Probability Tutorials: Notations

1. Tutorial 1

\( \triangleq \): equality which is true by definition, hence always true.
\( \Omega \): an arbitrary set.
\( P(\Omega) \): the power set of \( \Omega \), i.e. the set of all subsets of \( \Omega \).
\( D \): a set of subsets of \( \Omega \), also a Dynkin system on \( \Omega \).
\( F \): a set of subsets of \( \Omega \), also a \( \sigma \)-algebra on \( \Omega \).
\( \Omega \in D \): \( \Omega \) is an element of the set \( D \).
\( A, B \): arbitrary subsets of \( \Omega \).
\( (A_n)_{n \geq 1} \): a sequence of subsets of \( \Omega \).
\( A \subseteq B \): \( A \) is a subset of \( B \), i.e. \( x \in A \Rightarrow x \in B \).
\( B \setminus A \): set difference defined by \( B \setminus A = \{ x \in B : x \notin A \} \).
\( \bigcup_{n=1}^{\infty} A_n \): union of all \( A_n \)'s, \( \bigcup_{n=1}^{\infty} A_n = \{ x : \exists n \geq 1, x \in A_n \} \).
\( A^c \): the complement of \( A \) in \( \Omega \), \( A^c = \{ x \in \Omega : x \notin A \} \).
\( A \cup B \): union of \( A \) and \( B \), \( A \cup B = \{ x : x \in A \text{ or } x \in B \} \).
A \cap B : \text{intersection of } A \text{ and } B,\ A \cap B = \{x : x \in A \text{ and } x \in B\}.

(D_i)_{i \in I} : \text{a family of Dynkin systems on } \Omega, \text{ indexed by a set } I.

\cap_{i \in I} D_i : \text{intersection of all } D_i\text{'s, } \cap_{i \in I} D_i = \{A : \forall i \in I, A \in D_i\}.

(F_i)_{i \in I} : \text{a family of } \sigma\text{-algebras on } \Omega, \text{ indexed by a set } I.

\cap_{i \in I} F_i : \text{intersection of all } F_i\text{'s, } \cap_{i \in I} F_i = \{A : \forall i \in I, A \in F_i\}.

\mathcal{A} : \text{a set of subsets of } \Omega, \text{ a subset of } \mathcal{P}(\Omega).

\mathcal{D}(\mathcal{A}) : \text{the set of all Dynkin systems on } \Omega, \text{ containing } \mathcal{A}.

\mathcal{D}(\mathcal{A}) : \text{the Dynkin system on } \Omega, \text{ generated by } \mathcal{A}.

\sigma(\mathcal{A}) : \text{the } \sigma\text{-algebra on } \Omega, \text{ generated by } \mathcal{A}.

\mathcal{C} : \text{a set of subsets of } \Omega, \text{ also a } \pi\text{-system on } \Omega.

2. Tutorial 2

\Omega : \text{an arbitrary set.}

\mathcal{P}(\Omega) : \text{the power set of } \Omega, \text{ i.e. the set of all subsets of } \Omega.

\emptyset : \text{the empty set, i.e. the only set with no elements.}

B \setminus A : \text{set difference defined by } B \setminus A = \{x \in B : x \notin A\}.

\uplus : \text{union of pairwise disjoint sets.}
\(\mathcal{R}\) : a set of subsets of \(\Omega\), also a ring on \(\Omega\).

\((\mathcal{R}_i)_{i \in I}\) : a family of rings on \(\Omega\), indexed by a set \(I\).

\(\mathcal{A}\) : a set of subsets of \(\Omega\), a subset of \(\mathcal{P}(\Omega)\).

\(\mathcal{R}(\mathcal{A})\) : the set of all rings on \(\Omega\), containing \(\mathcal{A}\).

\(\mathcal{R}(\mathcal{A})\) : the ring on \(\Omega\), generated by \(\mathcal{A}\).

\(\mu\) : a measure defined on a set of subsets of \(\Omega\).

\([0, +\infty]\) : the set \(\mathbb{R}^+ \cup \{+\infty\}\).

\(\mathcal{R}(\mathcal{S})\) : the ring on \(\Omega\), generated by the semi-ring \(\mathcal{S}\).

\(\bar{\mu}, \bar{\mu}'\) : measures defined on the ring \(\mathcal{R}(\mathcal{S})\).

\(\bar{\mu}|_{\mathcal{S}}, \bar{\mu}'|_{\mathcal{S}}\) : the restrictions of \(\bar{\mu}\) and \(\bar{\mu}'\) to the smaller domain \(\mathcal{S}\).

\(\mu^*\) : an outer-measure on \(\Omega\).

\(\Sigma(\mu^*), \Sigma\) : the \(\sigma\)-algebra on \(\Omega\), associated with \(\mu^*\).

\(A, B, T\) : arbitrary subsets of \(\Omega\).

\(A^c\) : the complement of \(A\) in \(\Omega\), \(A^c = \{x \in \Omega : x \notin A\}\).

\(\mu^*_\Sigma\) : the restriction of \(\mu^*\) to the smaller domain \(\Sigma\).

\(\sigma(\mathcal{R}), \sigma(\mathcal{R}(\mathcal{S})), \sigma(\mathcal{S})\) : \(\sigma\)-algebras on \(\Omega\), generated by \(\mathcal{R}, \mathcal{R}(\mathcal{S}), \mathcal{S}\).

\(\mu'\) : a measure defined on \(\sigma(\mathcal{R})\), or \(\sigma(\mathcal{S})\).

\(\mu'|_{\mathcal{R}}, \mu'|_{\mathcal{S}}\) : the restrictions of \(\mu'\) to the smaller domains \(\mathcal{R}\) and \(\mathcal{S}\).
3. Tutorial 3

Ω : an arbitrary set.
\mathcal{P}(\Omega) : the power set of \Omega, i.e. the set of all subsets of \Omega.
\mathcal{A} : a set of subsets of \Omega.
\mu : a finitely additive map on \mathcal{A} or a measure on \mathcal{F}.
\psi : a union of pairwise disjoint sets.
A, A_1, A_n : arbitrary subsets of \Omega.
a \lor b : the largest of a and b, a \lor b = \max(a, b).
a \land b : the smallest of a and b, a \land b = \min(a, b).
\mathcal{S} : the semi-ring \mathcal{S} = \{[a, b], a, b \in \mathbb{R}\}, or a semi-ring on \Omega.
\mathcal{R}(\mathcal{S}) : the ring generated by \mathcal{S}.
\bar{\mu} : a finitely additive map defined on \mathcal{R}(\mathcal{S}).
F : a right-continuous and non-decreasing map defined on \mathbb{R} or \mathbb{R}^+.
T : a topology on \Omega.
(\Omega, T) : a topological space.
\mathcal{B}(\Omega) : the Borel \sigma-algebra on (\Omega, T).
\mathbb{R} : the real line \mathbb{R} = ]-\infty, +\infty[. 

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**R**^+ : the subset of **R**, **R**^+ = [0, +∞].

**T**_**R** : the usual topology on **R**.

**B**(**R**) : the Borel **σ**-algebra on **R**.

**B**(**R**^+): the Borel **σ**-algebra on **R**^+.

**Q** : the set of all rational numbers.

**σ**(**S**) : the **σ**-algebra generated by **S**.

**F** : a **σ**-algebra on Ω.

(Ω, **F**) : a measurable space.

(Ω, **F**, **μ**) : a measure space.

A_1 \uparrow A : for all n \geq 1, A_n \subseteq A_{n+1} and A = \bigcup_{n=1}^{+\infty} A_n.

A_1 \downarrow A : for all n \geq 1, A_{n+1} \subseteq A_n and A = \bigcap_{n=1}^{+\infty} A_n.

**D**_n : a Dynkin system on **R** or **R**^+.

**μ**_1, **μ**_2 : measures defined on **B**(**R**) or **B**(**R**^+).

d**F** : the Stieltjes measure on **B**(**R**) or **B**(**R**^+) associated with **F**.

d**x** : the Lebesgue measure on **B**(**R**).

**F**(x_0−) : the left limit of **F** at x = x_0.

Ω' : a subset of Ω.

**A**_Ω : the trace of **A** on Ω, **A**_{Ω'} = \{ A \cap Ω' : A \in **A** \}.
\( T' \): the topology on \( \Omega' \), induced by the topology \( T \) on \( \Omega \).
\( \sigma(\mathcal{A}) \): the \( \sigma \)-algebra on \( \Omega \) generated by \( \mathcal{A} \).
\( \sigma(\mathcal{A}|_{\Omega'}) \): the \( \sigma \)-algebra on \( \Omega' \) generated by \( \mathcal{A}|_{\Omega'} \).
\( \sigma(\mathcal{A})|_{\Omega'} \): the trace of \( \sigma(\mathcal{A}) \) on \( \Omega' \).
\( \mathcal{B}(\Omega)|_{\Omega'} \): the trace of \( \mathcal{B}(\Omega) \) on \( \Omega' \).
\( \mathcal{B}(\Omega') \): the Borel \( \sigma \)-algebra on \( (\Omega', T|_{\Omega'}) \).
\( \mathcal{F}|_{\Omega'} \): the trace of \( \mathcal{F} \) on \( \Omega' \).
\( \mu|_{\Omega'} \): the restriction of \( \mu \) to \( \mathcal{F}|_{\Omega'} \), when \( \Omega' \in \mathcal{F} \).

4. Tutorial 4

\( f: A \to B \): a map defined on \( A \) with values in \( B \).
\( f(A') \): direct image of \( A' \) by \( f \), \( f(A') = \{ f(x) : x \in A' \} \).
\( f^{-1}(B') \): inverse image of \( B' \) by \( f \), \( f^{-1}(B') = \{ x \in A : f(x) \in B' \} \).
\( \{ f \in B' \} \): same as \( f^{-1}(B') \).
\( (\Omega, T), (S, T_S) \): topological spaces.
\( (E, d), (F, \delta) \): metric spaces.
\( B(x, \epsilon) \): the open ball on \( E \), \( B(x, \epsilon) = \{ y \in E : d(x, y) < \epsilon \} \).
$T^d_E$: the metric topology on $E$, associated with the metric $d$.

$d|_F$: restriction of the metric $d$ to $F \times F$, when $F \subseteq E$.

$T_F$, $(T^d_E)|_F$: the topology on $F$, induced by the metric topology $T^d_E$.

$T_F$, $T^d|_F$: the metric topology on $F$, associated with the metric $d|_F$.

$\bar{\mathbb{R}}$: the extended real line, $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$.

$T_{\bar{\mathbb{R}}}$: the usual topology on $\bar{\mathbb{R}}$.

$T_\mathbb{R}$: the usual topology on $\mathbb{R}$.

$(T_{\bar{\mathbb{R}}})|_{\mathbb{R}}$: the topology on $\mathbb{R}$, induced by the usual topology on $\bar{\mathbb{R}}$.

$\mathcal{B}(\mathbb{R})$: the Borel $\sigma$-algebra on $\mathbb{R}$.

$\mathcal{B}(\mathbb{R})$: the Borel $\sigma$-algebra on $\mathbb{R}$.

$\mathcal{B}(\mathbb{R})|_R$: the trace of $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$.

$T^d_{\bar{\mathbb{R}}}$: the metric topology on $\bar{\mathbb{R}}$ associated with the metric $d$.

$(\Omega, \mathcal{F})$, $(S, \Sigma)$, $(S_1, \Sigma_1)$: measurable spaces.

$\Sigma'$, $\Sigma|_{S'}$: the trace of $\Sigma$ on $S'$.

$g \circ f$: the composition of $g$ and $f$, defined by $g \circ f(x) = g(f(x))$.

$\mathcal{A}$: a set of subsets of $S$.

$\sigma(\mathcal{A})$: the $\sigma$-algebra on $S$ generated by $\mathcal{A}$.
\(C_1, C_2, C_3, C_4\) : set of subsets of \(\overline{\mathbb{R}}\).
\(\{f \leq c\}\) : the inverse image of \([-\infty, c]\) by \(f\).
\(\{f < c\}\) : the inverse image of \([-\infty, c[\) by \(f\).
\(\{c \leq f\}\) : the inverse image of \([c, +\infty]\) by \(f\).
\(\{c < f\}\) : the inverse image of \([c, +\infty[\) by \(f\).
\(\inf_{n \geq 1} v_n\) : the greatest lower-bound of \(\{v_n : n \geq 1\}\).
\(\sup_{n \geq 1} v_n\) : the smallest upper-bound of \(\{v_n : n \geq 1\}\).
\(\liminf v_n\) : the lower limit of \((v_n)_{n \geq 1}\) as \(n \to +\infty\).
\(\limsup v_n\) : the upper limit of \((v_n)_{n \geq 1}\) as \(n \to +\infty\).
\(\lim v_n\) : the limit of \((v_n)_{n \geq 1}\) as \(n \to +\infty\).
\(f^+\) : the positive part of \(f\), \(f^+ = \max(f, 0)\).
\(f^-\) : the negative part of \(f\), \(f^- = \max(-f, 0)\).
\(\bar{A}\) : the closure of \(A\) in \((\Omega, \mathcal{T})\).
\(d(x, A)\) : the distance from \(x\) to \(A\), \(d(x, A) = \inf\{d(x, y) : y \in A\}\).
\(\lim f_n\) : simple limit of \((f_n)_{n \geq 1}\), defined by \((\lim f_n)(\omega) = \lim f_n(\omega)\).
\(\mathbb{C}\) : the set of complex numbers.
\(\text{Re}(f)\) : the real part of \(f\).
\(\text{Im}(f)\) : the imaginary part of \(f\).
5. Tutorial 5

\((\Omega, F, \mu)\) : an arbitrary measure space.

\(1_A\) : the characteristic function of \(A \subseteq \Omega\).

\(\cup\) : a union of pairwise disjoint sets.

\(I_\mu(s)\) : the integral w.r. to \(\mu\) of the simple function \(s\) on \((\Omega, F)\).

\(\int f d\mu\) : the Lebesgue integral of \(f\) with respect to \(\mu\).

\(v_n \uparrow v\) : for all \(n \geq 1\), \(v_n \leq v_{n+1}\) and \(v = \sup_{n \geq 1} v_n\).

\(f_n \uparrow f\) : for all \(\omega \in \Omega\), \(f_n(\omega) \uparrow f(\omega)\).

\(A_n \uparrow A\) : for all \(n \geq 1\), \(A_n \subseteq A_{n+1}\) and \(A = \cup_{n=1}^{+\infty} A_n\).

\(P(\omega), \mu\)-a.s. : the property \(P\) holds \(\mu\)-almost surely.

\(\mathcal{F}_A\) : the trace of \(\mathcal{F}\) on \(A \subseteq \Omega\).

\(\mu|_A\) : the restriction of \(\mu\) to \(\mathcal{F}_A\), when \(A \in \mathcal{F}\).

\(f|_A\) : the restriction of \(f\) to \(A\).

\(\mu^A\) : the measure defined on \(\mathcal{F}\) by \(\mu^A(E) = \mu(A \cap E)\).

\(\int_A f d\mu\) : the partial Lebesgue integral of \(f\) over \(A\) with respect to \(\mu\).

\(L^1_R(\Omega, F, \mu)\) : set of \(R\)-valued, measurable maps with \(\int |f| d\mu < +\infty\).

\(L^1_C(\Omega, F, \mu)\) : set of \(C\)-valued, measurable maps with \(\int |f| d\mu < +\infty\).
6. Tutorial 6

$I$ : an arbitrary non-empty set.

$(\Omega_i)_{i \in I}$ : a family of sets indexed by $I$.

$\prod_{i \in I} \Omega_i$ : the cartesian product of the family $(\Omega_i)_{i \in I}$.

$\Omega_i$ : the cartesian product when $\Omega_i = \Omega$, for all $i \in I$.

$\prod_{n=1}^{\infty} \Omega_n$ : the cartesian product when $I = \mathbb{N}^*$.

$\Omega_1 \times \ldots \times \Omega_n$ : the cartesian product when $I = \mathbb{N}_n$.

$\mathbb{N}$ : the set $\mathbb{N} = \{0, 1, 2, \ldots\}$.

$\mathbb{N}^*$ : the set $\mathbb{N}^* = \{1, 2, 3, \ldots\}$.

$\mathbb{N}_n$ : the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

$(\lambda_{\lambda \in \Omega})$ : a partition of the set $I$.

$(\mathcal{E}_i)_{i \in I}$ : a family, where each $\mathcal{E}_i$ is a set of subsets of $\Omega_i$.

$\prod_{i \in I} \mathcal{E}_i$ : a rectangle of the family $(\mathcal{E}_i)_{i \in I}$.

$\prod_{i \in I} \mathcal{E}_i$ : the set of all rectangles of the family $(\mathcal{E}_i)_{i \in I}$.

$\mathcal{E}_1 \times \ldots \times \mathcal{E}_n$ : the set of all rectangles when $I = \mathbb{N}_n$.

$(\Omega_i, \mathcal{F}_i)_{i \in I}$ : a family of measurable spaces indexed by $I$.

$\prod_{i \in I} \mathcal{F}_i$ : the set of measurable rectangles, the rectangles of $(\mathcal{F}_i)_{i \in I}$.
$\otimes_{i \in I} \mathcal{F}_i$: the product $\sigma$-algebra of $(\mathcal{F}_i)_{i \in I}$ on $\Pi_{i \in I} \Omega_i$.

$\sigma(\bigcap_{i \in I} \mathcal{F}_i)$: the $\sigma$-algebra generated by the measurable rectangles.

$\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$: the product $\sigma$-algebra when $I = \mathbb{N}_n$.

$\sigma(\mathcal{E}_i)$: the $\sigma$-algebra on $\Omega_i$, generated by $\mathcal{E}_i$.

$\otimes_{i \in I} \sigma(\mathcal{E}_i)$: the product $\sigma$-algebra of $(\sigma(\mathcal{E}_i))_{i \in I}$ on $\Pi_{i \in I} \Omega_i$.

$\prod_{i \in I} \sigma(\mathcal{E}_i)$: the set of measurable rectangles of $(\sigma(\mathcal{E}_i))_{i \in I}$.

$\mathcal{T}_R$: the usual topology on $\mathbb{R}$.

$\mathcal{T}_R \Pi \ldots \Pi \mathcal{T}_R$: set of rectangles when $I = \mathbb{N}_n$ and $\mathcal{E}_i = \mathcal{T}_R$.

$\mathcal{A}$: a set of subsets of $\Omega$.

$\mathcal{T}(\mathcal{A})$: the topology on $\Omega$, generated by $\mathcal{A}$.

$(\Omega_i, \mathcal{T}_i)_{i \in I}$: a family of topological spaces indexed by $I$.

$\prod_{i \in I} \mathcal{T}_i$: the set of rectangles of $(\mathcal{T}_i)_{i \in I}$.

$\otimes_{i \in I} \mathcal{T}_i$: the product topology of $(\mathcal{T}_i)_{i \in I}$ on $\Pi_{i \in I} \Omega_i$.

$\mathcal{B}(\Omega_i)$: the Borel $\sigma$-algebra on $(\Omega_i, \mathcal{T}_i)$.

$\otimes_{i \in I} \mathcal{B}(\Omega_i)$: product $\sigma$-algebra of $(\mathcal{B}(\Omega_i))_{i \in I}$ on $\Pi_{i \in I} \Omega_i$.

$\mathcal{H}$: a countable base of $(\Omega, \mathcal{T})$.

$\mathcal{B}(\Pi_{i \in I} \Omega_i)$: the Borel $\sigma$-algebra for the product topology.
7. Tutorial 7

\(E^{\omega_1}:\) \(\omega_1\)-section of a subset \(E\) of \(\Omega_1 \times \Omega_2\).
\(\mathcal{F}_1 \Pi \mathcal{F}_2:\) set of measurable rectangles of \(\mathcal{F}_1\) and \(\mathcal{F}_2\).
\(\mathcal{F}_1 \otimes \mathcal{F}_2:\) product \(\sigma\)-algebra of \(\mathcal{F}_1\) and \(\mathcal{F}_2\).
\(\mathcal{B}(E):\) Borel \(\sigma\)-algebra on a metric space \((E,d)\).
\(\Omega_n \uparrow \Omega: \) for all \(n \geq 1\), \(\Omega_n \subseteq \Omega_{n+1}\) and \(\Omega = \bigcup_{n=1}^{\infty} \Omega_n\).
\(\mu_1 \otimes \ldots \otimes \mu_n:\) product of \(\sigma\)-finite measures.
\(dx^n:\) the Lebesgue measure on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\).
\(\mathcal{N}_n:\) the set \(\{1, \ldots, n\}\).
\(\sigma:\) a permutation, i.e. a bijection \(\sigma: \mathcal{N}_n \to \mathcal{N}_n\).
\(f_p \uparrow f: \) For all \(p \geq 1\), \(f_p \leq f_{p+1}\) and \(f = \lim f_p\).
\(\int_{\Omega_2} f(\omega,x) d\mu_2(x):\) the integral of \(f(\omega,\cdot)\) w.r.t. \(\mu_2\), \(\omega \in \Omega_1\).

8. Tutorial 8

\(\lim_{x \uparrow x_0} \phi(x):\) the limit of \(\phi(x)\) as \(x \to x_0\) with \(x_0 < x\).
\(T|_K:\) the induced topology on \(K\).
\(\delta(A)\) : the diameter of a set \(A\).
\(\inf_{x\in\Omega} f(x)\) : the infimum of \(f(\Omega)\).
\(\sup_{x\in\Omega} f(x)\) : the supremum of \(f(\Omega)\).
\(f'(c)\) : the derivative of \(f\) evaluated at \(c\).
\(f^{(k)}(a)\) : the \(k\)th derivative of \(f\) evaluated at \(a\).
\(C^n\) : [of class] for all \(k \leq n\), \(f^{(k)}\) exists and is continuous.
\((\Omega, \mathcal{F}, P)\) : a probability space.
\((S, \Sigma)\) : a measurable space.
\(E[X]\) : the expectation of the random variable \(X\).
\(\phi \circ X\) : the composition \(\phi \circ X(\omega) = \phi(X(\omega))\).

9. Tutorial 9

\((\Omega, \mathcal{F}, \mu)\) : a measure space.
\(L^p_R(\Omega, \mathcal{F}, \mu)\) : set of \(\mathbb{R}\)-valued measurable maps \(f\), with \(\|f\|_p < +\infty\).
\(L^p_C(\Omega, \mathcal{F}, \mu)\) : set of \(\mathbb{C}\)-valued measurable maps \(f\), with \(\|f\|_p < +\infty\).
\(\|f\|_p\) : \(p\)-norm of \(f\). For \(p \in [1, +\infty[\), \(\|f\|_p = (\int |f|^p d\mu)^{1/p}\).
\(\|f\|_\infty\) : \(\infty\)-norm of \(f\). \(\|f\|_\infty = \inf \{M \in \mathbb{R}^+ : |f| \leq M, \mu\text{-a.s.}\}\).
\[ B(f, \epsilon) : \text{the open ball in } L^p(\Omega, \mathcal{F}, \mu) \text{ or } L^p_C(\Omega, \mathcal{F}, \mu). \]

\[ x_n \xrightarrow{T} x : (x_n)_{n \geq 1} \text{ converges to } x, \text{ with respect to the topology } T. \]

\[ f_n \xrightarrow{L^p} f : (f_n)_{n \geq 1} \text{ converges to } f \text{ in } L^p. \|f_n - f\|_p \to 0. \]

\[ f_n \rightarrow f : (f_n)_{n \geq 1} \text{ converges to } f, \text{ simply: } f_n(x) \to f(x) \text{ for all } x. \]

\[ f_n \rightarrow f, \mu\text{-a.s. : } f_n(x) \to f(x) \text{ for } \mu\text{-almost all } x. \]

\[ (f_n)_{k \geq 1} : \text{a sub-sequence of } (f_n)_{n \geq 1}. \]

### 10. Tutorial 10

\( K \): the field \( \mathbb{R} \) or \( \mathbb{C} \).

\( \mathbb{N}^* \): the set of positive integers, \( \mathbb{N}^* = \{1, 2, 3, \ldots\} \).

\( \mathcal{T}_{\mathbb{R}^n} \): usual topology on \( \mathbb{R}^n \).

\( \mathcal{T}_\mathbb{R} \): usual topology on \( \mathbb{R} \).

\[ x_n \xrightarrow{T} x : \text{convergence with respect to a topology } T. \]

\( d_{\mathbb{C}^n} \): usual metric on \( \mathbb{C}^n \).

\( d_{\mathbb{R}^n} \): usual metric on \( \mathbb{R}^n \).

\( \delta(A) \): diameter of \( A \), \( \delta(A) = \sup \{d(x, y) : x, y \in A\} \).

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\( \tilde{F} \): closure of the set \( F \).
\( \tilde{z} \): complex conjugate of \( z \). If \( z = a + ib \), \( \tilde{z} = a - ib \).
\( \langle \cdot, \cdot \rangle \): an inner-product on a \( K \)-vector space.
\( \| \cdot \| \): the norm induced by an inner product, \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \).
\( T(\cdot, \cdot) \): norm topology induced by an inner-product.
\( G^\perp \): orthogonal of a set \( G \) w.r. to some inner-product.
\([f]\): \( \mu \)-almost sure equivalence class of \( f \) in \( L_K^2(\Omega, \mathcal{F}, \mu) \).

11. Tutorial 11

\( N^* \): the set of positive integers \( N^* = \{1, 2, 3, \ldots\} \).
\( Z \): the set of integers \( Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \).
\( (\Omega, \mathcal{F}) \): a measurable space.
\( \sigma \): a bijection between \( N^* \) and itself.
\( \cup_{n \geq 1} \): a countable union of pairwise disjoint sets.
\( dx \): the Lebesgue measure on \( (\mathbb{R}^n, B(\mathbb{R}^n)) \).
\( M^1(\Omega, \mathcal{F}) \): set of complex measures on \( (\Omega, \mathcal{F}) \).
12. Tutorial 12

\( (\Omega, \mathcal{F}) \) : a measurable space.
\( \nu << \mu \) : the measure \( \nu \) is absolutely continuous w.r. to \( \mu \).
\( \limsup_{n \geq 1} E_n \) : the set \( \cap_{n \geq 1} \cup_{k \geq n} E_k \), also denoted \( \{ E_n : \text{i.o.} \} \).
\( M^1(\Omega, \mathcal{F}) \) : set of complex measures on \( (\Omega, \mathcal{F}) \).
\( |\nu| \) : total variation of complex measure \( \nu \).
\( E_n \uparrow E : E_n \subseteq E_{n+1} \) for all \( n \geq 1 \), and \( E = \cup_{n \geq 1} E_n \).
\( u^+ \) : positive part of function \( u \), \( u^+ = u \vee 0 = \max(u, 0) \).
\( \mu^+ \) : positive part of signed measure \( \mu \), \( \mu^+ = (|\mu| + \mu)/2 \).
\( \mathcal{F}|_A \) : trace of \( \sigma \)-algebra \( \mathcal{F} \) on \( A \), \( \mathcal{F}|_A = \{ A \cap E : E \in \mathcal{F} \} \).

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\(\mu|_{A}\): restriction of \(\mu\) to \(\mathcal{F}|_{A}\).
\(\mu^{A}\): the complex measure \(\mu(A \cap \cdot)\) on \((\Omega, \mathcal{F})\).
\(|\mu^{A}|\): total variation of the complex measure \(\mu^{A}\) on \((\Omega, \mathcal{F})\).
\(|\mu|_{A}\): total variation of the complex measure \(\mu|_{A}\) on \((A, \mathcal{F}|_{A})\).
\(|\mu^{A}|\): the measure \(|\mu|(A \cap \cdot)\).
\(|\mu|_{A}\): restriction of \(|\mu|\) to \(\mathcal{F}|_{A}\).
\(f|_{A}\): restriction of the map \(f\) to \(A\).
\(\int f|_{A}d\mu|_{A}\): integral of \(f|_{A}\) on the measure space \((A, \mathcal{F}|_{A}, \mu|_{A})\).
\(\mathcal{F}_{1} \otimes \ldots \otimes \mathcal{F}_{n}\): product of the \(\sigma\)-algebras \(\mathcal{F}_{1}, \ldots, \mathcal{F}_{n}\).
\(|\mu|\): total mass of total variation of \(\mu\), \(|\mu| = |\mu|(\Omega)\).

## 13. Tutorial 13

\(\mathbb{K}\): the field \(\mathbb{R}\) or \(\mathbb{C}\).
\(\mathcal{S}_{\mathbb{K}}(\Omega, \mathcal{F})\): set of \(\mathbb{K}\)-valued complex simple functions on \((\Omega, \mathcal{F})\).
\(\mathcal{C}_{\mathbb{K}}^{0}(\Omega)\): set of \(\mathbb{K}\)-valued continuous and bounded maps on \(\Omega\).
\(\mathcal{M}^{1}(\Omega, \mathcal{B}(\Omega))\): set of complex Borel measures on \(\Omega\).
\(d(x, A)\): distance from \(x\) to \(A\), \(d(x, A) = \inf\{d(x, y) : y \in A\}\).
\( \bar{A} \): closure of the set \( A \).
\( A^{\text{cl}} \): closure of the set \( A \), relative to the induced topology on \( \Omega' \).
\( B(x, \epsilon) \): open ball with center \( x \) and radius \( \epsilon \) in a metric space.
\( \text{supp}(\phi) \): support of \( \phi \), closure of \( \{ \phi \neq 0 \} \).
\( C^c_K(\Omega) \): set of \( K \)-valued continuous maps with compact support.

**14. Tutorial 14**

\( |b| \): total variation map of \( b : \mathbb{R}^+ \to \mathbb{C} \).
\( |b(t)| \): modulus of complex number \( b(t) \).
\( |b|(t) \): total variation of \( b \) evaluated at \( t \in \mathbb{R}^+ \).
\( |f(t)| \): modulus of complex number \( f(t) \).
\( B(\mathbb{R}^+), B(\mathbb{C}) \): Borel \( \sigma \)-algebras on \( \mathbb{R}^+ \) and \( \mathbb{C} \).
\( ds \): Lebesgue measure on \( (\mathbb{R}^+, B(\mathbb{R}^+)) \).
\( |b|^+ \): positive variation of \( b \).
\( |b|^− \): negative variation of \( b \).
\( db \): complex Stieltjes measure associated with \( b \).
\( \tilde{b}^T \): stopped map defined by \( \tilde{b}^T(t) = b(t \wedge T) \).
$C^c_C(\mathbb{R}^+)$: $\mathbb{C}$-valued continuous maps on $\mathbb{R}^+$ with compact support.

$C^b_C(\mathbb{R}^+)$: $\mathbb{C}$-valued continuous maps on $\mathbb{R}^+$ which are bounded.

$b(t-)$: left-limit of $b$ at $t$.

$\Delta b(t)$: jump of $b$ at $t$, $\Delta b(t) = b(t) - b(t-)$. 

15. Tutorial 15

$d|b|$: Stieltjes measure on $\mathbb{R}^+$ associated with total variation $|b|$.

$L^1_C(b)$: $\mathbb{C}$-valued, measurable maps $f$ with $\int_{\mathbb{R}^+} |f|d|b| < +\infty$.

$L^1_{C,loc}(b)$: measurable maps with $\int_0^t |f|d|b| < +\infty$ for all $t \in \mathbb{R}^+$.

$f_0^t$: partial Lebesgue integral on interval $[0,t]$.

$|db|$: total variation of complex Stieltjes measure $db$.

$t_n \downarrow t$: $t < t_{n+1} \leq t_n$ for all $n \geq 1$, and $t = \inf_{n \geq 1} t_n$.

$da$: Stieltjes measure on $\mathbb{R}^+$ associated with $a$.

$f.a$: the map defined by $(f.a)(t) = \int_0^t fda$.

$d(f.a)$: Stieltjes measure on $\mathbb{R}^+$ associated with $f.a$.

$a^T$: stopped map defined by $a^T = a(t \wedge T)$.

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\(d(f.a)^T\) : Stieltjes measure on \(\mathbb{R}^+\) associated with \((f.a)^T\).
\(|d(f.a)^T|\) : total variation of measure \(d(f.a)^T\).
\(\Delta a(t)\) : jump of \(a\) at \(t\), \(\Delta a(t) = a(t) - a(t^-)\).
\(d|b| << da\) : \(d|b|\) is absolutely continuous w.r. to \(da\).

16. Tutorial 16

\(\mathcal{B}(\Omega)\) : Borel \(\sigma\)-algebra on \(\Omega\).
\(L^1_\mathcal{R}(\Omega, \mathcal{B}(\Omega), \mu)\) : real valued Borel measurable \(f\)'s with \(\int |f|d\mu < +\infty\).
\(T_A\) : induced topology on \(A\), \(T_A = \{A \cap V : V \in T\}\).
\(\mathcal{T}_\mathcal{R}\) : usual topology on \(\mathbb{R}\).
\(|\mu|\) : total variation of complex measure \(\mu\).
\(M\mu\) : maximal function of complex measure \(\mu\).
\(\mathcal{B}(x, \epsilon)\) : open ball with center \(x\) and radius \(\epsilon\).
\(\mathcal{N}_p\) : the set \(\{1, \ldots, p\}\).
\(||\mu||\) : total mass of total variation, \(||\mu|| = |\mu|(\mathbb{R}^\mathbb{N})\).
\(Mf\) : maximal function of \(f\).

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$dx(B(x, \epsilon))$ : Lebesgue measure of open ball $B(x, \epsilon)$ in $\mathbb{R}^n$.

17. Tutorial 17

$K$ : the field $\mathbb{R}$ or $\mathbb{C}$.

$M_n(K)$ : set of $n \times n$ matrices with $K$-valued entries.

$e_1, \ldots, e_n$ : canonical basis of $K^n$.

$\mu^X, X(\mu)$ : law, distribution of $X$ under $\mu$, image measure of $\mu$ by $X$.

$X^{-1}(B), \{X \in B\}$ : inverse image of $B$ by $X$.

$Y \circ X$ : composition of $X$ and $Y$, $(Y \circ X)(\omega) = Y(X(\omega))$.

$\tau_a$ : translation mapping of vector $a$ in $\mathbb{R}^n$.

$\biguplus$ : union of pairwise disjoint sets.

$B(\mathbb{R}^n)$ : Borel $\sigma$-algebra on $\mathbb{R}^n$.

$\sigma(C)$ : $\sigma$-algebra on $\mathbb{R}^n$ generated by $C$.

$dx$ : Lebesgue measure on $\mathbb{R}^n$.

$\det \Sigma$ : determinant of matrix $\Sigma$.

$\dim V$ : dimension of linear subspace $V$ of $\mathbb{R}^n$.

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18. Tutorial 18

\( \mathbb{K} \): the field \( \mathbb{R} \) or \( \mathbb{C} \).

\( N \), \( \| \cdot \| \): norm on a \( \mathbb{K} \)-vector space.

\( E, F : \mathbb{K} \)-normed spaces.

\( \mathcal{L}_\mathbb{K}(E,F) \): set of continuous linear maps \( l : E \to F \).

\( d\phi(a) \): differential of \( \phi \) at \( a \).

\( d\phi \): differential mapping of \( \phi \).

\( \frac{\partial \phi}{\partial x_i}(a) \): \( i \)-th partial derivative of \( \phi \) at \( a \).

\( l|_U \): restriction of \( l \) to \( U \).

\( J(\phi)(a) \): jacobian of \( \phi \) at \( a \), determinant of \( d\phi(a) \).

\( B(\mathbb{R}^n) \): Borel \( \sigma \)-algebra on \( \mathbb{R}^n \).

\( dx|_\Omega \): Lebesgue measure on \( \Omega \in B(\mathbb{R}^n) \), restriction of \( dx \) to \( B(\Omega) \).

\( B(a, \epsilon) \): open ball with center \( a \) and radius \( \epsilon \).

\( \phi(dx|_\Omega) \): image measure of \( dx|_\Omega \) by \( \phi \), \( \phi(dx|_\Omega)(B) = dx|_\Omega(\phi^{-1}(B)) \).

\( \int |J(\psi)||dx|_\Omega \) : measure on \( \Omega' \) with density \( |J(\psi)| \) w.r. to \( dx|_\Omega \).
19. Tutorial 19

\( C^1(\mathbb{R}, \mathbb{R}) \) : real, continuously differentiable maps on \( \mathbb{R} \).

\( \mu_1 \ast \ldots \ast \mu_p \) : the convolution of \( \mu_1, \ldots, \mu_p \).

\( \mu \ast \nu \) : the convolution of \( \mu \) and \( \nu \).

\( \mu \otimes \nu \) : the product measure of \( \mu \) and \( \nu \).

\( B - x \) : the set \( \{ y \in \mathbb{R}^n : y + x \in B \} \).

\( \delta_a \) : Dirac probability measure on \( \mathbb{R}^n \), centered in \( a \in \mathbb{R}^n \).

\( \tau_a \) : translation mapping on \( \mathbb{R}^n \), \( \tau_a(x) = a + x \).

\( B(\mathbb{R}^n) \otimes B(\mathbb{R}^n) \) : product of Borel \( \sigma \)-algebras on \( \mathbb{R}^n \times \mathbb{R}^n \).

\( \mathcal{F} \mu \) : Fourier transform of complex measure \( \mu \).

\( C_b^0(\Omega) \) : set of real functions on \( \Omega \), which are continuous and bounded.

\(\mu_k \rightharpoonup \mu\), narrowly : for all \( f \in C_b^0(\Omega) \), \( \int f \, d\mu_k \to \int f \, d\mu \).

\( \phi_X \) : characteristic function of \( \mathbb{R}^n \)-valued random variable \( X \).

\( |\alpha| \) : for \( \alpha \in \mathbb{N}^n \), \( |\alpha| = \alpha_1 + \ldots + \alpha_n \).

\( x^\alpha \) : for \( \alpha \in \mathbb{N}^n \) and \( x \in \mathbb{R}^n \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

\( \partial^{\alpha} f \) : the \( |\alpha| \)-th order partial derivative of \( f \), \( \partial^{\alpha} f = \frac{\partial^{\left|\alpha\right|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \).

\( x^\alpha \mu \) : \( x^\alpha \mu = \int x^\alpha \, d\mu \), measure with density \( x^\alpha \) w.r. to \( \mu \).
20. Tutorial 20

\( \mathcal{M}_n(\mathbb{R}) \) : set of \( n \times n \) matrices with real entries.

\( M^t \) : transposed matrix of \( M \).

\( M^{-1} \) : inverse matrix of non-singular matrix \( M \).

\( \langle u, Mu \rangle \) : inner-product in \( \mathbb{R}^n \) of \( u \) and \( Mu \).

\( \Sigma \) : a symmetric and non-negative \( n \times n \) real matrix.

\( \phi(\mu) \) : image measure of \( \mu \) by \( \phi \), \( \phi(\mu)(B) = \mu(\phi^{-1}(B)) \).

\( \mathcal{F}P(u) \) : Fourier transform of probability \( P \), evaluated at \( u \).

\( N_n(m, \Sigma) \) : Gaussian measure on \( \mathbb{R}^n \) with mean \( m \) and covariance \( \Sigma \).

\( \mathcal{N}_1(0, 1) \) : reduced Gaussian measure on \( \mathbb{R} \).

\( x^\alpha \) : for \( \alpha \in \mathbb{N}^n \) and \( x \in \mathbb{R}^n \), \( x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \).

\( \text{cov}(X, Y) \) : covariance between square-integrable variables \( X \) and \( Y \).

\( \text{var}(X) \) : variance of square-integrable random variable \( X \).

\( \delta_0, \delta_1 \) : dirac probability measures on \( \mathbb{R} \), centered in \( 0 \) and \( 1 \).

\( \det \Sigma \) : determinant of matrix \( \Sigma \).

\( dx \) : Lebesgue measure on \( \mathbb{R}^n \).