Convexity Adjustment between Futures and Forward Rates Using a Martingale Approach

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1 Introduction

The purpose of this report is to describe the question of the *convexity adjustment* needed to convert a forward rate to its corresponding futures rate. Because of the marking to market of any profit and loss on a futures position, strictly speaking futures and forward contracts do not provide equal payoffs. It is therefore not surprising that futures and forward rates should be different.

The theoretical results presented in this report are due to Paul Doust [1]. However, we shall resort to a slightly different approach, making use of martingales as opposed to PDE's, and making small adjustments to distributional assumptions. It is reassuring to see that whichever way one looks at it, the same *convexity adjustment* is obtained.

This report can be divided in two parts. We shall first derive a theoretical formula for the *convexity adjustment*. A second part will show how to approximate such formula, and provide comments on the results obtained, after a simple spreadsheet implementation.

2 Theoretical Derivation

2.1 The Underlying Principle

Let T and $T+\Delta T$ be the starting and end dates of a forward period. We denote L_t the forward rate between T and $T+\Delta T$ at time t, and F_t the futures rate at time t corresponding to the same period. Note that both rates L_t and F_t will converge at time T to the then prevailing money market rate with maturity ΔT, so that $L_T = F_T$.

Let V_t denote the $T+\Delta T$ discount factor at time t. A forward contract struck at a rate K is a contingent claim with final payoff at time T equal to:

$$\Pi_T = \alpha V_T (L_T - K) \tag{1}$$

where α denotes the day count fraction between T and $T + \Delta T$.

The value today ($t = 0$) of such a forward contract is given by:

$$\Pi_0 = \alpha V_0 (L_0 - K) \tag{2}$$

and as we can see, this value is a function of the current discount factor V_0 and forward rate L_0. However, since $L_T = F_T$, the final payoff Π_T could also have been written as:

$$\Pi_T = \alpha V_T (F_T - K) \tag{3}$$

This important point, together with the fact that futures contracts can actually be traded, will enable to show that the current value Π_0 of our forward contract is also a function of V_0 and the current futures rate F_0, i.e.

$$\Pi_0 = f(V_0, F_0) \tag{4}$$

for some appropriate function f. It can therefore be seen from (2) and (4) that the current forward rate L_0 and its corresponding futures rate F_0 are linked together by:

$$\alpha V_0 (L_0 - K) = f(V_0, F_0) \tag{5}$$

In general, the function f is not given by $\alpha V_0 (F_0 - K)$, and L_0 is not equal to F_0. Determining the explicit form of the function f will enable us through (5), to determine the exact link between F_0 and L_0, which is the so called *convexity adjustment*.

2.2 Valuing a FRA Using Futures

Determining $f(V_0, F_0)$ amounts to valuing a forward contract viewed as a contingent claim with final payoff (3). In order to do that, we shall call $v_0 = f(V_0, F_0)$ the unknown premium to be determined. We consider an investor receiving an initial ($t = 0$) amount of cash equal to v_0, and engaging in a continuous trading strategy $\theta = (\theta_t)$ in the futures contract,[1] where all cash is reinvested in the discount bond V_t. If we call π_t the value of the investor's portfolio at time t, then the process $\pi = (\pi_t)$ is given by $\pi_0 = v_0$, and the stochastic differential equation: [2]

$$d\pi_t = \theta_t dF_t + \frac{\pi_t}{V_t} dV_t \tag{6}$$

In other words, a variation $d\pi_t$ in the portfolio's value arises due to variations dF_t and dV_t, and the two long positions θ_t and π_t / V_t in F_t and V_t respectively. The solution to (6), given the initial condition $\pi_0 = v_0$, can be expressed as:[3]

$$\pi_t = V_t \left(\frac{v_0}{V_0} + \int_0^t \hat{\theta}_t d\hat{F}_t \right) \tag{7}$$

[1] At time t, the investor has a long position θ_t in the rate F_t, which actually corresponds to a short position in terms of contracts.

[2] Note that by writing (6), we have neglected the effect of minimum margin requirements. In real life, an investor entering a futures contract could not reinvest the totality of his profits in the discount bond, since some of his cash has to be left on his margin account.

[3] See appendix A.

where:

$$\hat{F}_t \triangleq F_t/C_t \tag{8}$$

$$\hat{\theta}_t \triangleq \theta_t C_t/V_t \tag{9}$$

and the process $C = (C_t)$ has been defined as:

$$C_t \triangleq \exp\left(\int_0^t \frac{1}{F_s V_s} d\langle F, V\rangle_s\right) \tag{10}$$

In particular, our investor will have a final wealth at time T equal to:

$$\pi_T = \pi_T(v_0, \theta) = V_T\left(\frac{v_0}{V_0} + \int_0^T \hat{\theta}_t d\hat{F}_t\right) \tag{11}$$

This final wealth is obviously a function of the initial premium v_0 and trading strategy θ. Now, suppose for a moment that we could find v_0 and θ such that:

$$\pi_T(v_0, \theta) = \alpha V_T(F_T - K) \tag{12}$$

Then, an investor receiving an initial cash payment of v_0 and entering the strategy θ, will exactly generate a final wealth equal to the final payoff of our forward contract. In other words, an initial investment together with adequate trading, enables the exact replication of a forward contract payoff. To avoid any possibility of arbitrage, the value of this forward contract has to be the initial investment v_0. Hence, if we can find v_0 and θ satisfying (12), then we know that v_0 is exactly the premium that we are looking for.

Our problem of finding v_0 can now be rephrased in terms of the following questions:

1. Do there exist v_0 and θ such that (12) holds?

2. If so, how do we calculate v_0?

Of course, the answer to these questions will very much depend on the particular assumptions made on the processes $V = (V_t)$ and $F = (F_t)$. In general, it is not true that v_0 and θ always exist, and if they do, actually computing v_0 can be quite tedious. However, without (for now) being more specific on V and F, we can indicate the general procedure enabling to get answers to the above questions: firstly, comparing (12) with (11) shows that v_0 and θ should satisfy the equation:

$$\frac{v_0}{V_0} + \int_0^T \hat{\theta}_t d\hat{F}_t = \alpha(F_T - K) \tag{13}$$

Now, let us assume that there exists a probability measure Q, under which the process $\hat{F} = (\hat{F}_t)$ (as defined in (8)) is a martingale,[4] and furthermore, that

[4] See appendix C for the proof of such existence, (provided we make the right assumptions). Do not be put off by the terminology here: everything you need to know is recalled below.

the *martingale representation theorem* can actually be applied:[5] this theorem states the existence of a constant x_0 together with a process $\phi = (\phi_t)$ such that:

$$x_0 + \int_0^T \phi_t d\hat{F}_t = \alpha(F_T - K) \tag{14}$$

Of course, we do not know explicitly what x_0 and ϕ are. But we are only interested in their existence: for once we know that x_0 and ϕ do exist, then defining $v_0 \stackrel{\triangle}{=} x_0 V_0$ and $\theta_t \stackrel{\triangle}{=} V_t \phi_t / C_t$, equation (14) can be rewritten as (13), which shows the existence of a premium v_0 and a strategy θ satisfying equation (12). This is the answer to the above first question.

Having answered question 1, we are now left with the task of actually computing v_0. As we shall see, there is very little to it: indeed, the nice thing about $\hat{F} = (\hat{F}_t)$ being a martingale under Q, is that we can always write:[6]

$$E_Q\left[\int_0^T \hat{\theta}_t d\hat{F}_t\right] = 0 \tag{15}$$

and taking Q-expectation on both sides of (13), we therefore obtain:

$$v_0 = \alpha V_0 (E_Q[F_T] - K) \tag{16}$$

which shows that computing v_0 amounts to the computation of the Q-expectation $E_Q[F_T]$. In general, this expectation can be quite difficult to obtain explicitly. However, if the assumptions made on the processes F and V are such that the process $C = (C_t)$ as defined in (10) is actually deterministic,[7] then we have the following:[8]

$$E_Q[F_T] = E_Q[\hat{F}_T C_T] = C_T E_Q[\hat{F}_T] = C_T F_0 \tag{17}$$

which can be substituted into (16) in order to obtain:

$$f(V_0, F_0) \stackrel{\triangle}{=} v_0 = \alpha V_0 (C_T F_0 - K) \tag{18}$$

This completes our task of answering questions 1 and 2. It should be remembered however, that before deriving anything like (18), some assumptions had to be made. In other words, taking just any kind of diffusion for the processes F and V will inevitably lead to the collapse of the previous developments. When confronted with the task of designing our financial model, three fundamental points have to be kept in mind:[9]

[5] See appendix D for the proof of that.

[6] We are being slightly over optimistic here. In reality, some integrability condition has to be met by $\hat{\theta}$. See appendix D.

[7] This looks like we have an additional requirement on F and V. In fact, the assumption of C being deterministic is also needed to ensure that the *martingale representation theorem* can be applied. See appendix D

[8] $\hat{F}$ being a martingale under Q, (and F_0 being constant), $E_Q[\hat{F}_T] = E_Q[\hat{F}_0] = F_0$.

[9] As already mentioned, point 3 is in fact a prerequisite to point 2.

1. We need a probability measure Q, under which $\hat{F}$ is a martingale.
2. The *martingale representation theorem* must be applicable.
3. The process $C = (C_t)$ should be deterministic.

2.3 The Convexity Adjustment

In the previous section, we were able to explicitly determine $f(V_0, F_0)$ by equation (18). Looking back at (5), it appears that the forward rate L_0 and futures rate F_0 satisfy the equation:

$$\alpha V_0(L_0 - K) = \alpha V_0(C_T F_0 - K) \tag{19}$$

from which we conclude that:

$$L_0 = C_T F_0 \tag{20}$$

In other words, the forward rate L_0 is equal to the futures rate F_0 times a convexity adjustment C_T given by:[10]

$$C_T = \exp\left(\int_0^T \frac{1}{F_t V_t} d\langle F, V\rangle_t\right) \tag{21}$$

In order to give a more explicit formulation of C_T, it is now time to be more specific about the processes $F = (F_t)$ and $V = (V_t)$. As detailed in appendix B, the chosen diffusion for F and V are:

$$dF_t = \mu(t)F_t dt + \sigma_F(t)F_t dW_t \tag{22}$$

$$V_t \stackrel{\triangle}{=} \exp(-(T + \Delta T - t)R_t) \tag{23}$$

$$dR_t = \gamma(R_\infty - R_t)dt + \sigma_R(t)R_\infty dW''_t \tag{24}$$

with $F_0, R_0 > 0$, where γ, R_∞ are strictly positive constants, and all processes μ, σ_F, σ_R are deterministic. It is of course understood that W and W'' in (22) and (24) are standard brownian motions. Furthermore, we assume that W and W'' have deterministic correlation $\rho(t)$.

In appendix B, we show that given (22), (23) and (24), the convexity adjustment C_T can be expressed as:[11]

$$C_T = \exp\left(-R_\infty \int_0^T (T + \Delta T - t)\sigma_R(t)\sigma_F(t)\rho(t)dt\right) \tag{25}$$

[10] There is no particular reason to call C_T a *convexity adjustment*, apart from current practice.

[11] Paul Doust [1] assumes log-normal diffusion for both F and V, with deterministic correlation $\rho_{F,V}$. In this case we obtain:

$$C_T = \exp\left(\int_0^T \sigma_V(t)\sigma_F(t)\rho_{F,V}(t)dt\right)$$

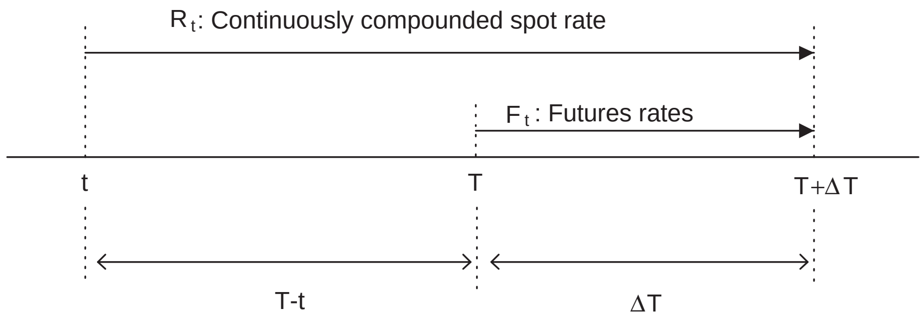

Figure 1: $\rho(t) = e^{-\delta(T-t)/\Delta T}$ is assumed to be the correlation between the futures rate F_t and continuously compounded spot rate R_t.

3 Practical Results

3.1 Approximating the Convexity Adjustment

In the previous section, we obtained formula (25), giving the convexity adjustment needed to convert a futures rate to its corresponding forward rate. As we can see, some additional assumptions have to be made on $\sigma_R(t)$,$\sigma_F(t)$ and $\rho(t)$ in order to compute the integral in (25) explicitly. Following Paul Doust in [2], we shall put:

$$\forall t \in \mathbf{R}^+ \ , \ \sigma_R(t) = \sigma_F(t) = \sigma \tag{26}$$

where σ is meant to represent some sort of *average volatility* for rates. This approximation could obviously be improved: it is widely acknowledged that volatilities for long rates are usually lower than short term volatilities. Hence, $\sigma_R(t)$ could be chosen to be an increasing function of time. As we shall see, given (26), the sensitivity of the convexity adjustment (25) with respect to the parameter σ (and indeed w.r. to R_∞), will not appear to be significant compared to the sensitivity with respect to our correlation input. The latter will be chosen to be of the form:

$$\rho(t) = \exp\left(-\delta\frac{(T-t)}{\Delta T}\right) \tag{27}$$

There is of course no *true* answer to the question of estimating the correlation ρ.[12] However, we believe that formula (27) displays some interesting features, which may be worth pointing out:

Firstly, assumption (27) has the simplicity of having only one parameter, the *decorrelation factor* δ, to describe the whole structure of correlation $\rho(t)$. Also, as t tends to the maturity T, $\rho(t)$ is increasing to 1, which is exactly what we should expect.[13] Furthermore, formula (27) ensures that the two rates F_t and

[12] Paul Doust in [2] assumes $\rho(t) = 1 - \delta(T-t)$.

[13] As t tends to T, the spot rate R_t is getting more and more *in line* with the futures rate F_t. In the limit, we have: $e^{-R_T\Delta T} = (1 + \alpha F_T)^{-1}$

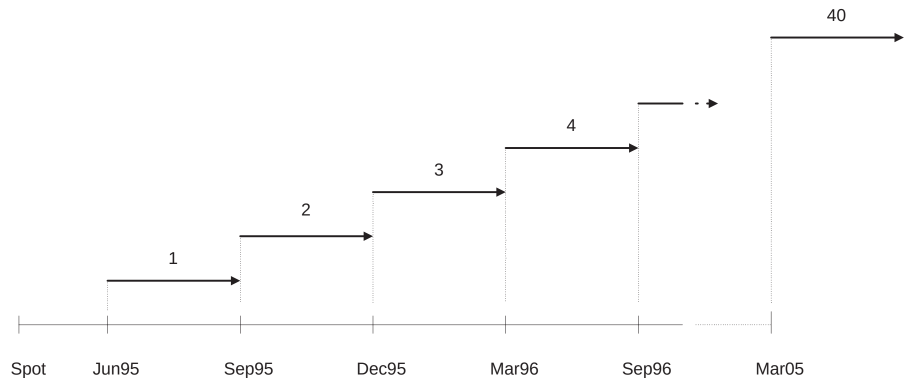

Figure 2: Each of the 40 forward periods is between two points of the IMM grid.

R_t are always positively correlated. Finally, as the forward interval ΔT goes to infinity, the relative weight of the period $T-t$ compared to $(T+\Delta T-t)$ is getting smaller and smaller. Hence, one would expect the corresponding correlation to increase to the value 1, as is indeed the case with formula (27).

Having made assumptions (26) and (27), the computation of the convexity adjustment (25) is just a simple exercise. We obtain:

$$C_T = \exp\left[-\frac{\sigma^2 R_\infty (\Delta T)^2}{\delta^2}\left((\delta+1)\left(1-e^{-\delta T/\Delta T}\right)-\left(\frac{\delta T}{\Delta T}\right)e^{-\delta T/\Delta T}\right)\right] \quad (28)$$

Note that in the limit case where F_t and R_t are perfectly correlated, i.e. where the *decorrelation factor* δ is zero, we have:

$$C_T = \exp\left[-\sigma^2 R_\infty (\Delta T)^2 \left(\frac{T}{\Delta T} + \frac{1}{2}\left(\frac{T}{\Delta T}\right)^2\right)\right] \quad (29)$$

Formulas (28)and (29) can easily be implemented on any spreadsheet. In the next section, we discuss the results following such implementation.

3.2 Spreadsheet Implementation for Eurodollars

We have applied formula (28) to the Eurodollars market. There are currently 40 futures contracts being traded, which gives 40 forward periods, as figure 2 indicates.

Each forward period is chosen to be an interval between two points of the IMM grid, the first point corresponding to the maturity of the futures contract. Note however that strictly speaking, a futures quote implies a futures rate corresponding to a period between the maturity of the contract, and this maturity +3 months. This period may not be exactly the one between two IMM

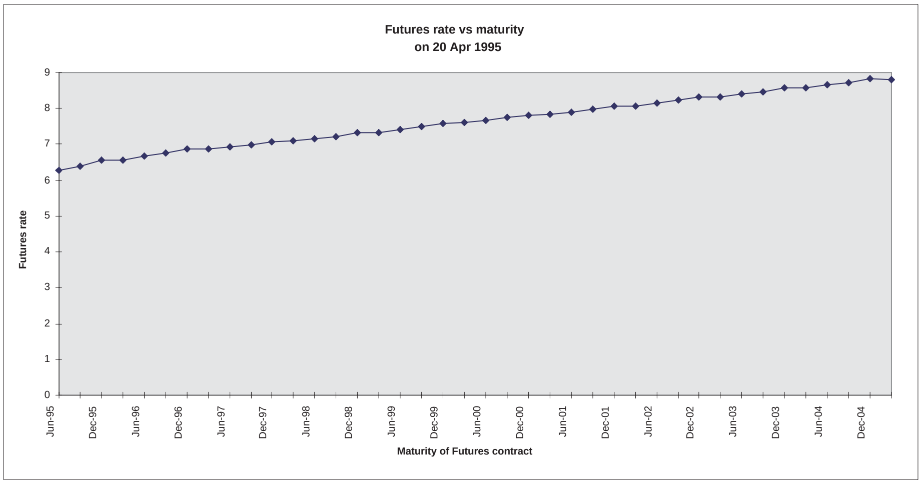

Figure 3: The futures rates of all 40 contracts as a function of their maturity, on spot 24 Apr 1995

points.[14] This problem is referred to as the *gap effect*, which hopefully should not be significant.

For each forward period, the convexity adjustment can be calculated using formula (28). A possible set of inputs to this formula is shown in figure 4. As expected, *Rate*, *Vol* and *Decorr* refer to R_∞, σ and the decorrelation δ respectively. However, the latter is not a very intuitive notion. It is easy to guess *sensible* values for R_∞ (e.g. 7%) or σ (e.g. 18%), but the same cannot be said for the decorrelation δ. Therefore, we have chosen to specify δ indirectly by the use of another input *correl*, more appealing to intuition: looking at (27), it appears that if the forward period is equal (in length) to the time left to maturity (see figure 5), then the corresponding correlation is given by:

$$\rho = \exp(-\delta) \tag{30}$$

The corresponding ρ is exactly the *correl* factor of figure 4. *It is the correlation between a spot and a forward*[15] *with same maturity, where the forward period is half the length of the period spanned by the spot rate.*

In figure 6, we show the results obtained for the inputs of figure 4. As we can see, the difference between a futures and its corresponding forward is limited to a few basis points. However, this is true for a correlation factor equal to 0.86.[16] As figure 7 shows, the effect of the correlation factor can be quite dramatic. As ρ tends to 1, the last contract of Mar 05 can have an adjustment of up to

[14] In other words, we want to know about forwards between IMM points, but we only know about futures between IMM and IMM+3m.

[15] Strictly speaking *futures rate*.

[16] The value of 0.86 is implied by an adjustment of 5 basis points on the Mar 00 contract, given $\sigma = 18\%$ and $R_\infty = 7\%$.

INPUT	
Spot	24-Apr-95
Rate	7.00
Vol	18.00
Correl	0.86
Decorr	0.15

Figure 4: These are the inputs needed by the spreadsheet. Note that *decorr* and *correl* are redundant information. It is however easier to *get a feel* for a *correlation* than it is for a *decorrelation*

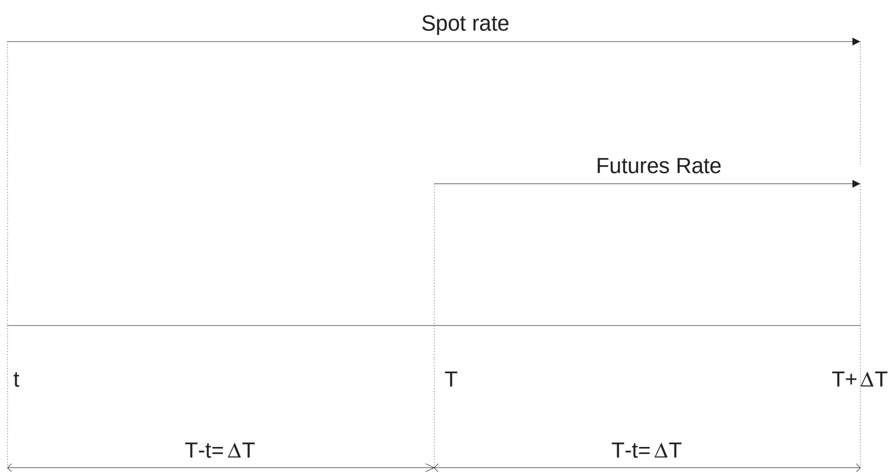

Figure 5: When $T - t = \Delta T$, the correlation between the spot rate and futures rate is $\rho = e^{-\delta}$. Inputting ρ is equivalent to inputting δ, but is a lot more intuitive.

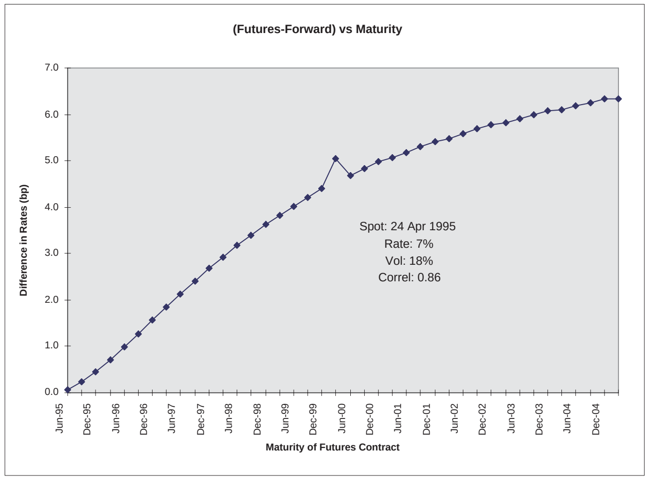

Figure 6: A futures rate is always larger than a forward rate. For a correlation factor equal to 0.86, the difference is of the order of a few basis points. Note that the *blip* on Mar 00 is due to the fact that the forward period is 5 weeks instead of 4.

100 basis points. In comparison, the effect of the volatility σ and rate R_∞ (see figure 8 and 9) is far less significant.

3.3 Conclusion

Using formula (28), we are theoretically able to explicitly determine the convexity adjustment between a forward and futures rate. However, it is extremely unfortunate that this adjustment should be particularly sensitive to the correlation input. If we estimate a rate volatility to be 14%, whether it is actually 16% or 12%, will not have a significant impact on the final result. In any case, the consequence for getting a wrong volatility estimate will be very little, compared with the consequence of assuming $\rho = .85$ when the *true* correlation is .95. It appears therefore that formula (28) is not sufficient in itself, to obtain both reliable and accurate estimate of the convexity adjustment. More information is needed on the correlation factor. One way forward could be to regard the SWAP market as a benchmark providing implied estimates. Another could be the use of historical data.[17] As we can see, further research appears to be necessary.

[17] Although one tends to prefer implied data, making sure that historical estimates are not *too far off* is surely worth investigating.

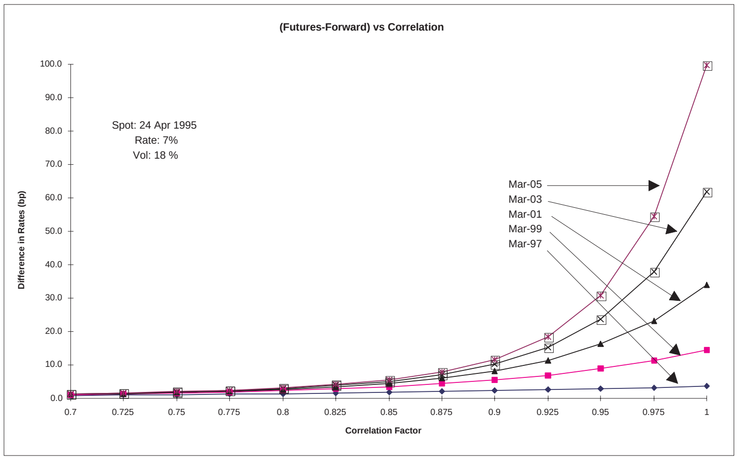

Figure 7: Unfortunately, the convexity adjustment is extremely sensitive to the correlation input, as it goes to 1. In practice,this means that the *true* Mar 05 adjustment could be anywhere between 10 and 100 basis points...

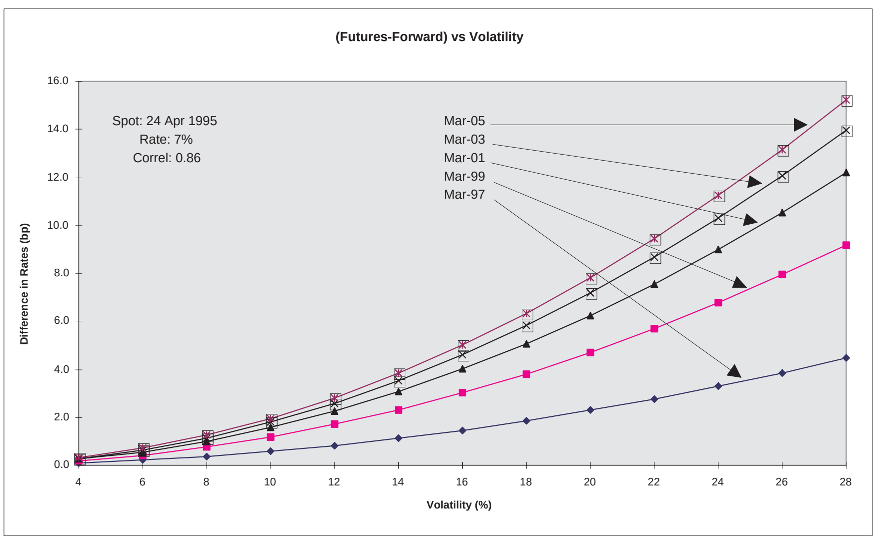

Figure 8: Estimating a *true* volatility for rates may be difficult. However, the consequence of getting it wrong is less dramatic than before.

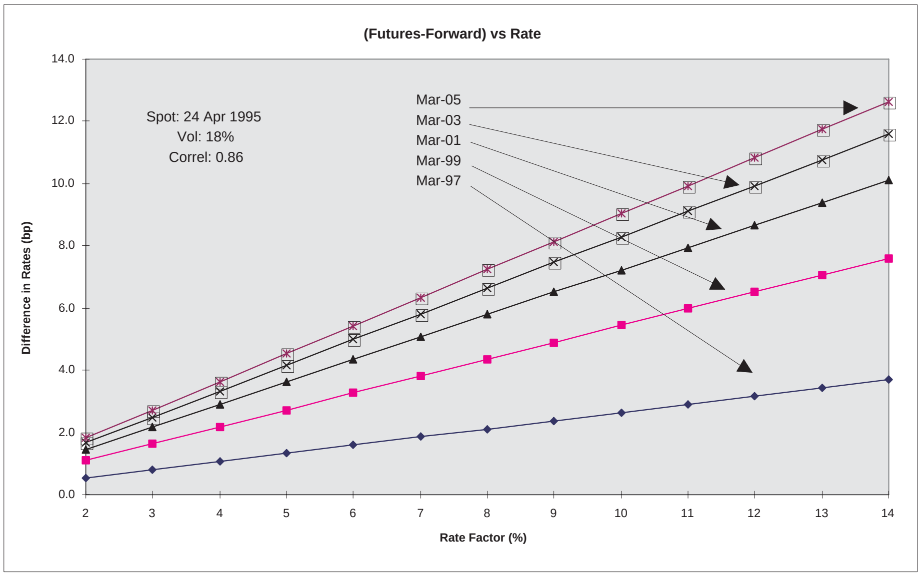

Figure 9: The effect of the R_∞ factor.

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[3] Karatzas,I. and Shreve,S.E.(1991). Brownian motion and stochastic calculus. 2nd Ed. Springer-Verlag

List of Figures

A Appendix

In this appendix, we solve the stochastic differential equation:

$$d\pi_t = \theta_t dF_t + \frac{\pi_t}{V_t} dV_t \tag{31}$$

given the initial condition $\pi_0 = v_0$.

We are given a complete probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$ satisfying the usual conditions. We assume that F and V are two strictly positive continuous semi-martingales, and that the process $\theta = (\theta_t)$ is integrable with respect to F: by this we mean that θ is a real valued progressive process satisfying:

$$\forall t \in \mathbf{R}^+ \ , \ \int_0^t |\theta_s| d|B|_s < +\infty \ , \ \text{P-a.s.}$$

$$\forall t \in \mathbf{R}^+ \ , \ \int_0^t \theta_s^2 d\langle M\rangle_s < +\infty \ , \ \text{P-a.s.}$$

where B and M are respectively the finite variations and local martingale parts of F.[18] Note that the integrability condition imposed on θ, together with the fact that all paths of π/V (when π is continuous) are bounded on any compact interval (i.e. π/V is integrable w.r. to V), ensures that the r.h.s. of (31) does make sense for any continuous semi-martingale π.

We are now in a position to state:

Proposition 1 *There is a unique (up to indistinguishability) continuous semi-martingale π satisfying equation (31) with $\pi_0 = v_0$, and it is given by equation (7), where $\hat{F}$, $\hat{\theta}$ and C are defined as in (8), (9) and (10) respectively.*

Proof
Before we check that π as defined in (7) is indeed a solution of (31), it may be worth pointing that all processes defined in (7), (8), (9) and (10) (including π itself) do actually make sense: having assumed F and V continuous and strictly positive, all paths of $1/FV$ are bounded on compact intervals, and the process C is therefore a well-defined strictly positive continuous process of finite variations. Furthermore, applying Ito's lemma[19]

$$\frac{dC_t}{C_t} = \frac{1}{F_t V_t} d\langle F, V\rangle_t \tag{32}$$

which shows that $\hat{F}$ is a continuous semi-martingale satisfying:

$$d\hat{F}_t = \frac{1}{C_t} dF_t - \frac{1}{C_t V_t} d\langle F, V\rangle_t \tag{33}$$

[18] Note that the quadratic variation process $\langle M\rangle$ will often be denoted $\langle F\rangle$, just as we have used the notation $\langle F, V\rangle$ in (10) , where strictly speaking we meant $\langle M, N\rangle$ where N is the local martingale part of the V.

[19] See e.g. [3], p. 149, Th. 3.3. Although there is no need to apply Ito's lemma here (everything is of finite variations), it is a good reference opportunity. See also p. 153, Th. 3.6 and p. 155, Pb. 3.12

Consequently, if B and M are respectively the finite variations and local martingale parts of F, then the finite variations and local martingale parts of $\hat{F}$, are given by:

$$\text{P-a.s. },\ \forall t \in \mathbf{R}^+ \ ,\ \hat{B}_t = \int_0^t \frac{1}{C_s} dB_s - \int_0^t \frac{1}{C_s V_s} d\langle F, V\rangle_s \tag{34}$$

$$\text{P-a.s. },\ \forall t \in \mathbf{R}^+ \ ,\ \hat{M}_t = \int_0^t \frac{1}{C_s} dM_s \tag{35}$$

and from the integrability of θ with respect to F, we deduce the integrability of $\hat{\theta}$ with respect to $\hat{F}$, the only may-be-delicate point being to show that:

$$\forall t \in \mathbf{R}^+ \ ,\ \int_0^t \frac{|\hat{\theta}_s|}{C_s V_s} d|\langle F, V\rangle|_s < +\infty \ ,\ \text{P-a.s.}$$

which is a consequence of the Kunita-Watanabe inequality.[20] Hence, the process π as defined in (7) is a well-defined continuous semi-martingale.

Checking that π is indeed solution of (31) is now straightforward: applying Ito's lemma to (7), we obtain:

$$d\pi_t = V_t \hat{\theta}_t d\hat{F}_t + \frac{\pi_t}{V_t} dV_t + \hat{\theta}_t d\langle \hat{F}, V\rangle_t \tag{36}$$

However, from (35), we have:

$$d\langle \hat{F}, V\rangle_t = \frac{1}{C_t} d\langle F, V\rangle_t \tag{37}$$

and substituting (33) and (37) into (36), we obtain equation (31).

We are now left with proving the uniqueness of π: suppose there are two continuous semi-martingales with v_0 as initial value and satisfying equation (31). Let X be their difference and define $Y = X/V$. Then $X_0 = 0$ and X satisfies the equation:

$$dX_t = \frac{X_t}{V_t} dV_t \tag{38}$$

In particular, we have:

$$\text{P-a.s. },\ \forall t \in \mathbf{R}^+ \ ,\ \langle X, V\rangle_t = \int_0^t \frac{X_s}{V_s} d\langle V\rangle_s \tag{39}$$

Furthermore, by Ito's lemma:

$$d\left(\frac{1}{V_t}\right) = -\frac{1}{V_t^2} dV_t + \frac{1}{V_t^3} d\langle V\rangle_t \tag{40}$$

[20]See e.g. [3], p. 142, prop. 2.14. Strictly speaking the result in [3] is not as general as the one used now, but extending it from square integrable martingales, to local martingales is not such a big step.

from which it is seen that:

$$dY_t = \frac{1}{V_t}dX_t - \frac{X_t}{V_t^2}dV_t + \frac{X_t}{V_t^3}d\langle V\rangle_t - \frac{1}{V_t^2}d\langle X,V\rangle_t \tag{41}$$

Substituting (38) and (39) into (41) shows that Y is indistinguishable from zero ($Y_0 = 0$). This completes the proof of the uniqueness property. **QED**

B Appendix

In this appendix, we explicitly determine the process C as defined in (10) , and describe the assumptions made on F and V. We are given a complete probability space $(\Omega, \mathcal{F}, P)$ together with a two-dimensional standard Brownian motion (W, W') and the corresponding augmented Brownian filtration $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$. Given a borel map $\rho : \mathbf{R}^+ \to [-1, 1]$, we define the Brownian motion: [21]

$$W''_t \triangleq \int_0^t \rho(s)dW_s + \int_0^t \sqrt{1-\rho^2(s)}dW'_s \tag{42}$$

We assume that the processes F and V are given by $F_0, V_0 > 0$ and the following:

$$dF_t = \mu(t)F_t dt + \sigma_F(t)F_t dW_t \tag{43}$$

$$V_t = \exp(-(T + \Delta T - t)R_t) \tag{44}$$

$$dR_t = \gamma(R_\infty - R_t)dt + \sigma_R(t)R_\infty dW''_t \tag{45}$$

where $\gamma, R_\infty > 0$ are constant, and μ, σ_F, σ_R are locally square integrable Borel maps on $\mathbf{R}^+$. We further assume that $|\sigma_F|$ is bounded away from zero, by a strictly positive constant. Note that F and R are explicitly given by:[22]

$$F_t = F_0 \exp\left(\int_0^t \sigma_F(s)dW_s - \frac{1}{2}\int_0^t \sigma_F^2(s)ds + \int_0^t \mu(s)ds\right) \tag{46}$$

$$R_t = R_0 e^{-\gamma t} + R_\infty(1 - e^{-\gamma t}) + R_\infty e^{-\gamma t}\int_0^t e^{\gamma s}\sigma_R(s)dW''_s \tag{47}$$

Moreover, F and V are two strictly positive continuous semi-martingales, which shows that appendix A can legitimately be applied to them.

It follows from (42) that the cross-variation process between W and W'' is equal to:

$$\text{P-a.s. },\ \forall t \in \mathbf{R}^+\ ,\ \langle W, W''\rangle_t = \int_0^t \rho(s)ds \tag{48}$$

[21] W'' is a continuous (local) martingale with quadratic variation $\langle W''\rangle_t = t$, hence it is a standard Brownian motion. See [3],p. 157,Th. 3.16

[22] The assumptions made on μ, σ_F and σ_R ensures that all integrals in (46) and (47) are meaningful. The reason for assuming $|\sigma_F|$ bounded away from zero, and μ locally square integrable (as opposed to just locally integrable) will appear in appendix C.

from which we see, using (43) and (45):

$$\text{P-a.s.}\ ,\ \forall t \in \mathbf{R}^+\ ,\ \langle F, R\rangle_t = R_\infty \int_0^t F_s \sigma_R(s)\sigma_F(s)\rho(s)ds \tag{49}$$

However, applying Ito's lemma to (44):

$$\frac{dV_t}{V_t} = R_t dt - (T + \Delta T - t)dR_t + \frac{1}{2}(T + \Delta T - t)^2 d\langle R\rangle_t \tag{50}$$

and therefore, using (49):

$$\langle F, V\rangle_t = -R_\infty \int_0^t F_s V_s (T + \Delta T - s)\sigma_R(s)\sigma_F(s)\rho(s)ds \tag{51}$$

We finally obtain from (10):

$$C_t = \exp\left(-R_\infty \int_0^t (T + \Delta T - s)\sigma_R(s)\sigma_F(s)\rho(s)ds\right) \tag{52}$$

C Appendix

In this appendix, we show the existence of a probability measure Q, such that $\hat{F}$ is a martingale under Q.[23] This will prove possible by Girsanov Theorem[24] and the assumptions described in appendix B. Looking at (8), (46) and (52), we have:

$$\hat{F}_t = F_0 \exp\left(\int_0^t \sigma_F(s)\beta(s)ds + \int_0^t \sigma_F(s)dW_s - \frac{1}{2}\int_0^t \sigma_F^2(s)ds\right) \tag{53}$$

where the map $\beta : \mathbf{R}^+ \to \mathbf{R}$ is defined as:

$$\beta(t) \triangleq \frac{\mu(t)}{\sigma_F(t)} + R_\infty (T + \Delta T - t)\sigma_R(t)\rho(t) \tag{54}$$

Let Q be defined as the probability measure on $(\Omega, \mathcal{F})$ with density Z_T with respect to P, where:[25]

$$Z_T \triangleq \exp\left(-\int_0^T \beta(s)dW_s - \frac{1}{2}\int_0^T \beta^2(s)ds\right) \tag{55}$$

By Girsanov theorem, the two-dimensional process $(\tilde{W}, \tilde{W}')$ defined by:[26]

$$\tilde{W}_t \triangleq W_t + \int_0^{t\wedge T} \beta(s)ds \tag{56}$$

$$\tilde{W}'_t \triangleq W'_t \tag{57}$$

[23]Strictly speaking, if $\hat{F}$ is viewed as a process indexed by the whole of $\mathbf{R}^+$, then it will not be a martingale under Q, but the stopped process $\hat{F}^T = (F_{t\wedge T})$ will.

[24]See e.g. [3] , p. 191, Th. 5.1

[25]Note that the assumptions made on μ, σ_F, σ_R and ρ in appendix B, ensure that β is a locally square integrable Borel map on $\mathbf{R}^+$. So Z_T is well-defined.

[26]Do not forget to stop your integral at T in (56).

is a standard two-dimensional Brownian motion on $(\Omega, \mathcal{F}, Q)$ endowed with the filtration $(\mathcal{F}_t)$. Looking back at (53), it appears that:[27]

$$\hat{F}_{t\wedge T} = F_0 \exp\left(\int_0^{t\wedge T} \sigma_F(s) d\tilde{W}_s - \frac{1}{2} \int_0^{t\wedge T} \sigma_F^2(s) ds \right) \tag{58}$$

from which we conclude that the stopped process $\hat{F}^T$ is a continuous martingale under Q.

D Appendix

In this appendix, we show that the *martingale representation theorem*[28] can actually be applied, to prove the existence of a constant x_0 together with a process ϕ such that:

$$x_0 + \int_0^T \phi_t d\hat{F}_t = \alpha(F_T - K) \tag{59}$$

We shall also give a justification for formula (15).

We first consider the complete probability space $(\Omega, \mathcal{F}, Q)$, together with the augmented filtration $(\mathcal{G}_t)_{t\in\mathbf{R}^+}$ generated by the one-dimensional Brownian motion $\tilde{W}$.[29] From equation (58), we have in particular:

$$\hat{F}_T = F_0 \exp\left(\int_0^T \sigma_F(s) d\tilde{W}_s - \frac{1}{2} \int_0^T \sigma_F^2(s) ds \right) \tag{60}$$

which shows that the random variable $\hat{F}_T$ is Q-square integrable, and measurable with respect to $\mathcal{G}_T$. If we assume that the process C is deterministic, then $F_T = \hat{F}_T C_T$ (and therefore $\alpha(F_T - K)$) is itself Q-square integrable and measurable with respect to $\mathcal{G}_T$.[30]

According to the *martingale representation theorem*, there exist a constant x_0 together with a $(\mathcal{G}_t)$-progressive process y satisfying:

$$E_Q\left[\int_0^T y_t^2 dt \right] < +\infty \tag{61}$$

[27] Beware, the following is NOT true for t greater than T:

$$\hat{F}_t = F_0 \exp\left(\int_0^t \sigma_F(s) d\tilde{W}_s - \frac{1}{2} \int_0^t \sigma_F^2(s) ds \right)$$

[28] See e.g. [3], p.182, Th.4.15. However, we shall more specifically use one of its corollaries: p.184, Pb. 4.17

[29] Working on the right filtered probability space is of crucial importance here. Refer to appendix C for unexplained notations.

[30] This is extremely important: if C_T is random, we may still have the square integrability, but the measurability with respect to $\mathcal{G}_T$ is lost for good. Note that we could relax slightly the assumption of C being deterministic, by just assuming C_T non-random.

such that:

$$\text{P-a.s.} \ , \ x_0 + \int_0^T y_t d\tilde{W}_t = \alpha(F_T - K) \tag{62}$$

Applying Ito's lemma to (58), we have:

$$d\hat{F}_t^T = \sigma_F(t)\hat{F}_t^T d\tilde{W}^T \tag{63}$$

from which we obtain (59), provided ϕ is defined as $\phi_t = y_t/\sigma_F(t)$.

Finally, if we put $v_0 = x_0 V_0$ and $\theta_t = V_t \phi_t / C_t$ [31], then $\phi_t = \hat{\theta}_t$ and therefore:

$$\frac{v_0}{V_0} + \int_0^T \hat{\theta}_t d\hat{F}_t = \alpha(F_T - K)$$

and by (61), we see that $t \to \int_0^{t\wedge T} y_s d\tilde{W}_s$ is a Q-square integrable martingale, from which we conclude:

$$E_Q\left[\int_0^T \hat{\theta}_t d\hat{F}_t\right] = E_Q\left[\int_0^T y_t d\tilde{W}_t\right] = 0$$

[31] Exercise: show that θ is integrable w.r. to F.