6. Product Spaces

In the following, $I$ is a non-empty set.

**Definition 50** Let $(\Omega_i)_{i \in I}$ be a family of sets, indexed by a non-empty set $I$. We call **Cartesian product** of the family $(\Omega_i)_{i \in I}$ the set, denoted $\prod_{i \in I} \Omega_i$, and defined by:

$$\prod_{i \in I} \Omega_i \triangleq \{ \omega : I \to \bigcup_{i \in I} \Omega_i, \ \omega(i) \in \Omega_i, \ \forall i \in I \}$$

In other words, $\prod_{i \in I} \Omega_i$ is the set of all maps $\omega$ defined on $I$, with values in $\bigcup_{i \in I} \Omega_i$, such that $\omega(i) \in \Omega_i$ for all $i \in I$.

**Theorem 25 (Axiom of choice)** Let $(\Omega_i)_{i \in I}$ be a family of sets, indexed by a non-empty set $I$. Then, $\prod_{i \in I} \Omega_i$ is non-empty, if and only if $\Omega_i$ is non-empty for all $i \in I$\(^1\).

\(^1\)When $I$ is finite, this theorem is traditionally derived from other axioms.
Exercise 1.

1. Let $\Omega$ be a set and suppose that $\Omega_i = \Omega, \forall i \in I$. We use the notation $\Omega^I$ instead of $\Pi_{i \in I} \Omega_i$. Show that $\Omega^I$ is the set of all maps $\omega: I \rightarrow \Omega$.

2. What are the sets $\mathbf{R}^{\mathbf{R}^+}$, $\mathbf{R}^\mathbf{N}$, $[0,1]^\mathbf{N}$, $\overline{\mathbf{R}}$?

3. Suppose $I = \mathbf{N}^*$. We sometimes use the notation $\Pi_{n=1}^{\infty} \Omega_n$ instead of $\Pi_{n \in \mathbf{N}} \Omega_n$. Let $S$ be the set of all sequences $(x_n)_{n \geq 1}$ such that $x_n \in \Omega_n$ for all $n \geq 1$. Is $S$ the same thing as the product $\Pi_{n=1}^{\infty} \Omega_n$?

4. Suppose $I = \mathbf{N}_n = \{1, \ldots, n\}$, $n \geq 1$. We use the notation $\Omega_1 \times \ldots \times \Omega_n$ instead of $\Pi_{i \in \{1, \ldots, n\}} \Omega_i$. For $\omega \in \Omega_1 \times \ldots \times \Omega_n$, it is customary to write $(\omega_1, \ldots, \omega_n)$ instead of $\omega$, where we have $\omega_i = \omega(i)$. What is your guess for the definition of sets such as $\mathbf{R}^n, \mathbf{R}^n, \mathbf{Q}^n, \mathbf{C}^n$.

5. Let $E, F, G$ be three sets. Define $E \times F \times G$. 

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Definition 51 Let $I$ be a non-empty set. We say that a family of sets $(I_\lambda)_{\lambda \in \Lambda}$, where $\Lambda \neq \emptyset$, is a partition of $I$, if and only if:

(i) $\forall \lambda \in \Lambda, \ I_\lambda \neq \emptyset$

(ii) $\forall \lambda, \lambda' \in \Lambda, \lambda \neq \lambda' \Rightarrow I_\lambda \cap I_{\lambda'} = \emptyset$

(iii) $I = \bigcup_{\lambda \in \Lambda} I_\lambda$

Exercise 2. Let $(\Omega_i)_{i \in I}$ be a family of sets indexed by $I$, and $(I_\lambda)_{\lambda \in \Lambda}$ be a partition of the set $I$.

1. For each $\lambda \in \Lambda$, recall the definition of $\Pi_{i \in I} \Omega_i$.

2. Recall the definition of $\Pi_{\lambda \in \Lambda} (\Pi_{i \in I} \Omega_i)$.

3. Define a natural bijection $\Phi : \Pi_{i \in I} \Omega_i \to \Pi_{\lambda \in \Lambda} (\Pi_{i \in I} \Omega_i)$.

4. Define a natural bijection $\psi : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^{p+n}$, for all $n, p \geq 1$. 

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**Definition 52** Let \((\Omega_i)_{i \in I}\) be a family of sets, indexed by a non-empty set \(I\). For all \(i \in I\), let \(\mathcal{E}_i\) be a set of subsets of \(\Omega_i\). We define a **rectangle** of the family \((\mathcal{E}_i)_{i \in I}\), as any subset \(A\) of \(\Pi_{i \in I} \Omega_i\), of the form \(A = \Pi_{i \in I} A_i\) where \(A_i \in \mathcal{E}_i \cup \{\Omega_i\}\) for all \(i \in I\), and such that \(A_i = \Omega_i\) except for a finite number of indices \(i \in I\). Consequently, the set of all rectangles, denoted \(\Pi_{i \in I} \mathcal{E}_i\), is defined as:

\[
\prod_{i \in I} \mathcal{E}_i \triangleq \left\{ \prod_{i \in I} A_i : A_i \in \mathcal{E}_i \cup \{\Omega_i\}, \ A_i \neq \Omega_i \text{ for finitely many } i \in I \right\}
\]

**Exercise 3.** \((\Omega_i)_{i \in I}\) and \((\mathcal{E}_i)_{i \in I}\) being as above:

1. Show that if \(I = \mathbb{N}_n\) and \(\Omega_i \in \mathcal{E}_i\) for all \(i = 1, \ldots, n\), then \(\mathcal{E}_1 \Pi \ldots \Pi \mathcal{E}_n = \{A_1 \times \ldots \times A_n : A_i \in \mathcal{E}_i, \ \forall i \in I\}\).

2. Let \(A\) be a rectangle. Show that there exists a finite subset \(J\) of \(I\) such that: \(A = \{\omega \in \Pi_{i \in I} \Omega_i : \omega(j) \in A_j, \ \forall j \in J\}\) for some \(A_j\)'s such that \(A_j \in \mathcal{E}_j\), for all \(j \in J\).
Definition 53  Let \((\Omega_i, \mathcal{F}_i)_{i \in I}\) be a family of measurable spaces, indexed by a non-empty set \(I\). We call measurable rectangle, any rectangle of the family \((\mathcal{F}_i)_{i \in I}\). The set of all measurable rectangles is given by\(^2\):

\[
\prod_{i \in I} \mathcal{F}_i \triangleq \left\{ \prod_{i \in I} A_i : A_i \in \mathcal{F}_i, A_i \neq \Omega_i \text{ for finitely many } i \in I \right\}
\]

Definition 54  Let \((\Omega_i, \mathcal{F}_i)_{i \in I}\) be a family of measurable spaces, indexed by a non-empty set \(I\). We define the product \(\sigma\)-algebra of \((\mathcal{F}_i)_{i \in I}\), as the \(\sigma\)-algebra on \(\prod_{i \in I} \Omega_i\), denoted \(\otimes_{i \in I} \mathcal{F}_i\), and generated by all measurable rectangles, i.e.

\[
\bigotimes_{i \in I} \mathcal{F}_i \triangleq \sigma \left( \prod_{i \in I} \mathcal{F}_i \right)
\]

\(^2\)Note that \(\Omega_i \in \mathcal{F}_i\) for all \(i \in I\).
Exercise 4.

1. Suppose $I = \mathbb{N}_n$. Show that $F_1 \otimes \ldots \otimes F_n$ is generated by all sets of the form $A_1 \times \ldots \times A_n$, where $A_i \in F_i$ for all $i = 1, \ldots, n$.

2. Show that $B(\mathbb{R}) \otimes B(\mathbb{R}) \otimes B(\mathbb{R})$ is generated by sets of the form $A \times B \times C$ where $A, B, C \in B(\mathbb{R})$.

3. Show that if $(\Omega, \mathcal{F})$ is a measurable space, $B(\mathbb{R}^+) \otimes \mathcal{F}$ is the $\sigma$-algebra on $\mathbb{R}^+ \times \Omega$ generated by sets of the form $B \times F$ where $B \in B(\mathbb{R}^+)$ and $F \in \mathcal{F}$.

Exercise 5. Let $(\Omega_i)_{i \in I}$ be a family of non-empty sets and $\mathcal{E}_i$ be a subset of the power set $\mathcal{P}(\Omega_i)$ for all $i \in I$.

1. Give a generator of the $\sigma$-algebra $\otimes_{i \in I} \sigma(\mathcal{E}_i)$ on $\prod_{i \in I} \Omega_i$.

2. Show that:

$$\sigma\left(\prod_{i \in I} \mathcal{E}_i\right) \subseteq \otimes_{i \in I} \sigma(\mathcal{E}_i)$$
3. Let $A$ be a rectangle of the family $(\sigma(E_i))_{i \in I}$. Show that if $A$ is not empty, then the representation $A = \prod_{i \in I} A_i$ with $A_i \in \sigma(E_i)$ is unique. Define $J_A = \{i \in I : A_i \neq \Omega_i\}$. Explain why $J_A$ is a well-defined finite subset of $I$.

4. If $A \in \prod_{i \in I} \sigma(E_i)$, Show that if $A = \emptyset$, or $A \neq \emptyset$ and $J_A = \emptyset$, then $A \in \sigma(\prod_{i \in I} E_i)$.

**Exercise 6.** Everything being as before, Let $n \geq 0$. We assume that the following induction hypothesis has been proved:

$$A \in \prod_{i \in I} \sigma(E_i), A \neq \emptyset, \text{card} J_A = n \Rightarrow A \in \sigma\left(\prod_{i \in I} E_i\right)$$

We assume that $A$ is a non empty measurable rectangle of $(\sigma(E_i))_{i \in I}$ with $\text{card} J_A = n + 1$. Let $J_A = \{i_1, \ldots, i_{n+1}\}$ be an extension of $J_A$. 

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For all $B \subseteq \Omega_{i_1}$, we define:

$$A^B \triangleq \prod_{i \in I} \bar{A}_i$$

where each $\bar{A}_i$ is equal to $A_i$ except $\bar{A}_{i_1} = B$. We define the set:

$$\Gamma \triangleq \left\{ B \subseteq \Omega_{i_1} : A^B \in \sigma \left( \prod_{i \in I} \mathcal{E}_i \right) \right\}$$

1. Show that $A^{\Omega_{i_1}} \neq \emptyset$, $\text{card}_{A^{\Omega_{i_1}}} = n$ and that $A^{\Omega_{i_1}} \in \Pi_{i \in I} \sigma (\mathcal{E}_i)$.

2. Show that $\Omega_{i_1} \in \Gamma$.

3. Show that for all $B \subseteq \Omega_{i_1}$, we have $A^{\Omega_{i_1}} \setminus B = A^{\Omega_{i_1}} \setminus A^B$.

4. Show that $B \in \Gamma \Rightarrow \Omega_{i_1} \setminus B \in \Gamma$.

5. Let $B_n \subseteq \Omega_{i_1}$, $n \geq 1$. Show that $A^{\cup B_n} = \cup_{n \geq 1} A^{B_n}$.

6. Show that $\Gamma$ is a $\sigma$-algebra on $\Omega_{i_1}$. 

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7. Let $B \in \mathcal{E}_{i_1}$, and for $i \in I$ define $\bar{B}_i = \Omega_i$ for all $i$’s except $\bar{B}_{i_1} = B$. Show that $A^B = A^{\Omega_{i_1}} \cap (\prod_{i \in I} \bar{B}_i)$.

8. Show that $\sigma(\mathcal{E}_{i_1}) \subseteq \Gamma$.

9. Show that $A = A^{A_{i_1}}$ and $A \in \sigma(\prod_{i \in I} \mathcal{E}_i)$.

10. Show that $\prod_{i \in I} \sigma(\mathcal{E}_i) \subseteq \sigma(\prod_{i \in I} \mathcal{E}_i)$.

11. Show that $\sigma(\prod_{i \in I} \mathcal{E}_i) = \otimes_{i \in I} \sigma(\mathcal{E}_i)$.

**Theorem 26**  Let $(\Omega_i)_{i \in I}$ be a family of non-empty sets indexed by a non-empty set $I$. For all $i \in I$, let $\mathcal{E}_i$ be a set of subsets of $\Omega_i$. Then, the product $\sigma$-algebra $\otimes_{i \in I} \sigma(\mathcal{E}_i)$ on the Cartesian product $\prod_{i \in I} \Omega_i$ is generated by the rectangles of $(\mathcal{E}_i)_{i \in I}$, i.e. :

$$ \otimes_{i \in I} \sigma(\mathcal{E}_i) = \sigma \left( \prod_{i \in I} \mathcal{E}_i \right) $$
EXERCISE 7. Let $\mathcal{T}_R$ denote the usual topology in $\mathbb{R}$. Let $n \geq 1$.

1. Show that $\mathcal{T}_R \times \ldots \times \mathcal{T}_R = \{ A_1 \times \ldots \times A_n : A_i \in \mathcal{T}_R \}$.

2. Show that $\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{T}_R \times \ldots \times \mathcal{T}_R)$.

3. Define $C_2 = \{ [a_1, b_1] \times \ldots \times [a_n, b_n] : a_i, b_i \in \mathbb{R} \}$. Show that $C_2 \subseteq \mathcal{S} \times \ldots \times \mathcal{S}$, where $\mathcal{S} = \{ [a, b] : a, b \in \mathbb{R} \}$, but that the inclusion is strict.

4. Show that $\mathcal{S} \times \ldots \times \mathcal{S} \subseteq \sigma(C_2)$.

5. Show that $\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = \sigma(C_2)$.

EXERCISE 8. Let $\Omega$ and $\Omega'$ be two non-empty sets. Let $\mathcal{A}$ be a subset of $\Omega$ such that $\emptyset \neq \mathcal{A} \neq \Omega$. Let $\mathcal{E} = \{ \mathcal{A} \} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{E}' = \emptyset \subseteq \mathcal{P}(\Omega')$.

1. Show that $\sigma(\mathcal{E}) = \{ \emptyset, \mathcal{A}, \mathcal{A}^c, \Omega \}$.

2. Show that $\sigma(\mathcal{E}') = \{ \emptyset, \Omega' \}$. 

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3. Define $\mathcal{C} = \{ E \times F, E \in \mathcal{E}, F \in \mathcal{E}' \}$ and show that $\mathcal{C} = \emptyset$.

4. Show that $\mathcal{E} \amalg \mathcal{E}' = \{ A \times \Omega', \Omega \times \Omega' \}$.

5. Show that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') = \{ \emptyset, A \times \Omega', A^c \times \Omega', \Omega \times \Omega' \}$.

6. Conclude that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') \neq \sigma(\mathcal{C}) = \{ \emptyset, \Omega \times \Omega' \}$.

**Exercise 9.** Let $n \geq 1$ and $p \geq 1$ be two positive integers.

1. Define $\mathcal{F} = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$, and $\mathcal{G} = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$.

   Explain why $\mathcal{F} \otimes \mathcal{G}$ can be viewed as a $\sigma$-algebra on $\mathbb{R}^{n+p}$.

2. Show that $\mathcal{F} \otimes \mathcal{G}$ is generated by sets of the form $A_1 \times \ldots \times A_{n+p}$ where $A_i \in \mathcal{B}(\mathbb{R}), i = 1, \ldots, n+p$. 

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3. Show that:
\[ \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = (\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})) \otimes (\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})) \]

**Exercise 10.** Let \((\Omega_i, \mathcal{F}_i)_{i \in I}\) be a family of measurable spaces. Let \((I_\lambda)_{\lambda \in \Lambda}\), where \(\Lambda \neq \emptyset\), be a partition of \(I\). Let \(\Omega = \Pi_{i \in I} \Omega_i\) and \(\Omega' = \Pi_{\lambda \in \Lambda} (\Pi_{i \in I_{\lambda}} \Omega_i)\).

1. Define a natural bijection between \(\mathcal{P}(\Omega)\) and \(\mathcal{P}(\Omega')\).
2. Show that through such bijection, \(A = \Pi_{i \in I} A_i \subseteq \Omega\), where \(A_i \subseteq \Omega_i\), is identified with \(A' = \Pi_{\lambda \in \Lambda} (\Pi_{i \in I_{\lambda}} A_i) \subseteq \Omega'\).
3. Show that \(\Pi_{i \in I} \mathcal{F}_i = \Pi_{\lambda \in \Lambda} (\Pi_{i \in I_{\lambda}} \mathcal{F}_i)\).
4. Show that \(\otimes_{i \in I} \mathcal{F}_i = \otimes_{\lambda \in \Lambda} (\otimes_{i \in I_{\lambda}} \mathcal{F}_i)\).
**Definition 55** Let $\Omega$ be a set and $\mathcal{A}$ be a set of subsets of $\Omega$. We call topology generated by $\mathcal{A}$, the topology on $\Omega$, denoted $T(\mathcal{A})$, equal to the intersection of all topologies on $\Omega$, which contain $\mathcal{A}$.

**Exercise 11.** Let $\Omega$ be a set and $\mathcal{A} \subseteq P(\Omega)$.

1. Explain why $T(\mathcal{A})$ is indeed a topology on $\Omega$.

2. Show that $T(\mathcal{A})$ is the smallest topology $T$ such that $\mathcal{A} \subseteq T$.

3. Show that the metric topology on a metric space $(E,d)$ is generated by the open balls $\mathcal{A} = \{B(x, \epsilon) : x \in E, \epsilon > 0\}$.

**Definition 56** Let $(\Omega_i, T_i)_{i \in I}$ be a family of topological spaces, indexed by a non-empty set $I$. We define the product topology of $(T_i)_{i \in I}$, as the topology on $\prod_{i \in I} \Omega_i$, denoted $\bigcirc_{i \in I} T_i$, and generated by
all rectangles of \((T_i)_{i \in I}\), i.e.
\[
\bigotimes_{i \in I} T_i \triangleq T \left( \prod_{i \in I} T_i \right)
\]

**Exercise 12.** Let \((\Omega_i, T_i)_{i \in I}\) be a family of topological spaces.

1. Show that \(U \in \bigotimes_{i \in I} T_i\), if and only if:
   \[
   \forall x \in U, \ \exists V \in \bigvee_{i \in I} T_i, \ x \in V \subseteq U
   \]
2. Show that \(\Pi_{i \in I} T_i \subseteq \bigotimes_{i \in I} T_i\).
3. Show that \(\otimes_{i \in I} \mathcal{B}(\Omega_i) = \sigma(\Pi_{i \in I} T_i)\).
4. Show that \(\otimes_{i \in I} \mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Pi_{i \in I} \Omega_i)\).
**Exercise 13.** Let $n \geq 1$ be a positive integer. For all $x, y \in \mathbb{R}^n$, let:

$$(x, y) \triangleq \sum_{i=1}^{n} x_i y_i$$

and we put $\|x\| = \sqrt{(x, x)}$.

1. Show that for all $t \in \mathbb{R}$, $\|x + ty\|^2 = \|x\|^2 + t^2\|y\|^2 + 2t(x, y)$.

2. From $\|x + ty\|^2 \geq 0$ for all $t$, deduce that $|(x, y)| \leq \|x\| \|y\|$.

3. Conclude that $\|x + y\| \leq \|x\| + \|y\|$.

**Exercise 14.** Let $(\Omega_1, \mathcal{T}_1), \ldots, (\Omega_n, \mathcal{T}_n), n \geq 1$, be metrizable topological spaces. Let $d_1, \ldots, d_n$ be metrics on $\Omega_1, \ldots, \Omega_n$, inducing the topologies $\mathcal{T}_1, \ldots, \mathcal{T}_n$ respectively. Let $\Omega = \Omega_1 \times \ldots \times \Omega_n$ and $\mathcal{T}$ be
the product topology on $\Omega$. For all $x, y \in \Omega$, we define:

$$d(x, y) \triangleq \sqrt{\sum_{i=1}^{n}(d_i(x_i, y_i))^2}$$

1. Show that $d : \Omega \times \Omega \to \mathbb{R}^+$ is a metric on $\Omega$.

2. Show that $U \subseteq \Omega$ is open in $\Omega$, if and only if, for all $x \in U$ there are open sets $U_1, \ldots, U_n$ in $\Omega_1, \ldots, \Omega_n$ respectively, such that:

$$x \in U_1 \times \ldots \times U_n \subseteq U$$

3. Let $U \in T$ and $x \in U$. Show the existence of $\epsilon > 0$ such that:

$$(\forall i = 1, \ldots, n \ d_i(x_i, y_i) < \epsilon) \Rightarrow y \in U$$

4. Show that $T \subseteq T_d$.

5. Let $U \in T_d$ and $x \in U$. Show the existence of $\epsilon > 0$ such that:

$$x \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon) \subseteq U$$

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6. Show that $\mathcal{T}^d_\Omega \subseteq \mathcal{T}$.

7. Show that the product topological space $(\Omega, \mathcal{T})$ is metrizable.

8. For all $x, y \in \Omega$, define:

\[
\begin{align*}
d'(x, y) & \triangleq \sum_{i=1}^{n} d_i(x_i, y_i) \\
d''(x, y) & \triangleq \max_{i=1, \ldots, n} d_i(x_i, y_i)
\end{align*}
\]

Show that $d', d''$ are metrics on $\Omega$.

9. Show the existence of $\alpha', \beta', \alpha''$ and $\beta'' > 0$, such that we have $\alpha'd' \leq d \leq \beta'd'$ and $\alpha''d'' \leq d \leq \beta''d''$.

10. Show that $d'$ and $d''$ also induce the product topology on $\Omega$.

**Exercise 15.** Let $(\Omega_n, \mathcal{T}_n)_{n \geq 1}$ be a sequence of metrizable topological spaces. For all $n \geq 1$, let $d_n$ be a metric on $\Omega_n$ inducing the topology
Let $\Omega = \Pi_{n=1}^{+\infty} \Omega_n$ be the Cartesian product and $T$ be the product topology on $\Omega$. For all $x, y \in \Omega$, we define:

$$d(x, y) \triangleq \sum_{n=1}^{+\infty} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n))$$

1. Show that for all $a, b \in \mathbb{R}^+$, we have $1 \wedge (a + b) \leq 1 \wedge a + 1 \wedge b$.

2. Show that $d$ is a metric on $\Omega$.

3. Show that $U \subseteq \Omega$ is open in $\Omega$, if and only if, for all $x \in U$, there is an integer $N \geq 1$ and open sets $U_1, \ldots, U_N$ in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

4. Show that $d(x, y) < 1/2^n \Rightarrow d_n(x_n, y_n) \leq 2^n d(x, y)$. 
5. Show that for all $U \in T$ and $x \in U$, there exists $\epsilon > 0$ such that $d(x, y) < \epsilon \Rightarrow y \in U$.

6. Show that $T \subseteq T_{\Omega^d}$.

7. Let $U \in T_{\Omega^d}$ and $x \in U$. Show the existence of $\epsilon > 0$ and $N \geq 1$, such that:

$$\sum_{n=1}^{N} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n)) < \epsilon \Rightarrow y \in U$$

8. Show that for all $U \in T_{\Omega^d}$ and $x \in U$, there is $\epsilon > 0$ and $N \geq 1$ such that:

$$x \in B(x_1, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

9. Show that $T_{\Omega^d} \subseteq T$.

10. Show that the product topological space $(\Omega, T)$ is metrizable.
Definition 57  Let $(\Omega, T)$ be a topological space. A subset $\mathcal{H}$ of $T$ is called a **countable base** of $(\Omega, T)$, if and only if $\mathcal{H}$ is at most countable, and has the property:

$$\forall U \in T, \exists \mathcal{H}' \subseteq \mathcal{H}, U = \bigcup_{V \in \mathcal{H}'} V$$

Exercise 16.

1. Show that $\mathcal{H} = \{[r, q]: r, q \in \mathbb{Q}\}$ is a countable base of $(\mathbb{R}, \mathcal{T}_\mathbb{R})$.

2. Show that if $(\Omega, T)$ is a topological space with countable base, and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', \mathcal{T}_{\Omega'})$ also has a countable base.

3. Show that $[-1, 1]$ has a countable base.

4. Show that if $(\Omega, T)$ and $(S, \mathcal{T}_S)$ are homeomorphic, then $(\Omega, T)$ has a countable base if and only if $(S, \mathcal{T}_S)$ has a countable base.
5. Show that \((\mathbb{R}, T_{\mathbb{R}})\) has a countable base.

**Exercise 17.** Let \((\Omega_n, T_n)_{n \geq 1}\) be a sequence of topological spaces with countable base. For \(n \geq 1\), let \(\{V^n_k : k \in I_n\}\) be a countable base of \((\Omega_n, T_n)\) where \(I_n\) is a finite or countable set. Let \(\Omega = \prod_{n=1}^{\infty} \Omega_n\) be the Cartesian product and \(T\) be the product topology on \(\Omega\). For all \(p \geq 1\), we define:

\[
H^p \triangleq \left\{ V^{k_1}_1 \times \ldots \times V^{k_p}_p \times \prod_{n=p+1}^{\infty} \Omega_n : (k_1, \ldots, k_p) \in I_1 \times \ldots \times I_p \right\}
\]

and we put \(H = \cup_{p \geq 1} H^p\).

1. Show that for all \(p \geq 1\), \(H^p \subseteq T\).
2. Show that \(H \subseteq T\).
3. For all \(p \geq 1\), show the existence of an injection \(j_p : H^p \rightarrow \mathbb{N}^p\).
4. Show the existence of a bijection \( \phi_2 : \mathbb{N}^2 \rightarrow \mathbb{N} \).

5. For \( p \geq 1 \), show the existence of a bijection \( \phi_p : \mathbb{N}^p \rightarrow \mathbb{N} \).

6. Show that \( \mathcal{H}^p \) is at most countable for all \( p \geq 1 \).

7. Show the existence of an injection \( j : \mathcal{H} \rightarrow \mathbb{N}^2 \).

8. Show that \( \mathcal{H} \) is a finite or countable set of open sets in \( \Omega \).

9. Let \( U \in \mathcal{T} \) and \( x \in U \). Show that there is \( p \geq 1 \) and \( U_1, \ldots, U_p \) open sets in \( \Omega_1, \ldots, \Omega_p \) such that:

\[
x \in U_1 \times \ldots \times U_p \times \prod_{n=p+1}^{+\infty} \Omega_n \subseteq U
\]

10. Show the existence of some \( V_x \in \mathcal{H} \) such that \( x \in V_x \subseteq U \).

11. Show that \( \mathcal{H} \) is a countable base of the topological space \( (\Omega, \mathcal{T}) \).

12. Show that \( \otimes_{n=1}^{+\infty} B(\Omega_n) \subseteq B(\Omega) \).
13. Show that \( H \subseteq \bigotimes_{n=1}^{+\infty} B(\Omega_n) \).

14. Show that \( B(\Omega) = \bigotimes_{n=1}^{+\infty} B(\Omega_n) \)

**Theorem 27** Let \( (\Omega_n, T_n)_{n \geq 1} \) be a sequence of topological spaces with countable base. Then, the product space \( (\prod_{n=1}^{+\infty} \Omega_n, \bigcirc_{n=1}^{+\infty} T_n) \) has a countable base and:

\[
B\left(\prod_{n=1}^{+\infty} \Omega_n\right) = \bigotimes_{n=1}^{+\infty} B(\Omega_n)
\]

**Exercise 18.**

1. Show that if \( (\Omega, T) \) has a countable base and \( n \geq 1 \):

\[
B(\Omega^n) = B(\Omega) \otimes \ldots \otimes B(\Omega)
\]
2. Show that $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$.

3. Show that $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

**Definition 58** We say that a metric space $(E, d)$ is **separable**, if and only if there exists a finite or countable dense subset of $E$, i.e. a finite or countable subset $A$ of $E$ such that $E = \overline{A}$, where $\overline{A}$ is the closure of $A$ in $E$.

**Exercise 19.** Let $(E, d)$ be a metric space.

1. Suppose that $(E, d)$ is separable. Let $\mathcal{H} = \{ B(x_n, \frac{1}{p}) : n, p \geq 1 \}$, where $\{ x_n : n \geq 1 \}$ is a countable dense subset in $E$. Show that $\mathcal{H}$ is a countable base of the metric topological space $(E, T_E^d)$.

2. Suppose conversely that $(E, T_E^d)$ has a countable base $\mathcal{H}$. For all $V \in \mathcal{H}$ such that $V \neq \emptyset$, take $x_V \in V$. Show that the set $\{ x_V : V \in \mathcal{H}, V \neq \emptyset \}$ is at most countable and dense in $E$. 

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3. For all \( x, y, x', y' \in E \), show that:
\[
|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')
\]

4. Let \( T_{E \times E} \) be the product topology on \( E \times E \). Show that the map \( d : (E \times E, T_{E \times E}) \to (\mathbb{R}^+, T^{\mathbb{R}^+}) \) is continuous.

5. Show that \( d : (E \times E, B(E \times E)) \to (\mathbb{R}, B(\mathbb{R})) \) is measurable.

6. Show that \( d : (E \times E, B(E) \otimes B(E)) \to (\mathbb{R}, B(\mathbb{R})) \) is measurable, whenever \((E, d)\) is a separable metric space.

7. Let \((\Omega, \mathcal{F})\) be a measurable space and \( f, g : (\Omega, \mathcal{F}) \to (E, B(E)) \) be measurable maps. Show that \( \Phi : (\Omega, \mathcal{F}) \to E \times E \) defined by \( \Phi(\omega) = (f(\omega), g(\omega)) \) is measurable with respect to the product \( \sigma \)-algebra \( B(E) \otimes B(E) \).

8. Show that if \((E, d)\) is separable, then \( \Psi : (\Omega, \mathcal{F}) \to (\mathbb{R}, B(\mathbb{R})) \) defined by \( \Psi(\omega) = d(f(\omega), g(\omega)) \) is measurable.

9. Show that if \((E, d)\) is separable then \( \{f = g\} \in \mathcal{F} \).
10. Let \((E_n, d_n)_{n \geq 1}\) be a sequence of separable metric spaces. Show that the product space \(\prod_{n=1}^{\infty} E_n\) is metrizable and separable.

**Exercise 20.** Prove the following theorem.

**Theorem 28** Let \((\Omega_i, \mathcal{F}_i)_{i \in I}\) be a family of measurable spaces and \((\Omega, \mathcal{F})\) be a measurable space. For all \(i \in I\), let \(f_i : \Omega \to \Omega_i\) be a map, and define \(f : \Omega \to \prod_{i \in I} \Omega_i\) by \(f(\omega) = (f_i(\omega))_{i \in I}\). Then, the map:

\[ f : (\Omega, \mathcal{F}) \to \left( \prod_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{F}_i \right) \]

is measurable, if and only if each \(f_i : (\Omega, \mathcal{F}) \to (\Omega_i, \mathcal{F}_i)\) is measurable.

**Exercise 21.**

1. Let \(\phi, \psi : \mathbb{R}^2 \to \mathbb{R}\) with \(\phi(x, y) = x + y\) and \(\psi(x, y) = x.y\). Show that both \(\phi\) and \(\psi\) are continuous.
2. Show that \( \phi, \psi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})) \rightarrow (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})) \) are measurable.

3. Let \((\Omega, \mathcal{F})\) be a measurable space, and \(f, g : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))\)
   be measurable maps. Using the previous results, show that \(f + g\) and \(f \cdot g\) are measurable with respect to \(\mathcal{F}\) and \(\mathcal{B}(\mathbb{R})\).