4. Measurability

**Definition 25** Let $A$ and $B$ be two sets, and $f : A \to B$ be a map. Given $A' \subseteq A$, we call **direct image* of $A'$ by $f$ the set denoted $f(A')$, and defined by $f(A') = \{ f(x) : x \in A' \}$.

**Definition 26** Let $A$ and $B$ be two sets, and $f : A \to B$ be a map. Given $B' \subseteq B$, we call **inverse image* of $B'$ by $f$ the set denoted $f^{-1}(B')$, and defined by $f^{-1}(B') = \{ x : x \in A, f(x) \in B' \}$.

**Exercise 1.** Let $A$ and $B$ be two sets, and $f : A \to B$ be a bijection from $A$ to $B$. Let $A' \subseteq A$ and $B' \subseteq B$.

1. Explain why the notation $f^{-1}(B')$ is potentially ambiguous.

2. Show that the inverse image of $B'$ by $f$ is in fact equal to the direct image of $B'$ by $f^{-1}$.

3. Show that the direct image of $A'$ by $f$ is in fact equal to the inverse image of $A'$ by $f^{-1}$.
**Definition 27** Let \((\Omega, T)\) and \((S, T_S)\) be two topological spaces. A map \(f : \Omega \to S\) is said to be **continuous** if and only if:

\[ \forall B \in T_S, f^{-1}(B) \in T \]

In other words, if and only if the inverse image of any open set in \(S\) is an open set in \(\Omega\).

We Write \(f : (\Omega, T) \to (S, T_S)\) is continuous, as a way of emphasizing the two topologies \(T\) and \(T_S\) with respect to which \(f\) is continuous.

**Definition 28** Let \(E\) be a set. A map \(d : E \times E \to [0, +\infty]\) is said to be a **metric** on \(E\), if and only if:

1. \((i)\) \(\forall x, y \in E, d(x, y) = 0 \iff x = y\)
2. \((ii)\) \(\forall x, y \in E, d(x, y) = d(y, x)\)
3. \((iii)\) \(\forall x, y, z \in E, d(x, y) \leq d(x, z) + d(z, y)\)
Definition 29  A metric space is an ordered pair \((E, d)\) where \(E\) is a set, and \(d\) is a metric on \(E\).

Definition 30  Let \((E, d)\) be a metric space. For all \(x \in E\) and \(\epsilon > 0\), we define the so-called open ball in \(E\):

\[
B(x, \epsilon) \triangleq \{ y : y \in E, \; d(x, y) < \epsilon \}
\]

We call metric topology on \(E\), associated with \(d\), the topology \(T^d_E\) defined by:

\[
T^d_E \triangleq \{ U \subseteq E, \; \forall x \in U, \exists \epsilon > 0, B(x, \epsilon) \subseteq U \}
\]

Exercise 2. Let \(T^d_E\) be the metric topology associated with \(d\), where \((E, d)\) is a metric space.

1. Show that \(T^d_E\) is indeed a topology on \(E\).

2. Given \(x \in E\) and \(\epsilon > 0\), show that \(B(x, \epsilon)\) is an open set in \(E\).
Exercise 3. Show that the usual topology on $\mathbb{R}$ is nothing but the metric topology associated with $d(x, y) = |x - y|$.

Exercise 4. Let $(E, d)$ and $(F, \delta)$ be two metric spaces. Show that a map $f : E \to F$ is continuous, if and only if for all $x \in E$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in E$:

$$d(x, y) < \eta \implies \delta(f(x), f(y)) < \epsilon$$

Definition 31 Let $(\Omega, T)$ and $(S, T_S)$ be two topological spaces. A map $f : \Omega \to S$ is said to be a homeomorphism, if and only if $f$ is a continuous bijection, such that $f^{-1}$ is also continuous.

Definition 32 A topological space $(\Omega, T)$ is said to be metrizable, if and only if there exists a metric $d$ on $\Omega$, such that the associated metric topology coincides with $T$, i.e. $T_d^T = T$. 

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**Definition 33** Let $(E,d)$ be a metric space and $F \subseteq E$. We call induced metric on $F$, denoted $d_F$, the restriction of the metric $d$ to $F \times F$, i.e. $d_F = d|_{F \times F}$.

**Exercise 5.** Let $(E,d)$ be a metric space and $F \subseteq E$. We define $T_F = (T_E^d)|_F$ as the topology on $F$ induced by the metric topology on $E$. Let $T_F' = T_F^{d_F}$ be the metric topology on $F$ associated with the induced metric $d_F$ on $F$.

1. Show that $T_F \subseteq T_F'$.
2. Given $A \in T_F'$, show that $A = (\cup_{x \in A} (B(x, \epsilon_x))) \cap F$ for some $\epsilon_x > 0$, $x \in A$, where $B(x, \epsilon_x)$ denotes the open ball in $E$.
3. Show that $T_F' \subseteq T_F$. 

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Theorem 12  Let $(E,d)$ be a metric space and $F \subseteq E$. Then, the topology on $F$ induced by the metric topology, is equal to the metric topology on $F$ associated with the induced metric, i.e. $(\mathcal{T}_d^E)_{|F} = \mathcal{T}_d^F$. 

Exercise 6. Let $\phi : \mathbb{R} \to ]-1,1[$ be the map defined by:

\[ \forall x \in \mathbb{R}, \quad \phi(x) \triangleq \frac{x}{|x| + 1} \]

1. Show that $[-1,0]$ is not open in $\mathbb{R}$.
2. Show that $[-1,0]$ is open in $[-1,1]$.
3. Show that $\phi$ is a homeomorphism between $\mathbb{R}$ and $]-1,1[$.
4. Show that $\lim_{x \to +\infty} \phi(x) = 1$ and $\lim_{x \to -\infty} \phi(x) = -1$.

Exercise 7. Let $\bar{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$. Let $\phi$ be defined.
as in exercise (6), and \( \overline{\phi} : \bar{\mathbb{R}} \to [-1, 1] \) be the map defined by:

\[
\overline{\phi}(x) = \begin{cases} 
\phi(x) & \text{if } x \in \mathbb{R} \\
1 & \text{if } x = +\infty \\
-1 & \text{if } x = -\infty
\end{cases}
\]

Define:

\[ T_{\bar{\mathbb{R}}} \triangleq \{ U \subseteq \bar{\mathbb{R}} , \overline{\phi}(U) \text{ is open in } [-1, 1] \} \]

1. Show that \( \overline{\phi} \) is a bijection from \( \bar{\mathbb{R}} \) to \([-1, 1]\), and let \( \overline{\psi} = \overline{\phi}^{-1} \).
2. Show that \( T_{\bar{\mathbb{R}}} \) is a topology on \( \bar{\mathbb{R}} \).
3. Show that \( \overline{\phi} \) is a homeomorphism between \( \bar{\mathbb{R}} \) and \([-1, 1]\).
4. Show that \([-\infty, 2[, ]3, +\infty[, ]3, +\infty[ \) are open in \( \bar{\mathbb{R}} \).
5. Show that if \( \phi' : \bar{\mathbb{R}} \to [-1, 1] \) is an arbitrary homeomorphism, then \( U \subseteq \bar{\mathbb{R}} \) is open, if and only if \( \phi'(U) \) is open in \([-1, 1]\).
Definition 34  The usual topology on $\bar{\mathbb{R}}$ is defined as:

$$T_{\bar{\mathbb{R}}} \triangleq \{ U \subseteq \bar{\mathbb{R}} , \ \tilde{\phi}(U) \text{ is open in } [-1, 1] \}$$

where $\tilde{\phi} : \bar{\mathbb{R}} \to [-1, 1]$ is defined by $\tilde{\phi}(-\infty) = -1$, $\tilde{\phi}(+\infty) = 1$ and:

$$\forall x \in \mathbb{R} \ , \ \tilde{\phi}(x) \triangleq \frac{x}{|x| + 1}$$

Exercise 8. Let $\phi$ and $\tilde{\phi}$ be as in exercise (7). Define:

$$T' \triangleq (T_{\bar{\mathbb{R}}})|_{\mathbb{R}} \triangleq \{ U \cap \mathbb{R} , \ U \in T_{\bar{\mathbb{R}}} \}$$

1. Recall why $T'$ is a topology on $\mathbb{R}$.

2. Show that for all $U \subseteq \bar{\mathbb{R}}$, $\phi(U \cap \mathbb{R}) = \tilde{\phi}(U) \cap [-1, 1]$.

3. Explain why if $U \in T_{\bar{\mathbb{R}}}$, $\phi(U \cap \mathbb{R})$ is open in $[-1, 1]$.

4. Show that $T' \subseteq T_{\mathbb{R}}$, (the usual topology on $\mathbb{R}$).
5. Let $U \in \mathcal{T}_R$. Show that $\bar{\phi}(U)$ is open in $]-1,1[\text{ and } [-1,1]$.

6. Show that $\mathcal{T}_R \subseteq \mathcal{T}_\bar{R}$

7. Show that $\mathcal{T}_R = \mathcal{T}'$, i.e. that the usual topology on $\bar{R}$ induces the usual topology on $R$.

8. Show that $\mathcal{B}(R) = \mathcal{B}(\bar{R})|_R = \{B \cap R, B \in \mathcal{B}(\bar{R})\}$

**Exercise 9.** Let $d: \bar{R} \times \bar{R} \to [0, +\infty]$ be defined by:

$$\forall (x, y) \in \bar{R} \times \bar{R}, \quad d(x, y) = |\phi(x) - \phi(y)|$$

where $\phi$ is an arbitrary homeomorphism from $\bar{R}$ to $[-1,1]$.

1. Show that $d$ is a metric on $R$.

2. Show that if $U \in \mathcal{T}_R$, then $\phi(U)$ is open in $[-1,1]$
3. Show that for all \( U \in \mathcal{T}_{\bar{\mathbb{R}}} \) and \( y \in \phi(U) \), there exists \( \epsilon > 0 \) such that:
\[
\forall z \in [-1, 1], |z - y| < \epsilon \Rightarrow z \in \phi(U)
\]

4. Show that \( \mathcal{T}_{\bar{\mathbb{R}}} \subseteq \mathcal{T}_{\bar{\mathbb{R}}^d} \).

5. Show that for all \( U \in \mathcal{T}_{\bar{\mathbb{R}}^d} \) and \( x \in U \), there is \( \epsilon > 0 \) such that:
\[
\forall y \in \bar{\mathbb{R}}, |\phi(x) - \phi(y)| < \epsilon \Rightarrow y \in U
\]

6. Show that for all \( U \in \mathcal{T}_{\bar{\mathbb{R}}^d} \), \( \phi(U) \) is open in \([-1, 1]\).

7. Show that \( \mathcal{T}_{\bar{\mathbb{R}}^d} \subseteq \mathcal{T}_{\bar{\mathbb{R}}} \).

8. Prove the following theorem.

**Theorem 13** The topological space \((\bar{\mathbb{R}}, \mathcal{T}_{\bar{\mathbb{R}}})\) is metrizable.
**Definition 35** Let \((\Omega, \mathcal{F})\) and \((S, \Sigma)\) be two measurable spaces. A map \(f : \Omega \to S\) is said to be **measurable** with respect to \(\mathcal{F}\) and \(\Sigma\), if and only if:

\[
\forall B \in \Sigma, \quad f^{-1}(B) \in \mathcal{F}
\]

We Write \(f : (\Omega, \mathcal{F}) \to (S, \Sigma)\) is measurable, as a way of emphasizing the two \(\sigma\)-algebras \(\mathcal{F}\) and \(\Sigma\) with respect to which \(f\) is measurable.

**Exercise 10.** Let \((\Omega, \mathcal{F})\) and \((S, \Sigma)\) be two measurable spaces. Let \(S'\) be a set and \(f : \Omega \to S\) be a map such that \(f(\Omega) \subseteq S' \subseteq S\). We define \(\Sigma'\) as the trace of \(\Sigma\) on \(S'\), i.e. \(\Sigma' = \Sigma_{|S'}\).

1. Show that for all \(B \in \Sigma\), we have \(f^{-1}(B) = f^{-1}(B \cap S')\)

2. Show that \(f : (\Omega, \mathcal{F}) \to (S, \Sigma)\) is measurable, if and only if \(f : (\Omega, \mathcal{F}) \to (S', \Sigma')\) is itself measurable.

3. Let \(f : \Omega \to \mathbb{R}^+\). Show that the following are equivalent:

   \[
   (i) \quad f : (\Omega, \mathcal{F}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+)) \text{ is measurable}
   \]
(ii) \( f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is measurable

(iii) \( f : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})) \) is measurable

**Exercise 11.** Let \((\Omega, \mathcal{F}), (S, \Sigma), (S_1, \Sigma_1)\) be three measurable spaces. Let \( f : (\Omega, \mathcal{F}) \to (S, \Sigma) \) and \( g : (S, \Sigma) \to (S_1, \Sigma_1) \) be two measurable maps.

1. For all \( B \subseteq S_1 \), show that \((g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))\)

2. Show that \( g \circ f : (\Omega, \mathcal{F}) \to (S_1, \Sigma_1) \) is measurable.

**Exercise 12.** Let \((\Omega, \mathcal{F})\) and \((S, \Sigma)\) be two measurable spaces. Let \( f : \Omega \to S \) be a map. We define:

\[ \Gamma \triangleq \{ B \in \Sigma : f^{-1}(B) \in \mathcal{F} \} \]

1. Show that \( f^{-1}(S) = \Omega \).
2. Show that for all $B \subseteq S$, $f^{-1}(B^c) = (f^{-1}(B))^c$.
3. Show that if $B_n \subseteq S, n \geq 1$, then $f^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$
4. Show that $\Gamma$ is a $\sigma$-algebra on $S$.
5. Prove the following theorem.

**Theorem 14** Let $(\Omega, \mathcal{F})$ and $(S, \Sigma)$ be two measurable spaces, and $\mathcal{A}$ be a set of subsets of $S$ generating $\Sigma$, i.e. such that $\Sigma = \sigma(\mathcal{A})$. Then $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if:

$$\forall B \in \mathcal{A}, \quad f^{-1}(B) \in \mathcal{F}$$
Exercise 13. Let \((\Omega, T)\) and \((S, T_S)\) be two topological spaces. Let \(f : \Omega \to S\) be a map. Show that if \(f : (\Omega, T) \to (S, T_S)\) is continuous, then \(f : (\Omega, \mathcal{B}(\Omega)) \to (S, \mathcal{B}(S))\) is measurable.

Exercise 14. We define the following subsets of the power set \(\mathcal{P}(\mathbb{R})\):

\[
\begin{align*}
\mathcal{C}_1 & \triangleq \{[-\infty, c], \ c \in \mathbb{R}\} \\
\mathcal{C}_2 & \triangleq \{[-\infty, c[, \ c \in \mathbb{R}\} \\
\mathcal{C}_3 & \triangleq \{[c, +\infty], \ c \in \mathbb{R}\} \\
\mathcal{C}_4 & \triangleq \{[c, +\infty[, \ c \in \mathbb{R}\}
\end{align*}
\]

1. Show that \(\mathcal{C}_2\) and \(\mathcal{C}_4\) are subsets of \(\mathcal{T}_{\mathbb{R}}\).

2. Show that the elements of \(\mathcal{C}_1\) and \(\mathcal{C}_3\) are closed in \(\mathbb{R}\).

3. Show that for all \(i = 1, 2, 3, 4\), \(\sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\mathbb{R})\).

4. Let \(U\) be open in \(\mathbb{R}\). Explain why \(U \cap \mathbb{R}\) is open in \(\mathbb{R}\).
5. Show that any open subset of $\mathbb{R}$ is a countable union of open bounded intervals in $\mathbb{R}$.

6. Let $a < b$, $a, b \in \mathbb{R}$. Show that we have:

$$\{a, b\} = \bigcup_{n=1}^{+\infty} [a, b - 1/n] = \bigcup_{n=1}^{+\infty} [a + 1/n, b]$$

7. Show that for all $i = 1, 2, 3, 4$, $\{a, b\} \in \sigma(C_i)$.

8. Show that for all $i = 1, 2, 3, 4$, $\{-\infty\}, \{+\infty\} \subseteq \sigma(C_i)$.

9. Show that for all $i = 1, 2, 3, 4$, $\sigma(C_i) = \mathcal{B}(\bar{\mathbb{R}})$

10. Prove the following theorem.
**Theorem 15** Let \((\Omega, \mathcal{F})\) be a measurable space, and \(f : \Omega \to \overline{\mathbb{R}}\) be a map. The following are equivalent:

(i) \(f : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))\) is measurable

(ii) \(\forall B \in \mathcal{B}(\overline{\mathbb{R}}), \{f \in B\} \in \mathcal{F}\)

(iii) \(\forall c \in \mathbb{R}, \{f \leq c\} \in \mathcal{F}\)

(iv) \(\forall c \in \mathbb{R}, \{f < c\} \in \mathcal{F}\)

(v) \(\forall c \in \mathbb{R}, \{c \leq f\} \in \mathcal{F}\)

(vi) \(\forall c \in \mathbb{R}, \{c < f\} \in \mathcal{F}\)

**Exercise 15.** Let \((\Omega, \mathcal{F})\) be a measurable space. Let \((f_n)_{n \geq 1}\) be a sequence of measurable maps \(f_n : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))\). Let \(g\) and \(h\) be the maps defined by \(g(\omega) = \inf_{n \geq 1} f_n(\omega)\) and \(h(\omega) = \sup_{n \geq 1} f_n(\omega)\), for all \(\omega \in \Omega\).

1. Let \(c \in \mathbb{R}\). Show that \(\{c \leq g\} = \bigcap_{n=1}^{+\infty} \{c \leq f_n\}\).

2. Let \(c \in \mathbb{R}\). Show that \(\{h \leq c\} = \bigcap_{n=1}^{+\infty} \{f_n \leq c\}\).
3. Show that \( g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})) \) are measurable.

**Definition 36** Let \((v_n)_{n \geq 1}\) be a sequence in \(\bar{\mathbb{R}}\). We define:

\[
u \triangleq \liminf_{n \to +\infty} v_n \triangleq \sup_{n \geq 1} \left( \inf_{k \geq n} v_k \right)
\]

and:

\[
w \triangleq \limsup_{n \to +\infty} v_n \triangleq \inf_{n \geq 1} \left( \sup_{k \geq n} v_k \right)
\]

Then, \(u, w \in \bar{\mathbb{R}}\) are respectively called **lower limit** and **upper limit** of the sequence \((v_n)_{n \geq 1}\).

**Exercise 16.** Let \((v_n)_{n \geq 1}\) be a sequence in \(\bar{\mathbb{R}}\). For \(n \geq 1\) we define \(u_n = \inf_{k \geq n} v_k\) and \(w_n = \sup_{k \geq n} v_k\). Let \(u\) and \(w\) be the lower limit and upper limit of \((v_n)_{n \geq 1}\), respectively.

1. Show that \(u_n \leq u_{n+1} \leq u\), for all \(n \geq 1\).
2. Show that $w \leq w_{n+1} \leq w_n$, for all $n \geq 1$.

3. Show that $u_n \to u$ and $w_n \to w$ as $n \to +\infty$.

4. Show that $u_n \leq v_n \leq w_n$, for all $n \geq 1$.

5. Show that $u \leq w$.

6. Show that if $u = w$ then $(v_n)_{n \geq 1}$ converges to a limit $v \in \overline{\mathbb{R}}$, with $u = v = w$.

7. Show that if $a, b \in \mathbb{R}$ are such that $u < a < b < w$ then for all $n \geq 1$, there exist $N_1, N_2 \geq n$ such that $v_{N_1} < a < b < v_{N_2}$.

8. Show that if $a, b \in \mathbb{R}$ are such that $u < a < b < w$ then there exist two strictly increasing sequences of integers $(n_k)_{k \geq 1}$ and $(m_k)_{k \geq 1}$ such that for all $k \geq 1$, we have $v_{n_k} < a < b < v_{m_k}$.

9. Show that if $(v_n)_{n \geq 1}$ converges to some $v \in \overline{\mathbb{R}}$, then $u = w$. 

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Theorem 16  Let \((v_n)_{n \geq 1}\) be a sequence in \(\bar{\mathbb{R}}\). Then, the following are equivalent:

(i) \(\liminf_{n \to +\infty} v_n = \limsup_{n \to +\infty} v_n\)

(ii) \(\lim_{n \to +\infty} v_n\) exists in \(\bar{\mathbb{R}}\).

in which case:

\[\lim_{n \to +\infty} v_n = \liminf_{n \to +\infty} v_n = \limsup_{n \to +\infty} v_n\]

Exercise 17. Let \(f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))\) be two measurable maps, where \((\Omega, \mathcal{F})\) is a measurable space.

1. Show that \(\{f < g\} = \bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{r < g\})\).

2. Show that the sets \(\{f < g\}, \{f > g\}, \{f = g\}, \{f \leq g\}, \{f \geq g\}\) belong to the \(\sigma\)-algebra \(\mathcal{F}\).
Exercise 18. Let $(\Omega, F)$ be a measurable space. Let $(f_n)_{n \geq 1}$ be a sequence of measurable maps $f_n : (\Omega, F) \rightarrow (\mathbb{R}, B(\mathbb{R}))$. We define $g = \lim\inf f_n$ and $h = \lim\sup f_n$ in the obvious way:
\[
\forall \omega \in \Omega, \quad g(\omega) \triangleq \lim_{n \to +\infty} f_n(\omega)
\]
\[
\forall \omega \in \Omega, \quad h(\omega) \triangleq \lim\sup_{n \to +\infty} f_n(\omega)
\]
1. Show that $g, h : (\Omega, F) \rightarrow (\mathbb{R}, B(\mathbb{R}))$ are measurable.
2. Show that $g \leq h$, i.e. $\forall \omega \in \Omega, \quad g(\omega) \leq h(\omega)$.
3. Show that $\{g = h\} \in F$.
4. Show that $\{\omega : \omega \in \Omega, \lim_{n \to +\infty} f_n(\omega) \text{ exists in } \mathbb{R}\} \in F$.
5. Suppose $\Omega = \{g = h\}$, and let $f(\omega) = \lim_{n \to +\infty} f_n(\omega)$, for all $\omega \in \Omega$. Show that $f : (\Omega, F) \rightarrow (\mathbb{R}, B(\mathbb{R}))$ is measurable.

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Exercise 19. Let $f, g : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space.

1. Show that $-f, |f|, f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$.

2. Let $a \in \mathbb{R}$. Explain why the map $a + f$ may not be well defined.

3. Show that $(a + f) : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, whenever $a \in \mathbb{R}$.

4. Show that $(a.f) : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, for all $a \in \mathbb{R}$. (Recall the convention $0.\infty = 0$).

5. Explain why the map $f + g$ may not be well defined.

6. Suppose that $f \geq 0$ and $g \geq 0$, i.e. $f(\Omega) \subseteq [0, +\infty]$ and also $g(\Omega) \subseteq [0, +\infty]$. Show that $\{f + g < c\} = \{f < c - g\}$, for all $c \in \mathbb{R}$. Show that $f + g : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable.
7. Show that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ is measurable whenever $f + g$ is well-defined, i.e. when the following condition holds:
\[
(f = +\infty) \cap (g = -\infty) \cup (f = -\infty) \cap (g = +\infty) = \emptyset
\]

8. Show that $1/f : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ is measurable, in the case when $f(\Omega) \subseteq \mathbb{R} \setminus \{0\}$.

9. Suppose that $f$ is $\mathbb{R}$-valued. Show that $\bar{f}$ defined by $\bar{f}(\omega) = f(\omega)$ if $f(\omega) \neq 0$ and $\bar{f}(\omega) = 1$ if $f(\omega) = 0$, is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbb{R}})$.

10. Suppose $f$ and $g$ take values in $\mathbb{R}$. Let $\bar{f}$ be defined as in 9. Show that for all $c \in \mathbb{R}$, the set $\{fg < c\}$ can be expressed as:
\[
(\{f > 0\} \cap \{g < c/\bar{f}\}) \cup (\{f < 0\} \cap \{g > c/\bar{f}\}) \cup (\{f = 0\} \cap \{f < c\})
\]

11. Show that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ is measurable, in the case when $f$ and $g$ take values in $\mathbb{R}$. 

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Exercise 20. Let $f, g : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space. Let $\bar{f}, \bar{g}$, be defined by:

$$\bar{f}(\omega) \triangleq \begin{cases} f(\omega) & \text{if } f(\omega) \not\in \{-\infty, +\infty\} \\ 1 & \text{if } f(\omega) \in \{-\infty, +\infty\} \end{cases}$$

$\bar{g}(\omega)$ being defined in a similar way. Consider the partitions of $\Omega$, $\Omega = A_1 \uplus A_2 \uplus A_3 \uplus A_4 \uplus A_5$ and $\Omega = B_1 \uplus B_2 \uplus B_3 \uplus B_4 \uplus B_5$, where $A_1 = \{f \in [0, +\infty]\}$, $A_2 = \{f \in ]-\infty, 0]\}$, $A_3 = \{f = 0\}$, $A_4 = \{f = -\infty\}$, $A_5 = \{f = +\infty\}$ and $B_1, B_2, B_3, B_4, B_5$ being defined in a similar way with $g$. Recall the conventions $0 \times (+\infty) = 0$, $(-\infty) \times (+\infty) = (-\infty)$, etc.

1. Show that $\bar{f}$ and $\bar{g}$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$.

2. Show that all $A_i$'s and $B_j$'s are elements of $\mathcal{F}$. 

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3. Show that for all \( B \in \mathcal{B}(\bar{\mathbb{R}}) \):
\[
\{ fg \in B \} = \bigcup_{i,j=1}^{5} (A_i \cap B_j \cap \{ fg \in B \})
\]

4. Show that \( A_i \cap B_j \cap \{ fg \in B \} = A_i \cap B_j \cap \{ \bar{f} \bar{g} \in B \} \), in the case when \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq 3 \).

5. Show that \( A_i \cap B_j \cap \{ fg \in B \} \) is either equal to \( \emptyset \) or \( A_i \cap B_j \), in the case when \( i \geq 4 \) or \( j \geq 4 \).

6. Show that \( fg : (\Omega, \mathcal{F}) \rightarrow (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})) \) is measurable.

**Definition 37** Let \((\Omega, T)\) be a topological space, and \( A \subseteq \Omega \). We call closure of \( A \) in \( \Omega \), denoted \( \bar{A} \), the set defined by:
\[
\bar{A} \triangleq \{ x \in \Omega : x \in U \in T \Rightarrow U \cap A \neq \emptyset \}
\]
**Exercise 21.** Let \((E, T)\) be a topological space, and \(A \subseteq E\). Let \(\bar{A}\) be the closure of \(A\).

1. Show that \(A \subseteq \bar{A}\) and that \(\bar{A}\) is closed.
2. Show that if \(B\) is closed and \(A \subseteq B\), then \(\bar{A} \subseteq B\).
3. Show that \(\bar{A}\) is the smallest closed set in \(E\) containing \(A\).
4. Show that \(A\) is closed if and only if \(A = \bar{A}\).
5. Show that if \((E, T)\) is metrizable, then:
   \[
   \bar{A} = \{ x \in E : \forall \epsilon > 0, \ B(x, \epsilon) \cap A \neq \emptyset \}
   \]
   where \(B(x, \epsilon)\) is relative to any metric \(d\) such that \(T_E^d = T\).

**Exercise 22.** Let \((E, d)\) be a metric space. Let \(A \subseteq E\). For all \(x \in E\), we define:

\[
d(x, A) \triangleq \inf \{ d(x, y) : y \in A \} \triangleq \Phi_A(x)
\]

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where it is understood that $\inf \emptyset = +\infty$.

1. Show that for all $x \in E$, $d(x, A) = d(x, \bar{A})$.

2. Show that $d(x, A) = 0$, if and only if $x \in \bar{A}$.

3. Show that for all $x, y \in E$, $d(x, A) \leq d(x, y) + d(y, A)$.

4. Show that if $A \neq \emptyset$, $|d(x, A) - d(y, A)| \leq d(x, y)$.

5. Show that $\Phi_A : (E, T^d_E) \to (\bar{R}, T_{\bar{R}})$ is continuous.

6. Show that if $A$ is closed, then $A = \Phi_A^{-1}(\{0\})$

**Exercise 23.** Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n \geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where $(E, d)$ is a metric space. We assume that for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n \geq 1}$ converges to some $f(\omega) \in E$. 
1. Explain why $\liminf f_n$ and $\limsup f_n$ may not be defined in an arbitrary metric space $E$.

2. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable, if and only if $f^{-1}(A) \in \mathcal{F}$ for all closed subsets $A$ of $E$.

3. Show that for all $A$ closed in $E$, $f^{-1}(A) = (\Phi_A \circ f)^{-1}(\{0\})$, where the map $\Phi_A : E \to \bar{\mathbb{R}}$ is defined as in exercise (22).

4. Show that $\Phi_A \circ f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ is measurable.

5. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable.

**Theorem 17** Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n \geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where $(E, d)$ is a metric space. Then, if the limit $f = \lim f_n$ exists on $\Omega$, the map $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is itself measurable.
Definition 38  The **usual topology** on \( \mathbb{C} \), the set of complex numbers, is defined as the **metric topology** associated with \( d(z, z') = |z - z'| \).

**Exercise 24.** Let \( f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) be a measurable map, where \( (\Omega, \mathcal{F}) \) is a measurable space. Let \( u = Re(f) \) and \( v = Im(f) \). Show that \( u, v, |f| : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) are all measurable.

**Exercise 25.** Define the subset of the power set \( \mathcal{P}(\mathbb{C}) \):

\[
C \triangleq \{ |a, b| \times |c, d|, \ a, b, c, d \in \mathbb{R} \}
\]

where it is understood that:

\[
|a, b| \times |c, d| \triangleq \{ z = x + iy \in \mathbb{C}, \ (x, y) \in |a, b| \times |c, d| \}
\]

1. Show that any element of \( C \) is open in \( \mathbb{C} \).
2. Show that \( \sigma(C) \subseteq \mathcal{B}(\mathbb{C}) \).
3. Let \( z = x + iy \in \mathbb{C} \). Show that if \( |x| < \eta \) and \( |y| < \eta \) then we have \( |z| < \sqrt{2}\eta \).

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4. Let \( U \) be open in \( \mathbb{C} \). Show that for all \( z \in U \), there are rational numbers \( a_z, b_z, c_z, d_z \) such that \( z \in [a_z, b_z] \times [c_z, d_z] \subseteq U \).

5. Show that \( U \) can be written as \( U = \bigcup_{n=1}^{+\infty} A_n \) where \( A_n \in \mathcal{C} \).

6. Show that \( \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{C}) \).

7. Let \( (\Omega, \mathcal{F}) \) be a measurable space, and \( u, v : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) be two measurable maps. Show that \( u + iv : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) is measurable.