9. L^p-spaces, $p \in [1, +\infty]$

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

EXERCISE 1. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be non-negative and measurable maps. Let $p, q \in \mathbf{R}^+$, such that $1/p + 1/q = 1$.

1. Show that $1 < p < +\infty$ and $1 < q < +\infty$.

2. For all $\alpha \in]0, +\infty[$, we define $\phi^\alpha : [0, +\infty] \to [0, +\infty]$ by:

$$\phi^\alpha(x) \stackrel{\triangle}{=} \begin{cases} x^\alpha & \text{if} \quad x \in \mathbf{R}^+ \\ +\infty & \text{if} \quad x = +\infty \end{cases}$$

   Show that ϕ^α is a continuous map.

3. Define $A = (\int f^p d\mu)^{1/p}$, $B = (\int g^q d\mu)^{1/q}$ and $C = \int f g d\mu$. Explain why A, B and C are well defined elements of $[0, +\infty]$.

4. Show that if $A = 0$ or $B = 0$ then $C \leq AB$.

5. Show that if $A = +\infty$ or $B = +\infty$ then $C \leq AB$.

6. We assume from now on that $0 < A < +\infty$ and $0 < B < +\infty$. Define $F = f/A$ and $G = g/B$. Show that:

$$\int_\Omega F^p d\mu = \int_\Omega G^p d\mu = 1$$

7. Let $a, b \in]0, +\infty[$. Show that:

$$\ln(a) + \ln(b) \leq \ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)$$

and:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

Prove this last inequality for all $a, b \in [0, +\infty]$.

8. Show that for all $\omega \in \Omega$, we have:

$$F(\omega)G(\omega) \leq \frac{1}{p}(F(\omega))^p + \frac{1}{q}(G(\omega))^q$$

9. Show that $C \leq AB$.

Theorem 41 (Hölder's inequality) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p, q \in \mathbf{R}^+$ be such that $1/p + 1/q = 1$. Then:*

$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}} \left(\int_\Omega g^q d\mu\right)^{\frac{1}{q}}$$

Theorem 42 (Cauchy-Schwarz's inequality:first) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Then:*

$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^2 d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2 d\mu\right)^{\frac{1}{2}}$$

Exercise 2. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p \in]1, +\infty[$. Define $A = (\int f^p d\mu)^{1/p}$ and

$B = (\int g^p d\mu)^{1/p}$ and $C = (\int (f+g)^p d\mu)^{1/p}$.

1. Explain why A, B and C are well defined elements of $[0, +\infty]$.

2. Show that for all $a, b \in]0, +\infty[$, we have:

   $$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

   Prove this inequality for all $a, b \in [0, +\infty]$.

3. Show that if $A = +\infty$ or $B = +\infty$ or $C = 0$ then $C \leq A + B$.

4. Show that if $A < +\infty$ and $B < +\infty$ then $C < +\infty$.

5. We assume from now that $A, B \in [0, +\infty[$ and $C \in]0, +\infty[$. Show the existence of some $q \in \mathbf{R}^+$ such that $1/p + 1/q = 1$.

6. Show that for all $a, b \in [0, +\infty]$, we have:

   $$(a+b)^p = (a+b).(a+b)^{p-1}$$

7. Show that:

$$\begin{aligned} \int_\Omega f.(f+g)^{p-1} d\mu &\leq AC^{\frac{p}{q}} \\ \int_\Omega g.(f+g)^{p-1} d\mu &\leq BC^{\frac{p}{q}} \end{aligned}$$

8. Show that:

$$\int_\Omega (f+g)^p d\mu \leq C^{\frac{p}{q}}(A+B)$$

9. Show that $C \leq A + B$.

10. Show that the inequality still holds if we assume that $p = 1$.

Theorem 43 (Minkowski's inequality) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p \in [1, +\infty[$. Then:*

$$\left(\int_\Omega (f+g)^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega g^p d\mu\right)^{\frac{1}{p}}$$

Definition 73 *The* **L^p-spaces**, *$p \in [1, +\infty[$, on $(\Omega, \mathcal{F}, \mu)$, are:*

$$L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \int_\Omega |f|^p d\mu < +\infty \right\}$$

$$L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \int_\Omega |f|^p d\mu < +\infty \right\}$$

For all $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we put:

$$\|f\|_p \stackrel{\triangle}{=} \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$$

EXERCISE 3. Let $p \in [1, +\infty[$. Let $f, g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ , \ f(\Omega) \subseteq \mathbf{R}\}$.
2. Show that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
3. Show that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.
4. Show that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.
5. Show that $\|f\|_p = 0 \ \Leftrightarrow \ f = 0$ μ-a.s.
6. Show that for all $\alpha \in \mathbf{C}$, $\|\alpha f\|_p = |\alpha|.\|f\|_p$.
7. Explain why $(f, g) \to \|f - g\|_p$ is not a metric on $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$

Definition 74 *For all $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable, Let:*

$$\|f\|_\infty \stackrel{\triangle}{=} \inf\{M \in \mathbf{R}^+ \ , \ |f| \leq M \ \ \mu\text{-}a.s.\}$$

The L^∞**-spaces** *on a measure space* $(\Omega, \mathcal{F}, \mu)$ *are:*

$$L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \{f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \|f\|_\infty < +\infty\}$$
$$L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \{f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \|f\|_\infty < +\infty\}$$

EXERCISE 4. Let $f, g \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f(\Omega) \subseteq \mathbf{R}\}$.
2. Show that $|f| \leq \|f\|_\infty$ μ-a.s.
3. Show that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
4. Show that $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
5. Show that $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.
6. Show that $\|f\|_\infty = 0 \ \Leftrightarrow\ f = 0$ μ-a.s..
7. Show that for all $\alpha \in \mathbf{C}$, $\|\alpha f\|_\infty = |\alpha|.\|f\|_\infty$.

8. Explain why $(f, g) \to \|f - g\|_\infty$ is not a metric on $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$

Definition 75 *Let $p \in [1, +\infty]$. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. For all $\epsilon > 0$ and $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, we define the so-called* **open ball** *in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$:*

$$B(f, \epsilon) \stackrel{\triangle}{=} \{g : g \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu), \|f - g\|_p < \epsilon\}$$

We call **usual topology** *in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, the set $\mathcal{T}$ defined by:*

$$\mathcal{T} \stackrel{\triangle}{=} \{U : U \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu), \forall f \in U, \exists \epsilon > 0, B(f, \epsilon) \subseteq U\}$$

Note that if $(f, g) \to \|f - g\|_p$ was a metric, the usual topology in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, would be nothing but the *metric* topology.

EXERCISE 5. Let $p \in [1, +\infty]$. Suppose there exists $N \in \mathcal{F}$ with $\mu(N) = 0$ and $N \neq \emptyset$. Put $f = 1_N$ and $g = 0$

1. Show that $f, g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f \neq g$.

2. Show that any open set containing f also contains g.
3. Show that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ are not Hausdorff.

Exercise 6. Show that the usual topology on $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is induced by the usual topology on $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where $p \in [1, +\infty]$.

Definition 76 *Let $(E, \mathcal{T})$ be a topological space. A sequence $(x_n)_{n\geq 1}$ in E is said to* **converge** *to $x \in E$, and we write $x_n \xrightarrow{\mathcal{T}} x$, if and only if, for all $U \in \mathcal{T}$ such that $x \in U$, there exists $n_0 \geq 1$ such that:*

$$n \geq n_0 \;\Rightarrow\; x_n \in U$$

When $E = L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ or $E = L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we write $x_n \xrightarrow{L^p} x$.

Exercise 7. Let $(E, \mathcal{T})$ be a topological space and $E' \subseteq E$. Let $\mathcal{T}' = \mathcal{T}_{|E'}$ be the induced topology on E'. Show that if $(x_n)_{n\geq 1}$ is a sequence in E' and $x \in E'$, then $x_n \xrightarrow{\mathcal{T}} x$ is equivalent to $x_n \xrightarrow{\mathcal{T}'} x$.

Exercise 8. Let $f, g, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $p \in [1, +\infty]$.

1. Recall what the notation $f_n \to f$ means.

2. Show that $f_n \xrightarrow{L^p} f$ is equivalent to $\|f_n - f\|_p \to 0$.

3. Show that if $f_n \xrightarrow{L^p} f$ and $f_n \xrightarrow{L^p} g$ then $f = g$ μ-a.s.

Exercise 9. Let $p \in [1, +\infty]$. Suppose there exists some $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $N \neq \emptyset$. Find a sequence $(f_n)_{n \geq 1}$ in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and f, g in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $f \neq g$ such that $f_n \xrightarrow{L^p} f$ and $f_n \xrightarrow{L^p} g$.

Definition 77 *Let $(f_n)_{n\geq 1}$ be a sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a measure space and $p \in [1, +\infty]$. We say that $(f_n)_{n\geq 1}$ is a* **Cauchy sequence***, if and only if, for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:*

$$n, m \geq n_0 \;\Rightarrow\; \|f_n - f_m\|_p \leq \epsilon$$

Exercise 10. Let $f, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $p \in [1, +\infty]$. Show that if $f_n \stackrel{L^p}{\to} f$, then $(f_n)_{n\geq 1}$ is a Cauchy sequence.

Exercise 11. Let $(f_n)_{n\geq 1}$ be Cauchy in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $p \in [1, +\infty]$.

1. Show the existence of $n_1 \geq 1$ such that:

$$n \geq n_1 \ \Rightarrow \ \|f_n - f_{n_1}\|_p \leq \frac{1}{2}$$

2. Suppose we have found $n_1 < n_2 < \ldots < n_k,\ k \geq 1$, such that:

$$\forall j \in \{1, \ldots, k\}\ ,\ n \geq n_j \ \Rightarrow \ \|f_n - f_{n_j}\|_p \leq \frac{1}{2^j}$$

   Show the existence of n_{k+1}, $n_k < n_{k+1}$ such that:

$$n \geq n_{k+1} \ \Rightarrow \ \|f_n - f_{n_{k+1}}\|_p \leq \frac{1}{2^{k+1}}$$

3. Show that there exists a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ with:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

Exercise 12. Let $p \in [1, +\infty]$, and $(f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, with:

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

We define:

$$g \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} |f_{n+1} - f_n|$$

1. Show that $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable.
2. If $p = +\infty$, show that $g \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty$ μ-a.s.

3. If $p \in [1, +\infty[$, show that for all $N \geq 1$, we have:

$$\left\| \sum_{n=1}^{N} |f_{n+1} - f_n| \right\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

4. If $p \in [1, +\infty[$, show that:

$$\left(\int_\Omega g^p d\mu \right)^{\frac{1}{p}} \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

5. Show that for $p \in [1, +\infty]$, we have $g < +\infty$ μ-a.s.

6. Define $A = \{g < +\infty\}$. Show that for all $\omega \in A$, $(f_n(\omega))_{n \geq 1}$ is a Cauchy sequence in $\mathbf{C}$. We denote $z(\omega)$ its limit.

7. Define $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, by:

$$f(\omega) \stackrel{\triangle}{=} \begin{cases} z(\omega) & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \notin A \end{cases}$$

Show that f is measurable and $f_n \to f$ μ-a.s.

8. if $p = +\infty$, show that for all $n \geq 1$, $|f_n| \leq |f_1| + g$ and conclude that $f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

9. If $p \in [1, +\infty[$, show the existence of $n_0 \geq 1$, such that:

$$n \geq n_0 \;\Rightarrow\; \int_\Omega |f_n - f_{n_0}|^p d\mu \leq 1$$

Deduce from Fatou's lemma that $f - f_{n_0} \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

10. Show that for $p \in [1, +\infty]$, $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

11. Suppose that $f_n \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, for all $n \geq 1$. Show the existence of $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ μ-a.s.

Exercise 13. Let $p \in [1, +\infty]$, and $(f_n)_{n \geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, with:

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

1. Does there exist $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ μ-a.s.

2. Suppose $p = +\infty$. Show that for all $n < m$, we have:

$$|f_{m+1} - f_n| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty \ \mu\text{-a.s.}$$

3. Suppose $p = +\infty$. Show that for all $n \geq 1$, we have:

$$\|f - f_n\|_\infty \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_\infty$$

4. Suppose $p \in [1, +\infty[$. Show that for all $n < m$, we have:

$$\left(\int_\Omega |f_{m+1} - f_n|^p d\mu \right)^{\frac{1}{p}} \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_p$$

5. Suppose $p \in [1, +\infty[$. Show that for all $n \geq 1$, we have:

$$\|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p$$

6. Show that for $p \in [1, +\infty]$, we also have $f_n \stackrel{L^p}{\rightarrow} f$.

7. Suppose conversely that $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is such that $f_n \stackrel{L^p}{\rightarrow} g$. Show that $f = g$ μ-a.s.. Conclude that $f_n \rightarrow g$ μ-a.s..

Theorem 44 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $p \in [1, +\infty]$, and $(f_n)_{n\geq 1}$ be a sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

Then, there exists $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ μ-a.s. Moreover, for all $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, the convergence $f_n \to g$ μ-a.s. and $f_n \xrightarrow{L^p} g$ are equivalent.

EXERCISE 14. Let $f, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \xrightarrow{L^p} f$, where $p \in [1, +\infty]$.

1. Show that there exists a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, with:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

2. Show that there exists $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_{n_k} \to g$ μ-a.s.

3. Show that $f_{n_k} \stackrel{L^p}{\to} g$ and $g = f$ μ-a.s.

4. Conclude with the following:

Theorem 45 *Let $(f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \stackrel{L^p}{\to} f$, where $p \in [1, +\infty]$. Then, we can extract a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ such that $f_{n_k} \to f$ μ-a.s.*

EXERCISE 15. Prove the last theorem for $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

EXERCISE 16. Let $(f_n)_{n\geq 1}$ be Cauchy in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $p \in [1, +\infty]$.

1. Show that there exists a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ and f belonging to $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that $f_{n_k} \stackrel{L^p}{\to} f$.

2. Using the fact that $(f_n)_{n\geq 1}$ is Cauchy, show that $f_n \xrightarrow{L^p} f$.

Theorem 46 *Let* $p \in [1, +\infty]$. *Let* $(f_n)_{n\geq 1}$ *be a Cauchy sequence in* $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. *Then, there exists* $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ *such that* $f_n \xrightarrow{L^p} f$.

EXERCISE 17. Prove the last theorem for $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.