dB_t - \psi_t \hat{Y}_t dB_t \tag{69}$$

which in light of $\hat{X} = X/B$ and $\hat{Y} = Y/B$, shows that SDE (8) is satisfied by π.

A.2 SDE for Futures-Tradable Asset and one Numeraire

In this appendix, we show how Ito's lemma can be used, to check that equation (11) is indeed a solution of the SDE (10). Equation (11) can be written as $\pi_t = B_t C_t$, where C is the semi-martingale defined by:

$$C_t = \frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d\hat{X}_s \tag{70}$$

¿From Ito's lemma, we have:

$$d\pi_t = C_t dB_t + B_t dC_t + d\langle B, C\rangle_t \tag{71}$$

where $\langle B, C\rangle$ is the cross-variation between B and C. From (70), we obtain:

$$d\langle B, C\rangle_t = \hat{\theta}_t d\langle B, \hat{X}\rangle_t \tag{72}$$

and furthermore:

$$B_t dC_t = \hat{\theta}_t B_t d\hat{X}_t \tag{73}$$

Applying Ito's lemma to $X = \hat{X} e^{[X,B]}$:

$$dX_t = X_t d[X, B]_t + e^{[X,B]_t} d\hat{X}_t \tag{74}$$

and in particular:

$$d\langle B, \hat{X}\rangle_t = e^{-[X,B]_t} d\langle B, X\rangle_t \tag{75}$$

From (72), (75) and the fact that $\hat{\theta} = (\theta e^{[X,B]})/B$, we obtain:[93]

$$d\langle B, C\rangle_t = \frac{\theta}{B_t} d\langle B, X\rangle_t = \theta_t X_t d[X, B]_t \tag{76}$$

Furthermore from (73):

$$B_t dC_t = \theta_t e^{[X,B]_t} d\hat{X}_t \tag{77}$$

Finally, from (77), (76) and (74):

$$B_t dC_t + d\langle B, C\rangle_t = \theta_t dX_t \tag{78}$$

We conclude from (78) and (71) that:

$$d\pi_t = \frac{\pi_t}{B_t} dB_t + \theta_t dX_t \tag{79}$$

which is exactly the SDE (10).

[93] Recall that the bracket $[X, B]$ is defined as $[X, B]_t = \int_0^t X_s^{-1} B_s^{-1} d\langle X, B\rangle_s$.

A.3 SDE for Funded Asset and one Numeraire

In this appendix, we show that equation (9) reduces to equation (13) in the case when $\psi = -\theta X/Y$. Equation (9) can be written as:

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \int_0^t \theta_s d(X/B)_s - \int_0^t \frac{\theta_s X_s}{Y_s} d(Y/B)_s \right) \tag{80}$$

Using the fact that $X/B = (X/Y) \times (Y/B) = X'/B'$ and $\hat{X} = X' e^{-[X',B']}$:

$$\begin{aligned} d(X/B)_t - \frac{X_t}{Y_t} d(Y/B)_t &= \frac{1}{B'_t} dX'_t + d\langle X', 1/B' \rangle_t \\ &= \frac{1}{B'_t} \left(dX'_t - \frac{1}{B'_t} d\langle X', B' \rangle_t \right) \\ &= \frac{1}{B'_t} (dX'_t - X'_t d[X', B']_t) \\ &= \frac{1}{B'_t} e^{[X',B']_t} d\hat{X}_t \end{aligned}$$

and from $\hat{\theta} = (\theta e^{[X',B']})/B'$, we conclude from (80) that:

$$\pi_t = B_t \left(\frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d\hat{X}_s \right) \tag{81}$$

which is the same as equation (13).

A.4 SDE for Funded Asset and one Collapsing Numeraire

In this appendix, we outline the proof that equation (22) is a solution of the SDE (21). A major difficulty in doing so, is the use of stochastic calculus within the framework of potentially discontinuous semi-martingales. Equation (22) can be written as $\pi_t = B_t C_t$, where C is the continuous semi-martingale defined by:

$$C_t = \frac{\pi_0}{B_0} + \int_0^t \hat{\theta}_s d\hat{X}_s \tag{82}$$

Applying Ito's lemma, we have:[94]

$$d\pi_t = C_{t-} dB_t + B_{t-} dC_t + 1_{\{t \le D\}} d\langle B^*, C \rangle_t + \Delta B_t \Delta C_t \tag{83}$$

Since C is continuous, $\Delta B_t \Delta C_t = 0$. Moreover, since $dB_t = 0$ for $t > D$[95] and $C_{t-} = \pi_{t-}/B^*_t$ for $t \le D$, we can replace C_{t-} by π_{t-}/B^*_t in (83). Furthermore,

[94] See [2], Th. (38.3) p.392. See also Def. (37.6) p. 389. Do note confuse the notation $[X, Y]$ in this reference, with *our* bracket. Note that the continuous part of B, as understood by this reference, is the stopped process $B^*_{t \wedge D}$, which explains the $1_{\{t \le D\}}$ in equation (83).

[95] A casual way of saying that $\int_0^t \psi_s dB_s = \int_0^D \psi_s dB_s$ for all $t > D$ and all ψ.

since $B_{t-} = 0$ for $t > D$, and $B_{t-} = B^*_t$ for $t \leq D$, we can replace B_{t-} by $B^*_t 1_{\{t\leq D\}}$ in (83). We obtain:

$$d\pi_t = \frac{\pi_{t-}}{B^*_t} dB_t + B^*_t 1_{\{t\leq D\}} dC_t + 1_{\{t\leq D\}} d\langle B^*, C\rangle_t \tag{84}$$

From (82), we have $dC_t = \hat{\theta}_t d\hat{X}_t$ and consequently:

$$\begin{aligned} B^*_t 1_{\{t\leq D\}} dC_t &= B^*_t \hat{\theta}_t 1_{\{t\leq D\}} d\hat{X}_t \\ &= Y_t \theta_t e^{[X',B']_t} 1_{\{t\leq D\}} d\hat{X}_t \end{aligned} \tag{85}$$

Furthermore:

$$1_{\{t\leq D\}} d\langle B^*, C\rangle_t = \hat{\theta}_t 1_{\{t\leq D\}} d\langle B^*, \hat{X}\rangle_t \tag{86}$$

¿From $\hat{X} = X' e^{-[X',B']}$, we obtain:

$$d\hat{X}_t = -\hat{X}_t d[X', B']_t + e^{-[X',B']_t} dX'_t \tag{87}$$

It follows that:

$$d\langle B^*, \hat{X}\rangle_t = e^{-[X',B']_t} d\langle B^*, X'\rangle_t \tag{88}$$

Combining (88) with (86), we obtain:

$$1_{\{t\leq D\}} d\langle B^*, C\rangle_t = \hat{\theta}_t e^{-[X',B']_t} 1_{\{t\leq D\}} d\langle B^*, X'\rangle_t \tag{89}$$

Furthermore, combining (87) with (85), we obtain:

$$\begin{aligned} B^*_t 1_{\{t\leq D\}} dC_t &= -\hat{X}_t Y_t \theta_t e^{[X',B']_t} 1_{\{t\leq D\}} d[X', B']_t + \theta_t Y_t 1_{\{t\leq D\}} dX'_t \\ &= -\frac{\theta_t Y_t}{B'_t} 1_{\{t\leq D\}} d\langle X', B'\rangle_t + \theta_t Y_t 1_{\{t\leq D\}} dX'_t \end{aligned} \tag{90}$$

Applying Ito's lemma to $B^* = B'Y$, we have:

$$dB^*_t = B'_t dY_t + Y_t dB'_t + d\langle B', Y\rangle_t \tag{91}$$

and in particular:

$$d\langle B^*, X'\rangle_t = B'_t d\langle Y, X'\rangle_t + Y_t d\langle X', B'\rangle_t \tag{92}$$

Combining (92) and (89), we obtain:

$$1_{\{t\leq D\}} d\langle B^*, C\rangle_t = \theta_t 1_{\{t\leq D\}} d\langle Y, X'\rangle_t + \frac{\theta_t Y_t}{B'_t} 1_{\{t\leq D\}} d\langle X', B'\rangle_t \tag{93}$$

Adding (90) with (93), we obtain:

$$\begin{aligned} B^*_t 1_{\{t\leq D\}} dC_t + 1_{\{t\leq D\}} d\langle B^*, C\rangle_t &= \theta_t Y_t 1_{\{t\leq D\}} dX'_t + \theta_t 1_{\{t\leq D\}} d\langle Y, X'\rangle_t \\ &= \theta_t 1_{\{t\leq D\}} \left(Y_t dX'_t + d\langle Y, X'\rangle_t\right) \\ &= \theta_t 1_{\{t\leq D\}} \left(dX_t - \frac{X_t}{Y_t} dY_t\right) \\ &= \theta_t dX^D_t - \frac{\theta_t X_t}{Y_t} dY^D_t \end{aligned} \tag{94}$$

Comparing (94) with (84), we conclude that:

$$d\pi_t = \frac{\pi_{t-}}{B_t^*} dB_t + \theta_t dX_t^D - \frac{\theta_t X_t}{Y_t} dY_t^D \tag{95}$$

which is exactly the SDE (21).

A.5 SDE for Collapsing Asset and Numeraire

In this appendix, we outline the proof that equation (46) is a solution of the SDE (44). Equation (46) can be written as $\pi_t = B_t C_t$, where C is the continuous semi-martingale defined by:

$$C_t = \frac{\pi_0}{B_0} + \int_0^t \theta_s d\hat{X}_s \tag{96}$$

Applying Ito's lemma, we have:

$$d\pi_t = C_{t-} dB_t + B_{t-} dC_t + 1_{\{t \leq D\}} d\langle B^*, C\rangle_t + \Delta B_t \Delta C_t \tag{97}$$

Since C is continuous, $\Delta B_t \Delta C_t = 0$. Moreover, since $dB_t = 0$ for $t > D$ and $C_{t-} = \pi_{t-}/B_t^*$ for $t \leq D$, we can replace C_{t-} by π_{t-}/B_t^* in (97). Furthermore, since $B_{t-} = 0$ for $t > D$ and $B_{t-} = B_t^*$ for $t \leq D$, we can replace B_{t-} by $B_t^* 1_{\{t \leq D\}}$ in (97). We obtain:

$$d\pi_t = \frac{\pi_{t-}}{B_t^*} dB_t + B_t^* 1_{\{t \leq D\}} dC_t + 1_{\{t \leq D\}} d\langle B^*, C\rangle_t \tag{98}$$

¿From (96), we have $dC_t = \theta_t d\hat{X}_t$, and consequently:

$$B_t^* 1_{\{t \leq D\}} dC_t = B_t^* \theta_t 1_{\{t \leq D\}} d\hat{X}_t \tag{99}$$

Furthermore:

$$1_{\{t \leq D\}} d\langle B^*, C\rangle_t = \theta_t 1_{\{t \leq D\}} d\langle B^*, \hat{X}\rangle_t \tag{100}$$

Applying Ito's lemma to $X^* = \hat{X} B^*$, we obtain:

$$dX_t^* - \frac{X_t^*}{B_t^*} dB_t^* = B_t^* d\hat{X}_t + d\langle B^*, \hat{X}\rangle_t \tag{101}$$

Adding (99) and (100), and comparing with (101):

$$B_t^* 1_{\{t \leq D\}} dC_t + 1_{\{t \leq D\}} d\langle B^*, C\rangle_t = \theta_t 1_{\{t \leq D\}} \left(dX_t^* - \frac{X_t^*}{B_t^*} dB_t^* \right) \tag{102}$$

¿From (102) and (98), we obtain:

$$d\pi_t = \frac{\pi_{t-}}{B_t^*} dB_t + \theta_t 1_{\{t \leq D\}} \left(dX_t^* - \frac{X_t^*}{B_t^*} dB_t^* \right) \tag{103}$$

Defining $I_t = 1_{\{t<D\}}$ and applying Ito's lemma to $X = X^* I$:

$$dX_t = I_{t-} dX_t^* + X_{t-}^* dI_t = 1_{\{t \le D\}} dX_t^* + X_t^* dI_t \tag{104}$$

and similarly, since $B = B^* I$:

$$dB_t = 1_{\{t \le D\}} dB_t^* + B_t^* dI_t \tag{105}$$

¿From (104) and (105), we obtain:

$$1_{\{t \le D\}} \left(dX_t^* - \frac{X_t^*}{B_t^*} dB_t^* \right) = dX_t - \frac{X_t^*}{B_t^*} dB_t \tag{106}$$

and comparing (106) and (103):

$$d\pi_t = \frac{\pi_{t-}}{B_t^*} dB_t + \theta_t \left(dX_t - \frac{X_t^*}{B_t^*} dB_t \right) \tag{107}$$

Since $dB_t = 0$ for $t > D$ and $X_t^* = X_{t-}$ for $t \le D$, we can replace X_t^* by X_{t-} in equation (107), obtaining the SDE (44).

A.6 Change of Measure and New SDE for Risky Swaption

We assume that $C' F e^{-[C'F,P]}$ and $C' e^{-[C',P]}$ are martingales under Q' and define $dQ = Z_T dQ'$, where $Z_T = (V_0 e^{-[C',P]_T} / C_0) C_T$. We claim that $F e^{-[F,P]}$ is a martingale under Q. Let $(\mathcal{F}_t)_{t \ge 0}$ be our filtration, let $s \le t$ and $A \in \mathcal{F}_s$. Since $V_T = 1$, we have:

$$\begin{aligned}
E_Q[1_A F_t] &= \frac{V_0}{C_0} E_{Q'}[1_A F_t C_T' e^{-[C',P]_T}] && (108)\\
&= \frac{V_0}{C_0} E_{Q'}[1_A F_t C_t' e^{-[C',P]_t}] && (109)\\
&= \frac{V_0}{C_0} e^{[F,P]_t} E_{Q'}[1_A C_t' F_t e^{-[C'F,P]_t}] && (110)\\
&= \frac{V_0}{C_0} e^{[F,P]_t} E_{Q'}[1_A C_s' F_s e^{-[C'F,P]_s}] && (111)\\
&= \frac{V_0}{C_0} e^{([F,P]_t - [F,P]_s)} E_{Q'}[1_A F_s C_T e^{-[C',P]_T}] && (112)\\
&= e^{([F,P]_t - [F,P]_s)} E_Q[1_A F_s] && (113)
\end{aligned}$$

where (109) was obtained from the fact that $1_A F_t$ is measurable w.r. to $\mathcal{F}_t$ and $C' e^{-[C',P]}$ is a martingale under Q', (111) was obtained from the fact that 1_A is measurable w.r. to $\mathcal{F}_s$ and $C' F e^{-[C'F,P]}$ is a martingale under Q', and (112) was obtained from the fact that $1_A F_s$ is measurable w.r. to $\mathcal{F}_s$ and $C' e^{-[C',P]}$ is a martingale under Q'. We conclude that $E_Q[F_t e^{-[F,P]_t} | \mathcal{F}_s] = F_s e^{-[F,P]_s}$, and $F e^{-[F,P]}$ is indeed a martingale under Q.

The fact that equation (58) is a solution of the SDE (57), can be seen from the following equation:[96]

$$\frac{dB_t}{B_t^*} + \frac{dC_t^D}{C_t} - \frac{dV_t^D}{V_t} = \frac{V_t}{B_t^* C_t} d\left(\frac{BC}{V}\right)_t - 1_{\{t \le D\}} d\left[\frac{C}{V}, \frac{B^*}{V}\right]_t \tag{114}$$

which allows (57) to be re-expressed as:

$$d\pi_t + 1_{\{t \le D\}} \pi_{t-} d[C', P]_t = \theta_t d(CF)_t^D - \theta_t F_t dC_t^D + \frac{V_t \pi_{t-}}{B_t^* C_t} d(BC/V)_t \tag{115}$$

or equivalently:

$$d\hat{\pi}_t = \theta_t^* d(CF)_t^D - \theta_t^* F_t dC_t^D + \frac{V_t \hat{\pi}_{t-}}{B_t^* C_t} d(BC/V)_t \tag{116}$$

where we have put $\hat{\pi}_t = \pi_t e^{[C',P]_{t\wedge D}}$ and $\theta_t^* = \theta_t e^{[C',P]_{t\wedge D}}$. Equation (116) being formally identical to (21), we see from (22) that:

$$\hat{\pi}_t = \frac{B_t C_t}{V_t} \left(\frac{\pi_0 V_0}{B_0 C_0} + \int_0^t \hat{\theta}_s d\hat{F}_s \right) \tag{117}$$

where the semi-martingale $\hat{F}$ is defined as $\hat{F} = Fe^{-[F,P]}$,[97] and the process $\hat{\theta}$ is defined as $\hat{\theta} = (\theta^* e^{[F,P]})/P$. Finally, we obtain:

$$\pi_t = \frac{B_t C_t}{V_t} \left(\frac{\pi_0 V_0}{B_0 C_0} + \int_0^t \hat{\theta}_s d\hat{F}_s \right) e^{-[C',P]_{t\wedge D}} \tag{118}$$

and since $B_t = 0$ for $t > D$, we can *drop* the $t \wedge D$ in $[C', P]_{t\wedge D}$, yielding (58).

References

[1] Karatzas, I. and Shreve, S.E. (1991). Brownian motion and stochastic calculus. 2nd Ed. Springer-Verlag

[2] Rogers, L.C.G and Williams, D. (1987). Diffusions, markov processes, and martingales. (Volume 2) Ed. Wiley

[96] Derived from Ito's lemma.
[97] Since $(B^*C/V)/C = B^*/V = P$.