18. The Jacobian Formula

In the following, $K$ denotes $\mathbb{R}$ or $\mathbb{C}$.

**Definition 125** We call **$K$-normed space**, an ordered pair $(E, N)$, where $E$ is a $K$-vector space, and $N : E \rightarrow \mathbb{R}^+$ is a norm on $E$.

See definition (89) for **vector space**, and definition (95) for **norm**.

**Exercise 1.** Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $K$-vector space $\mathcal{H}$.

1. Show that $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ is a norm on $\mathcal{H}$.
2. Show that $(\mathcal{H}, \| \cdot \|)$ is a $K$-normed space.

**Exercise 2.** Let $(E, \| \cdot \|)$ be a $K$-normed space:

1. Show that $d(x, y) = \|x - y\|$ defines a metric on $E$.
2. Show that for all $x, y \in E$, we have $\|x\| - \|y\| \leq \|x - y\|$.
Definition 126  Let $(E, \| \cdot \|)$ be a $K$-normed space, and $d$ be the metric defined by $d(x, y) = \| x - y \|$. We call norm topology on $E$, denoted $\mathcal{T}_{\| \cdot \|}$, the topology on $E$ associated with $d$.

Note that this definition is consistent with definition (82) of the norm topology associated with an inner-product.

Exercise 3. Let $E, F$ be two $K$-normed spaces, and $l : E \to F$ be a linear map. Show that the following are equivalent:

(i) $l$ is continuous (w.r. to the norm topologies)
(ii) $l$ is continuous at $x = 0$.
(iii) $\exists K \in \mathbb{R}^+, \forall x \in E, \|l(x)\| \leq K\|x\|$ 
(iv) $\sup\{\|l(x)\| : x \in E, \|x\| = 1\} < +\infty$

Definition 127  Let $E, F$ be $K$-normed spaces. The $K$-vector space of all continuous linear maps $l : E \to F$ is denoted $\mathcal{L}_K(E, F)$.
**Exercise 4.** Show that $\mathcal{L}_K(E, F)$ is indeed a $K$-vector space.

**Exercise 5.** Let $E, F$ be $K$-normed spaces. Given $l \in \mathcal{L}_K(E, F)$, let:

$$\|l\| \triangleq \sup \{ \|l(x)\| : x \in E, \|x\| = 1 \} < +\infty$$

1. Show that:

$$\|l\| = \sup \{ \|l(x)\| : x \in E, \|x\| \leq 1 \}$$

2. Show that:

$$\|l\| = \sup \left\{ \frac{\|l(x)\|}{\|x\|} : x \in E, x \neq 0 \right\}$$

3. Show that $\|l(x)\| \leq \|l\| \cdot \|x\|$, for all $x \in E$.

4. Show that $\|l\|$ is the smallest $K \in \mathbb{R}^+$, such that:

$$\forall x \in E, \|l(x)\| \leq K \|x\|$$

5. Show that $l \to \|l\|$ is a norm on $\mathcal{L}_K(E, F)$.
6. Show that \((L_K(E, F), \|\cdot\|)\) is a \(K\)-normed space.

**Definition 128** Let \(E, F\) be \(\mathbb{R}\)-normed spaces and \(U\) be an open subset of \(E\). We say that a map \(\phi : U \to F\) is **differentiable** at some \(a \in U\), if and only if there exists \(l \in L_{\mathbb{R}}(E, F)\) such that, for all \(\epsilon > 0\), there exists \(\delta > 0\), such that for all \(h \in E\):

\[
\|h\| \leq \delta \implies a + h \in U \text{ and } \|\phi(a + h) - \phi(a) - l(h)\| \leq \epsilon \|h\|
\]

**Exercise 6.** Let \(E, F\) be two \(\mathbb{R}\)-normed spaces, and \(U\) be open in \(E\). Let \(\phi : U \to F\) be a map and \(a \in U\).

1. Suppose that \(\phi : U \to F\) is differentiable at \(a \in U\), and that \(l_1, l_2 \in L_{\mathbb{R}}(E, F)\) satisfy the requirement of definition (128). Show that for all \(\epsilon > 0\), there exists \(\delta > 0\) such that:

\[
\forall h \in E, \|h\| \leq \delta \implies \|l_1(h) - l_2(h)\| \leq \epsilon \|h\|
\]

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2. Conclude that $||l_1 - l_2|| = 0$ and finally that $l_1 = l_2$.

**Definition 129** Let $E, F$ be $\mathbb{R}$-normed spaces and $U$ be an open subset of $E$. Let $\phi : U \to F$ be a map and $a \in U$. If $\phi$ is differentiable at $a$, we call differential of $\phi$ at $a$, the unique element of $\mathcal{L}_{\mathbb{R}}(E, F)$, denoted $d\phi(a)$, satisfying the requirement of definition (128). If $\phi$ is differentiable at all points of $U$, the map $d\phi : U \to \mathcal{L}_{\mathbb{R}}(E, F)$ is also called the differential of $\phi$.

**Definition 130** Let $E, F$ be $\mathbb{R}$-normed spaces and $U$ be an open subset of $E$. A map $\phi : U \to F$ is said to be of class $C^1$, if and only if $d\phi(a)$ exists for all $a \in U$, and the differential $d\phi : U \to \mathcal{L}_{\mathbb{R}}(E, F)$ is a continuous map.

**Exercise 7.** Let $E, F$ be two $\mathbb{R}$-normed spaces and $U$ be open in $E$. Let $\phi : U \to F$ be a map, and $a \in U$. 

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1. Show that $\phi$ differentiable at $a \Rightarrow \phi$ continuous at $a$.

2. If $\phi$ is of class $C^1$, explain with respect to which topologies the differential $d\phi : U \to \mathcal{L}_\mathbb{R}(E, F)$ is said to be continuous.

3. Show that if $\phi$ is of class $C^1$, then $\phi$ is continuous.

4. Suppose that $E = \mathbb{R}$. Show that for all $a \in U$, $\phi$ is differentiable at $a \in U$, if and only if the derivative:

$$\phi'(a) \triangleq \lim_{t \neq 0, t \to 0} \frac{\phi(a + t) - \phi(a)}{t}$$

exists in $F$, in which case $d\phi(a) \in \mathcal{L}_\mathbb{R}(\mathbb{R}, F)$ is given by:

$$\forall t \in \mathbb{R}, \quad d\phi(a)(t) = t.\phi'(a)$$

In particular, $\phi'(a) = d\phi(a)(1)$.

**Exercise 8.** Let $E, F, G$ be three $\mathbb{R}$-normed spaces. Let $U$ be open in $E$ and $V$ be open in $F$. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps
such that \( \phi(U) \subseteq V \). We assume that \( \phi \) is differentiable at \( a \in U \), and we put:

\[
l_1 \triangleq d\phi(a) \in \mathcal{L}_R(E, F)
\]

We assume that \( \psi \) is differentiable at \( \phi(a) \in V \), and we put:

\[
l_2 \triangleq d\psi(\phi(a)) \in \mathcal{L}_R(F, G)
\]

1. Explain why \( \psi \circ \phi : U \to G \) is a well-defined map.

2. Given \( \epsilon > 0 \), show the existence of \( \eta > 0 \) such that:

\[
\eta(\eta + \|l_1\| + \|l_2\|) \leq \epsilon
\]

3. Show the existence of \( \delta_2 > 0 \) such that for all \( h_2 \in F \) with \( \|h_2\| \leq \delta_2 \), we have \( \phi(a) + h_2 \in V \) and:

\[
\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2(h_2)\| \leq \eta\|h_2\|
\]

4. Show that if \( h_2 \in F \) and \( \|h_2\| \leq \delta_2 \), then for all \( h \in E \), we have:

\[
\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| \leq \eta\|h_2\| + \|l_2\| \cdot \|h_2 - l_1(h)\|
\]
5. Show the existence of $\delta > 0$ such that for all $h \in E$ with $\|h\| \leq \delta$, we have $a + h \in U$ and $\|\phi(a + h) - \phi(a) - l_1(h)\| \leq \eta \|h\|$, together with $\|\phi(a + h) - \phi(a)\| \leq \delta_2$.

6. Show that if $h \in E$ is such that $\|h\| \leq \delta$, then $a + h \in U$ and:

$$
\|\psi \circ \phi(a+h) - \psi \circ \phi(a) - l_2 \circ l_1(h)\|
\leq \eta \|\phi(a+h) - \phi(a)\| + \eta \|l_2\| \|h\|
\leq \eta (\eta + \|l_1\| + \|l_2\|) \|h\|
\leq \epsilon \|h\|
$$

7. Show that $l_2 \circ l_1 \in L(E, G)$

8. Conclude with the following:
**Theorem 110**  Let $E, F, G$ be three $\mathbb{R}$-normed spaces, $U$ be open in $E$ and $V$ be open in $F$. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. Let $a \in U$. Then, if $\phi$ is differentiable at $a \in U$, and $\psi$ is differentiable at $\phi(a) \in V$, then $\psi \circ \phi$ is differentiable at $a \in U$, and furthermore:
\[ d(\psi \circ \phi)(a) = d\psi(\phi(a)) \circ d\phi(a) \]

**Exercise 9.** Let $(\Omega', T')$ and $(\Omega, T)$ be two topological spaces, and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a set of subsets of $\Omega$ generating the topology $T$, i.e. such that $T = T(\mathcal{A})$ as defined in (55). Let $f : \Omega' \to \Omega$ be a map, and define:
\[ \mathcal{U} \triangleq \{ A \subseteq \Omega : f^{-1}(A) \in T' \} \]
1. Show that $\mathcal{U}$ is a topology on $\Omega$.
2. Show that $f : (\Omega', T') \to (\Omega, T)$ is continuous, if and only if:
\[ \forall A \in \mathcal{A}, f^{-1}(A) \in T' \]
Exercise 10. Let \((\Omega', T')\) be a topological space, and \((\Omega_i, \mathcal{T}_i)_{i \in I}\) be a family of topological spaces, indexed by a non-empty set \(I\). Let \(\Omega\) be the Cartesian product \(\Omega = \prod_{i \in I} \Omega_i\) and \(\mathcal{T} = \bigotimes_{i \in I} \mathcal{T}_i\) be the product topology on \(\Omega\). Let \((f_i)_{i \in I}\) be a family of maps \(f_i : \Omega' \to \Omega_i\) indexed by \(I\), and let \(f : \Omega' \to \Omega\) be the map defined by \(f(\omega) = (f_i(\omega))_{i \in I}\) for all \(\omega \in \Omega'\). Let \(p_i : \Omega \to \Omega_i\) be the canonical projection mapping.

1. Show that \(p_i : (\Omega, \mathcal{T}) \to (\Omega_i, \mathcal{T}_i)\) is continuous for all \(i \in I\).

2. Show that \(f : (\Omega', T') \to (\Omega, \mathcal{T})\) is continuous, if and only if each coordinate mapping \(f_i : (\Omega', T') \to (\Omega_i, \mathcal{T}_i)\) is continuous.

Exercise 11. Let \(E, F, G\) be three \(\mathbb{R}\)-normed spaces. Let \(U\) be open in \(E\) and \(V\) be open in \(F\). Let \(\phi : U \to F\) and \(\psi : V \to G\) be two maps of class \(C^1\) such that \(\phi(U) \subseteq V\).

1. For all \((l_1, l_2) \in \mathcal{L}_\mathbb{R}(F, G) \times \mathcal{L}_\mathbb{R}(E, F)\), we define:
\[
N_1(l_1, l_2) \triangleq \|l_1\| + \|l_2\|
\]

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\[
N_2(l_1, l_2) \triangleq \sqrt{\|l_1\|^2 + \|l_2\|^2}
\]

\[
N_\infty(l_1, l_2) \triangleq \max(\|l_1\|, \|l_2\|)
\]

Show that \(N_1, N_2, N_\infty\) are all norms on \(\mathcal{L}_R(F, G) \times \mathcal{L}_R(E, F)\).

2. Show they induce the product topology on \(\mathcal{L}_R(F, G) \times \mathcal{L}_R(E, F)\).

3. We define the map \(H : \mathcal{L}_R(F, G) \times \mathcal{L}_R(E, F) \to \mathcal{L}_R(E, G)\) by:

\[\forall (l_1, l_2) \in \mathcal{L}_R(F, G) \times \mathcal{L}_R(E, F), \quad H(l_1, l_2) \triangleq l_1 \circ l_2\]

Show that \(\|H(l_1, l_2)\| \leq \|l_1\| \cdot \|l_2\|\), for all \(l_1, l_2\).

4. Show that \(H\) is continuous.

5. We define \(K : U \to \mathcal{L}_R(F, G) \times \mathcal{L}_R(E, F)\) by:

\[\forall a \in U, \quad K(a) \triangleq (d\psi(\phi(a)), d\phi(a))\]

Show that \(K\) is continuous.
6. Show that $\psi \circ \phi$ is differentiable on $U$.

7. Show that $d(\psi \circ \phi) = H \circ K$.

8. Conclude with the following:

**Theorem 111** Let $E,F,G$ be three $\mathbb{R}$-normed spaces, $U$ be open in $E$ and $V$ be open in $F$. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class $C^1$ such that $\phi(U) \subseteq V$. Then, $\psi \circ \phi : U \to G$ is of class $C^1$.

**Exercise 12.** Let $E$ be an $\mathbb{R}$-normed space. Let $a,b \in \mathbb{R}$, $a < b$. Let $f : [a,b] \to E$ and $g : [a,b] \to \mathbb{R}$ be two continuous maps which are differentiable at every point of $[a,b]$. We assume that:

$$\forall t \in [a,b], \quad \|f'(t)\| \leq g'(t)$$

1. Given $\epsilon > 0$, we define $\phi_\epsilon : [a,b] \to \mathbb{R}$ by:

$$\phi_\epsilon(t) \overset{\Delta}{=} \|f(t) - f(a)\| - g(t) + g(a) - \epsilon(t - a)$$
for all $t \in [a, b]$. Show that $\phi_\epsilon$ is continuous.

2. Define $E_\epsilon = \{ t \in [a, b] : \phi_\epsilon(t) \leq \epsilon \}$, and $c = \sup E_\epsilon$. Show that:
   
   $c \in [a, b]$ and $\phi_\epsilon(c) \leq \epsilon$

3. Show the existence of $h > 0$, such that:

   $\forall t \in [a, a + h] \cap [a, b]$,
   $\phi_\epsilon(t) \leq \epsilon$

4. Show that $c \in ]a, b[$.

5. Suppose that $c \in ]a, b[$. Show the existence of $t_0 \in ]c, b]$ such that:

   $\left\| \frac{f(t_0) - f(c)}{t_0 - c} \right\| \leq \|f'(c)\| + \epsilon/2$ and $g'(c) \leq \frac{g(t_0) - g(c)}{t_0 - c} + \epsilon/2$

6. Show that $\|f(t_0) - f(c)\| \leq g(t_0) - g(c) + \epsilon(t_0 - c)$.

7. Show that $\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c - a) + \epsilon$.

8. Show that $\|f(t_0) - f(a)\| \leq g(t_0) - g(a) + \epsilon(t_0 - a) + \epsilon$. 

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9. Show that $c \in \]a, b\]$ leads to a contradiction.

10. Show that $\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b - a) + \epsilon$.

11. Conclude with the following:

**Theorem 112** Let $E$ be an $\mathbb{R}$-normed space. Let $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \to E$ and $g : [a, b] \to \mathbb{R}$ be two continuous maps which are differentiable at every point of $[a, b]$, and such that:

$$\forall t \in ]a, b[, \|f'(t)\| \leq g'(t)$$

Then:

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$
Definition 131 Let \( n \geq 1 \) and \( U \) be open in \( \mathbb{R}^n \). Let \( \phi : U \to E \) be a map, where \( E \) is an \( \mathbb{R} \)-normed space. For all \( i = 1, \ldots, n \), we say that \( \phi \) has an \( i \)th partial derivative at \( a \in U \), if and only if the limit:

\[
\frac{\partial \phi}{\partial x_i}(a) \triangleq \lim_{h \neq 0, h \to 0} \frac{\phi(a + he_i) - \phi(a)}{h}
\]

exists in \( E \), where \( (e_1, \ldots, e_n) \) is the canonical basis of \( \mathbb{R}^n \).

Exercise 13. Let \( n \geq 1 \) and \( U \) be open in \( \mathbb{R}^n \). Let \( \phi : U \to E \) be a map, where \( E \) is an \( \mathbb{R} \)-normed space.

1. Suppose \( \phi \) is differentiable at \( a \in U \). Show that for all \( i \in \mathbb{N}_n \):

\[
\lim_{h \neq 0, h \to 0} \frac{1}{\|he_i\|} \left\| \phi(a + he_i) - \phi(a) - d\phi(a)(he_i) \right\| = 0
\]

2. Show that for all \( i \in \mathbb{N}_n \), \( \frac{\partial \phi}{\partial x_i}(a) \) exists, and:

\[
\frac{\partial \phi}{\partial x_i}(a) = d\phi(a)(e_i)
\]

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3. Conclude with the following:

**Theorem 113** Let $n \geq 1$ and $U$ be open in $\mathbb{R}^n$. Let $\phi : U \to E$ be a map, where $E$ is an $\mathbb{R}$-normed space. Then, if $\phi$ is differentiable at $a \in U$, for all $i = 1, \ldots, n$, $\frac{\partial \phi}{\partial x_i}(a)$ exists and we have:

$$\forall h \triangleq (h_1, \ldots, h_n) \in \mathbb{R}^n, \ d\phi(a)(h) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i$$

**Exercise 14.** Let $n \geq 1$ and $U$ be open in $\mathbb{R}^n$. Let $\phi : U \to E$ be a map, where $E$ is an $\mathbb{R}$-normed space.

1. Show that if $\phi$ is differentiable at $a, b \in U$, then for all $i \in \mathbb{N}_n$:

$$\left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \| d\phi(b) - d\phi(a) \|$$
2. Conclude that if \( \phi \) is of class \( C^1 \) on \( U \), then \( \frac{\partial \phi}{\partial x_i} \) exists and is continuous on \( U \), for all \( i \in \mathbb{N}_n \).

**Exercise 15.** Let \( n \geq 1 \) and \( U \) be open in \( \mathbb{R}^n \). Let \( \phi : U \to E \) be a map, where \( E \) is an \( \mathbb{R} \)-normed space. We assume that \( \frac{\partial \phi}{\partial x_i} \) exists on \( U \), and is continuous at \( a \in U \), for all \( i \in \mathbb{N}_n \). We define \( l : \mathbb{R}^n \to E \):

\[
\forall h \triangleq (h_1, \ldots, h_n) \in \mathbb{R}^n, \ l(h) \triangleq \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a)h_i
\]

1. Show that \( l \in \mathcal{L}_{\mathbb{R}}(\mathbb{R}^n, E) \).

2. Given \( \epsilon > 0 \), show the existence of \( \eta > 0 \) such that for all \( h \in \mathbb{R}^n \) with \( \| h \| < \eta \), we have \( a + h \in U \), and:

\[
\forall i = 1, \ldots, n, \quad \left\| \frac{\partial \phi}{\partial x_i}(a + h) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \epsilon
\]
3. Let $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ be such that $\|h\| < \eta$. $(e_1, \ldots, e_n)$ being the canonical basis of $\mathbb{R}^n$, we define $k_0 = a$ and for $i \in \mathbb{N}_n$:

$$k_i \triangleq a + \sum_{j=1}^{i} h_j e_j$$

Show that $k_0, \ldots, k_n \in U$, and that we have:

$$\phi(a + h) - \phi(a) - l(h) = \sum_{i=1}^{n} \left( \phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right)$$

4. Let $i \in \mathbb{N}_n$. Assume that $h_i > 0$. We define $f_i : [0, h_i] \rightarrow E$ by:

$$\forall t \in [0, h_i], \quad f_i(t) \triangleq \phi(k_{i-1} + t e_i) - \phi(k_{i-1}) - t \frac{\partial \phi}{\partial x_i}(a)$$

Show $f_i$ is well-defined, $f'_i(t)$ exists for all $t \in [0, h_i]$, and:

$$\forall t \in [0, h_i], \quad f'_i(t) = \frac{\partial \phi}{\partial x_i}(k_{i-1} + te_i) - \frac{\partial \phi}{\partial x_i}(a)$$
5. Show \( f_i \) is continuous on \([0, h_i], \) differentiable on \(]0, h_i[\), with:
\[
\forall t \in ]0, h_i[, \quad \|f'_i(t)\| \leq \epsilon
\]

6. Show that:
\[
\|\phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a)\| \leq \epsilon|h_i|
\]

7. Show that the previous inequality still holds if \( h_i \leq 0 \).

8. Conclude that for all \( h \in \mathbb{R}^n \) with \( \|h\| < \eta \), we have:
\[
\|\phi(a + h) - \phi(a) - l(h)\| \leq \epsilon \sqrt{n} \|h\|
\]

9. Prove the following:

**Theorem 114** Let \( n \geq 1 \) and \( U \) be open in \( \mathbb{R}^n \). Let \( \phi : U \rightarrow E \) be a map, where \( E \) is an \( \mathbb{R} \)-normed space. If, for all \( i \in \mathbb{N} \) \( \frac{\partial \phi}{\partial x_i} \) exists on \( U \) and is continuous at \( a \in U \), then \( \phi \) is differentiable at \( a \in U \).
Exercise 16. Let $n \geq 1$ and $U$ be open in $\mathbb{R}^n$. Let $\phi : U \to E$ be a map, where $E$ is an $\mathbb{R}$-normed space. We assume that for all $i \in \mathbb{N}_n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on $U$.

1. Show that $\phi$ is differentiable on $U$.

2. Show that for all $a, b \in U$ and $h \in \mathbb{R}^n$:
   \[
   \| (d\phi(b) - d\phi(a))(h) \| \leq \left( \sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{1/2} \| h \|
   \]

3. Show that for all $a, b \in U$:
   \[
   \| d\phi(b) - d\phi(a) \| \leq \left( \sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{1/2}
   \]

4. Show that $d\phi : U \to \mathcal{L}_{\mathbb{R}}(\mathbb{R}^n, E)$ is continuous.

5. Prove the following:

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**Theorem 115** Let \( n \geq 1 \) and \( U \) be open in \( \mathbb{R}^n \). Let \( \phi : U \to E \) be a map, where \( E \) is an \( \mathbb{R} \)-normed space. Then, \( \phi \) is of class \( C^1 \) on \( U \), if and only if for all \( i = 1, \ldots, n \), \( \frac{\partial \phi}{\partial x_i} \) exists and is continuous on \( U \).

**Exercise 17.** Let \( E, F \) be two \( \mathbb{R} \)-normed spaces and \( l \in \mathcal{L}_\mathbb{R}(E, F) \). Let \( U \) be open in \( E \) and \( l|_U \) be the restriction of \( l \) to \( U \). Show that \( l|_U \) is of class \( C^1 \) on \( U \), and that we have:

\[ \forall x \in U \, , \, d(l|_U)(x) = l \]

**Exercise 18.** Let \( E_1, \ldots, E_n, n \geq 1 \), be \( n \) \( K \)-normed spaces. Let \( E = E_1 \times \ldots \times E_n \). Let \( p \in [1, +\infty] \), and for all \( x = (x_1, \ldots, x_n) \in E \):

\[
\begin{align*}
\|x\|_p &\triangleq \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \\
\|x\|_\infty &\triangleq \max_{i=1,\ldots,n} \|x_i\|
\end{align*}
\]
1. Using theorem (43), show that $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are norms on $E$.

2. Show $\|\cdot\|_p$ and $\|\cdot\|_\infty$ induce the product topology on $E$.

3. Conclude that $E$ is also an $K$-normed space, and that the norm topology on $E$ is exactly the product topology on $E$.

**Exercise 19.** Let $E$ and $F$ be two $R$-normed spaces. Let $U$ be open in $E$ and $\phi, \psi : U \to F$ be two maps. We assume that both $\phi$ and $\psi$ are differentiable at $a \in U$. Given $\alpha \in R$, show that $\phi + \alpha \psi$ is differentiable at $a \in U$ and:

$$d(\phi + \alpha \psi)(a) = d\phi(a) + \alpha d\psi(a)$$

**Exercise 20.** Let $E$ and $F$ be $K$-normed spaces. Let $U$ be open in $E$ and $\phi : U \to F$ be a map. Let $N_E$ and $N_F$ be two norms on $E$ and $F$, inducing the same topologies as the norm topologies of $E$ and $F$.

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respectively. For all \( l \in \mathcal{L}_K(E, F) \), we define:

\[
N(l) = \sup \{ N_F(l(x)) : x \in E, N_E(x) = 1 \}
\]

1. Explain why the set \( \mathcal{L}_K(E, F) \) is unambiguously defined.

2. Show that the identity \( \text{id}_E : (E, \| \cdot \|) \rightarrow (E, N_E) \) is continuous.

3. Show the existence of \( m_E, M_E > 0 \) such that:

\[
\forall x \in E, \quad m_E \| x \| \leq N_E(x) \leq M_E \| x \|
\]

4. Show the existence of \( m, M > 0 \) such that:

\[
\forall l \in \mathcal{L}_K(E, F), \quad m \| l \| \leq N(l) \leq M \| l \|
\]

5. Show that \( \| \cdot \| \) and \( N \) induce the same topology on \( \mathcal{L}_K(E, F) \).

6. Show that if \( K = \mathbb{R} \) and \( \phi \) is differentiable at \( a \in U \), then \( \phi \) is also differentiable at \( a \) with respect to the norms \( N_E \) and \( N_F \), and the differential \( d\phi(a) \) is unchanged.

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7. Show that if $K = \mathbb{R}$ and $\phi$ is of class $C^1$ on $U$, then $\phi$ is also of class $C^1$ on $U$ with respect to the norms $N_E$ and $N_F$.

**Exercise 21.** Let $E$ and $F_1, \ldots, F_p$, $p \geq 1$, be $p+1 \mathbb{R}$-normed spaces. Let $U$ be open in $E$ and $F = F_1 \times \cdots \times F_p$. Let $\phi : U \to F$ be a map. For all $i \in \mathbb{N}_p$, we denote $p_i : F \to F_i$ the canonical projection and we define $\phi_i = p_i \circ \phi$. We also consider $u_i : F_i \to F$, defined as:

$$\forall x_i \in F_i, \ u_i(x_i) \triangleq (0, \ldots, x_i, \ldots, 0)$$

1. Given $i \in \mathbb{N}_p$, show that $p_i \in \mathcal{L}_\mathbb{R}(F, F_i)$.

2. Given $i \in \mathbb{N}_p$, show that $u_i \in \mathcal{L}_\mathbb{R}(F_i, F)$ and $\phi = \sum_{i=1}^{p} u_i \circ \phi_i$.

3. Show that if $\phi$ is differentiable at $a \in U$, then for all $i \in \mathbb{N}_p$, $\phi_i : U \to F_i$ is differentiable at $a \in U$ and $d\phi_i(a) = p_i \circ d\phi(a)$. 

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4. Show that if \( \phi_i \) is differentiable at \( a \in U \) for all \( i \in \mathbb{N}_p \), then \( \phi \) is differentiable at \( a \in U \) and:
\[
d\phi(a) = \sum_{i=1}^{p} u_i \circ d\phi_i(a)
\]

5. Suppose that \( \phi \) is differentiable at \( a, b \in U \). Let \( F \) be given the norm \( \| \cdot \|_2 \) of exercise 18. Show that for all \( i \in \mathbb{N}_p \):
\[
\|d\phi_i(b) - d\phi_i(a)\| \leq \|d\phi(b) - d\phi(a)\|
\]

6. Show that:
\[
\|d\phi(b) - d\phi(a)\| \leq \left( \sum_{i=1}^{p} \|d\phi_i(b) - d\phi_i(a)\|^2 \right)^{1/2}
\]

7. Show that \( \phi \) is of class \( C^1 \) \( \iff \) \( \phi_i \) is of class \( C^1 \) for all \( i \in \mathbb{N}_p \).

8. Conclude with theorem (116)
Theorem 116  Let $E, F_1, \ldots, F_p, \ (p \geq 1)$, be $p+1$ $\mathbb{R}$-normed spaces and $U$ be open in $E$. Let $F$ be the $\mathbb{R}$-normed space $F = F_1 \times \ldots \times F_p$ and $\phi = (\phi_1, \ldots, \phi_p) : U \to F$ be a map. Then, $\phi$ is differentiable at $a \in U$, if and only if $d\phi_i(a)$ exists for all $i \in \mathbb{N}_p$, in which case:

$$\forall h \in E, \ d\phi(a)(h) = (d\phi_1(a)(h), \ldots, d\phi_p(a)(h))$$

Also, $\phi$ is of class $C^1$ on $U \iff \phi_i$ is of class $C^1$ on $U$, for all $i \in \mathbb{N}_p$.

Theorem 117  Let $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$ be a map, where $n \geq 1$ and $U$ is open in $\mathbb{R}^n$. We assume that $\phi$ is differentiable at $a \in U$. Then, for all $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}(a)$ exists, and we have:

$$d\phi(a) = \begin{pmatrix}
\frac{\partial \phi_1}{\partial x_1}(a) & \cdots & \frac{\partial \phi_n}{\partial x_1}(a) \\
\vdots & \ddots & \vdots \\
\frac{\partial \phi_1}{\partial x_n}(a) & \cdots & \frac{\partial \phi_n}{\partial x_n}(a)
\end{pmatrix}$$
Moreover, $\phi$ is of class $C^1$ on $U$, if and only if for all $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}$ exists and is continuous on $U$.

**Exercise 22.** Prove theorem (117)

**Definition 132** Let $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$ be a map, where $n \geq 1$ and $U$ is open in $\mathbb{R}^n$. We assume that $\phi$ is differentiable at $a \in U$. We call *Jacobian* of $\phi$ at $a$, denoted $J(\phi)(a)$, the determinant of the differential $d\phi(a)$ at $a$, i.e.

$$J(\phi)(a) = \det \left( \begin{array}{ccc}
\frac{\partial \phi_1}{\partial x_1}(a) & \cdots & \frac{\partial \phi_1}{\partial x_n}(a) \\
\vdots & \ddots & \vdots \\
\frac{\partial \phi_n}{\partial x_1}(a) & \cdots & \frac{\partial \phi_n}{\partial x_n}(a)
\end{array} \right)$$

**Definition 133** Let $n \geq 1$ and $\Omega, \Omega'$ be open in $\mathbb{R}^n$. A bijection $\phi : \Omega \to \Omega'$ is called a $C^1$-diffeomorphism between $\Omega$ and $\Omega'$, if and only if $\phi : \Omega \to \mathbb{R}^n$ and $\phi^{-1} : \Omega' \to \mathbb{R}^n$ are both of class $C^1$. 

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**Exercise 23.** Let $\Omega$ and $\Omega'$ be open in $\mathbb{R}^n$. Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism, $\psi = \phi^{-1}$, and $I_n$ be the identity mapping of $\mathbb{R}^n$.

1. Explain why $J(\psi) : \Omega' \to \mathbb{R}$ and $J(\phi) : \Omega \to \mathbb{R}$ are continuous.
2. Show that $d\phi(\psi(x)) \circ d\psi(x) = I_n$, for all $x \in \Omega'$.
3. Show that $d\psi(\phi(x)) \circ d\phi(x) = I_n$, for all $x \in \Omega$.
4. Show that $J(\psi)(x) \neq 0$ for all $x \in \Omega'$.
5. Show that $J(\phi)(x) \neq 0$ for all $x \in \Omega$.
6. Show that $J(\psi) = 1/(J(\phi) \circ \psi)$ and $J(\phi) = 1/(J(\psi) \circ \phi)$. 


**Definition 134** Let $n \geq 1$ and $\Omega \in \mathcal{B}(\mathbb{R}^n)$, be a Borel set in $\mathbb{R}^n$. We define the **Lebesgue measure** on $\Omega$, denoted $dx|_{\Omega}$, as the restriction to $\mathcal{B}(\Omega)$ of the Lebesgue measure on $\mathbb{R}^n$, i.e. the measure on $(\Omega, \mathcal{B}(\Omega))$ defined by:

$$\forall B \in \mathcal{B}(\Omega), \ dx|_{\Omega}(B) \triangleq dx(B)$$

**Exercise 24.** Show that $dx|_{\Omega}$ is a well-defined measure on $(\Omega, \mathcal{B}(\Omega))$.

**Exercise 25.** Let $n \geq 1$ and $\Omega, \Omega'$ be open in $\mathbb{R}^n$. Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We assume that $d\psi(a) = I_n$, (identity mapping on $\mathbb{R}^n$), and given $\epsilon > 0$, we denote:

$$B(a, \epsilon) \triangleq \{ x \in \mathbb{R}^n : \|a - x\| < \epsilon \}$$

where $\| . \|$ is the usual norm in $\mathbb{R}^n$.

1. Why are $dx|_{\Omega'}$, $\phi(dx|_{\Omega})$ well-defined measures on $(\Omega', \mathcal{B}(\Omega'))$.

2. Show that for $\epsilon > 0$ sufficiently small, $B(a, \epsilon) \in \mathcal{B}(\Omega')$.  

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3. Show that it makes sense to investigate whether the limit:

$$\lim_{\epsilon \downarrow 0} \frac{\phi(dx_{|\Omega}) (B(a, \epsilon))}{dx_{|\Omega} (B(a, \epsilon))}$$

does exists in $\mathbb{R}$.

4. Given $r > 0$, show the existence of $\epsilon_1 > 0$ such that for all $h \in \mathbb{R}^n$ with $\|h\| \leq \epsilon_1$, we have $a + h \in \Omega'$, and:

$$\|\psi(a + h) - \psi(a) - h\| \leq r \|h\|$$

5. Show for all $h \in \mathbb{R}^n$ with $\|h\| \leq \epsilon_1$, we have $a + h \in \Omega'$, and:

$$\|\psi(a + h) - \psi(a)\| \leq (1 + r)\|h\|$$

6. Show that for all $\epsilon \in [0, \epsilon_1]$, we have $B(a, \epsilon) \subseteq \Omega'$, and:

$$\psi(B(a, \epsilon)) \subseteq B(\psi(a), \epsilon(1 + r))$$

7. Show that $d\phi(\psi(a)) = I_n$. 

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8. Show the existence of $\epsilon_2 > 0$ such that for all $k \in \mathbb{R}^n$ with $\|k\| \leq \epsilon_2$, we have $\psi(a) + k \in \Omega$, and:

$$\|\phi(\psi(a) + k) - a - k\| \leq r\|k\|$$

9. Show for all $k \in \mathbb{R}^n$ with $\|k\| \leq \epsilon_2$, we have $\psi(a) + k \in \Omega$, and:

$$\|\phi(\psi(a) + k) - a\| \leq (1 + r)\|k\|$$

10. Show for all $\epsilon \in [0, \epsilon_2(1 + r)]$, we have $B(\psi(a), \frac{\epsilon}{1 + r}) \subseteq \Omega$, and:

$$B(\psi(a), \frac{\epsilon}{1 + r}) \subseteq \{\phi \in B(a, \epsilon)\}$$

11. Show that if $B(a, \epsilon) \subseteq \Omega'$, then $\psi(B(a, \epsilon)) = \{\phi \in B(a, \epsilon)\}$.

12. Show if $0 < \epsilon < \epsilon_0 = \epsilon_1 \wedge \epsilon_2(1 + r)$, then $B(a, \epsilon) \subseteq \Omega'$, and:

$$B(\psi(a), \frac{\epsilon}{1 + r}) \subseteq \{\phi \in B(a, \epsilon)\} \subseteq B(\psi(a), \epsilon(1 + r))$$
13. Show that for all $\epsilon \in ]0, \epsilon_0[$:

(i) $dx(B(\psi(a), \frac{\epsilon}{1+r})) = (1+r)^{-n}dx_{|\Omega'}(B(a, \epsilon))$

(ii) $dx(B(\psi(a), \epsilon(1+r))) = (1+r)^n dx_{|\Omega'}(B(a, \epsilon))$

(iii) $dx(\{\phi \in B(a, \epsilon)\}) = \phi(dx_{|\Omega}(B(a, \epsilon))$

14. Show that for all $\epsilon \in ]0, \epsilon_0[$, $B(a, \epsilon) \subseteq \Omega'$, and:

$(1+r)^{-n} \leq \frac{\phi(dx_{|\Omega})(B(a, \epsilon))}{dx_{|\Omega'}(B(a, \epsilon))} \leq (1+r)^n$

15. Conclude that:

$$\lim_{\epsilon \to 0} \frac{\phi(dx_{|\Omega})(B(a, \epsilon))}{dx_{|\Omega'}(B(a, \epsilon))} = 1$$

**Exercise 26.** Let $n \geq 1$ and $\Omega, \Omega'$ be open in $\mathbb{R}^n$. Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We put $A = d\psi(a)$. 

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1. Show that $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection.

2. Define $\Omega'' = A^{-1}(\Omega)$. Show that this definition does not depend on whether $A^{-1}(\Omega)$ is viewed as inverse, or direct image.

3. Show that $\Omega''$ is an open subset of $\mathbb{R}^n$.

4. We define $\tilde{\phi} : \Omega'' \to \Omega'$ by $\tilde{\phi}(x) = \phi \circ A(x)$. Show that $\tilde{\phi}$ is a $C^1$-diffeomorphism with $\tilde{\psi} = \tilde{\phi}^{-1} = A^{-1} \circ \psi$.

5. Show that $d\tilde{\psi}(a) = I_n$.

6. Show that:
$$\lim_{\epsilon \to 0} \frac{\tilde{\phi}(dx_{\Omega''})(B(a, \epsilon))}{dx_{\Omega'}(B(a, \epsilon))} = 1$$

7. Let $\epsilon > 0$ with $B(a, \epsilon) \subseteq \Omega'$. Justify each of the following steps:
$$\tilde{\phi}(dx_{\Omega''})(B(a, \epsilon)) = dx_{\Omega'}(\{\tilde{\phi} \in B(a, \epsilon)\})$$
$$= dx(\{\tilde{\phi} \in B(a, \epsilon)\})$$
\begin{align*}
&= dx(\{x \in \Omega^n : \phi \circ A(x) \in B(a, \epsilon)\}) \\
&= dx(\{x \in \Omega^n : A(x) \in \phi^{-1}(B(a, \epsilon))\}) \\
&= dx(\{x \in \mathbb{R}^n : A(x) \in \phi^{-1}(B(a, \epsilon))\}) \\
&= A(dx)(\{\phi \in B(a, \epsilon)\}) \\
&= |\det A|^{-1} dx(\{\phi \in B(a, \epsilon)\}) \\
&= |\det A|^{-1} dx_{\Omega}(\{\phi \in B(a, \epsilon)\}) \\
&= |\det A|^{-1} \phi(dx_{\Omega})(B(a, \epsilon))
\end{align*}

8. Show that:
\[\lim_{\epsilon \downarrow 0} \frac{\phi(dx_{\Omega})(B(a, \epsilon))}{dx_{\Omega}(B(a, \epsilon))} = |\det A|\]

9. Conclude with the following:
Theorem 118  Let $n \geq 1$ and $\Omega, \Omega'$ be open in $\mathbb{R}^n$. Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism and $\psi = \phi^{-1}$. Then, for all $a \in \Omega'$, we have:

$$
\lim_{\epsilon \to 0} \frac{\phi(dx_{\Omega})(B(a, \epsilon))}{dx_{\Omega'}(B(a, \epsilon))} = |J(\psi)(a)|
$$

where $J(\psi)(a)$ is the Jacobian of $\psi$ at $a$, $B(a, \epsilon)$ is the open ball in $\mathbb{R}^n$, and $dx_{\Omega}$, $dx_{\Omega'}$ are the Lebesgue measures on $\Omega$ and $\Omega'$ respectively.

Exercise 27. Let $n \geq 1$ and $\Omega, \Omega'$ be open in $\mathbb{R}^n$. Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism and $\psi = \phi^{-1}$.

1. Let $K \subseteq \Omega'$ be a non-empty compact subset of $\Omega'$ such that $dx_{\Omega'}(K) = 0$. Given $\epsilon > 0$, show the existence of $V$ open in $\Omega'$ such that $K \subseteq V \subseteq \Omega'$, and $dx_{\Omega'}(V) \leq \epsilon$.

2. Explain why $V$ is also open in $\mathbb{R}^n$.

3. Show that $M \overset{\triangle}{=} \sup_{x \in K} \|d\psi(x)\| \in \mathbb{R}^+$. 

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4. For all $x \in K$, show there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subseteq V$, and for all $h \in \mathbb{R}^n$ with $\|h\| \leq 3\epsilon_x$, we have $x + h \in \Omega'$, and:
\[
\|\psi(x + h) - \psi(x)\| \leq (M + 1)\|h\|
\]

5. Show that for all $x \in K$, $B(x, 3\epsilon_x) \subseteq \Omega'$, and:
\[
\psi(B(x, 3\epsilon_x)) \subseteq B(\psi(x), 3(M + 1)\epsilon_x)
\]

6. Show that $\psi(B(x, 3\epsilon_x)) = \{\phi \in B(x, 3\epsilon_x)\}$, for all $x \in K$.

7. Show the existence of $\{x_1, \ldots, x_p\} \subseteq K$, ($p \geq 1$), such that:
\[
K \subseteq B(x_1, \epsilon_{x_1}) \cup \ldots \cup B(x_p, \epsilon_{x_p})
\]

8. Show the existence of $S \subseteq \{1, \ldots, p\}$ such that the $B(x_i, \epsilon_{x_i})$'s are pairwise disjoint for $i \in S$, and:
\[
K \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_{x_i})
\]
9. Show that $\{ \phi \in K \} \subseteq \cup_{i \in S} B(\psi(x_i), 3(M + 1)\epsilon_{x_i})$.

10. Show that $\phi(d\pi_{\Omega})(K) \leq \sum_{i \in S} 3^n(M + 1)^n d\pi(B(x_i, \epsilon_{x_i}))$.

11. Show that $\phi(d\pi_{\Omega})(K) \leq 3^n(M + 1)^n d\pi(V)$.

12. Show that $\phi(d\pi_{\Omega})(K) \leq 3^n(M + 1)^n \epsilon$.

13. Conclude that $\phi(d\pi_{\Omega})(K) = 0$.

14. Show that $\phi(d\pi_{\Omega})$ is a locally finite measure on $(\Omega', \mathcal{B}(\Omega'))$.

15. Show that for all $B \in \mathcal{B}(\Omega')$:

$$\phi(d\pi_{\Omega})(B) = \sup \{ \phi(d\pi_{\Omega})(K) : K \subseteq B, K \text{ compact} \}$$

16. Show that for all $B \in \mathcal{B}(\Omega')$:

$$d\pi_{\Omega'}(B) = 0 \Rightarrow \phi(d\pi_{\Omega})(B) = 0$$

17. Conclude with the following:
Theorem 119  Let $n \geq 1$, $\Omega$, $\Omega'$ be open in $\mathbb{R}^n$, and $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism. Then, the image measure $\phi(dx|\Omega)$, by $\phi$ of the Lebesgue measure on $\Omega$, is absolutely continuous with respect to $dx|\Omega'$, the Lebesgue measure on $\Omega'$, i.e.:

$$\phi(dx|\Omega) << dx|\Omega'$$

Exercise 28. Let $n \geq 1$ and $\Omega$, $\Omega'$ be open in $\mathbb{R}^n$. Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism and $\psi = \phi^{-1}$.

1. Explain why there exists a sequence $(V_p)_{p \geq 1}$ of open sets in $\Omega'$, such that $V_p \uparrow \Omega'$ and for all $p \geq 1$, the closure of $V_p$ in $\Omega'$, i.e. $\bar{V}_p$, is compact.

2. Show that each $V_p$ is also open in $\mathbb{R}^n$, and that $\bar{V}_p = \bar{V}_p$.

3. Show that $\phi(dx|\Omega)(V_p) < +\infty$, for all $p \geq 1$. 

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4. Show that $dx|_{\Omega'}$ and $\phi(dx|_{\Omega})$ are two $\sigma$-finite measures on $\Omega'$.

5. Show there is $h : (\Omega', \mathcal{B}(\Omega')) \rightarrow (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ measurable, with:

$$\forall B \in \mathcal{B}(\Omega'), \phi(dx|_{\Omega})(B) = \int_B hdx|_{\Omega'}$$

6. For all $p \geq 1$, we define $h_p = h1_{V_p}$, and we put:

$$\forall x \in \mathbb{R}^n, \tilde{h}_p(x) \triangleq \begin{cases} h_p(x) & \text{if } x \in \Omega' \\ 0 & \text{if } x \notin \Omega' \end{cases}$$

Show that:

$$\int_{\mathbb{R}^n} \tilde{h}_p dx = \int_{\Omega'} h_p dx|_{\Omega'} = \phi(dx|_{\Omega})(V_p) < +\infty$$

and conclude that $\tilde{h}_p \in L^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$.

7. Show the existence of some $N \in \mathcal{B}(\mathbb{R}^n)$, such that $dx(N) = 0$
and for all $x \in N^c$ and $p \geq 1$, we have:

$$
\tilde{h}_p(x) = \lim_{\epsilon \downarrow 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} \tilde{h}_p d\epsilon
$$

8. Put $N' = N \cap \Omega'$. Show that $N' \in B(\Omega')$ and $dx|_{\Omega'}(N') = 0$.

9. Let $x \in \Omega'$ and $p \geq 1$ be such that $x \in V_p$. Show that if $\epsilon > 0$ is such that $B(x, \epsilon) \subseteq V_p$, then $dx(B(x, \epsilon)) = dx|_{\Omega'}(B(x, \epsilon))$, and:

$$
\int_{B(x, \epsilon)} \tilde{h}_p d\epsilon = \int_{\mathbb{R}^n} 1_{B(x, \epsilon)} \tilde{h}_p d\epsilon = \int_{\Omega'} 1_{B(x, \epsilon)} h_p d\epsilon|_{\Omega'}
$$

10. Show that:

$$
\int_{\Omega'} 1_{B(x, \epsilon)} h_p d\epsilon|_{\Omega'} = \int_{\Omega'} 1_{B(x, \epsilon)} h d\epsilon|_{\Omega'} = \phi(dx|_{\Omega})(B(x, \epsilon))
$$

11. Show that for all $x \in \Omega' \setminus N'$, we have:

$$
h(x) = \lim_{\epsilon \downarrow 0} \frac{\phi(dx|_{\Omega})(B(x, \epsilon))}{dx|_{\Omega'}(B(x, \epsilon))}
$$
12. Show that $h = |J(\psi)| \, dx_{\Omega'}$-a.s. and conclude with the following:

**Theorem 120** Let $n \geq 1$ and $\Omega, \Omega'$ be open in $\mathbb{R}^n$. Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism and $\psi = \phi^{-1}$. Then, the image measure by $\phi$ of the Lebesgue measure on $\Omega$, is equal to the measure on $(\Omega', \mathcal{B}(\Omega'))$ with density $|J(\psi)|$ with respect to the Lebesgue measure on $\Omega'$, i.e.:

$$\phi(dx_{\Omega}) = \int |J(\psi)| \, dx_{\Omega'}$$

**Exercise 29.** Prove the following:

**Theorem 121 (Jacobian Formula 1)** Let $n \geq 1$ and $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism where $\Omega, \Omega'$ are open in $\mathbb{R}^n$. Let $\psi = \phi^{-1}$. Then, for all $f : (\Omega', \mathcal{B}(\Omega')) \to [0, +\infty]$ non-negative and measurable:

$$\int_{\Omega} f \circ \phi \, dx_{\Omega} = \int_{\Omega'} f|J(\psi)| \, dx_{\Omega'}$$
and:

\[ \int_{\Omega} (f \circ \phi) |J(\phi)| dx_{|\Omega} = \int_{\Omega'} f dx_{|\Omega'} \]

**Exercise 30.** Prove the following:

**Theorem 122 (Jacobian Formula 2)** Let \( n \geq 1 \) and \( \phi : \Omega \to \Omega' \) be a \( C^1 \)-diffeomorphism where \( \Omega, \Omega' \) are open in \( \mathbb{R}^n \). Let \( \psi = \phi^{-1} \). Then, for all measurable map \( f : (\Omega', \mathcal{B}(\Omega')) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \), we have the equivalence:

\[ f \circ \phi \in L^1_C(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \iff |J(\psi)| \in L^1_C(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'}) \]

in which case:

\[ \int_{\Omega} f \circ \phi dx_{|\Omega} = \int_{\Omega'} f|J(\psi)| dx_{|\Omega'} \]

and, furthermore:

\[ (f \circ \phi)|J(\phi)| \in L^1_C(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \iff f \in L^1_C(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'}) \]
in which case:
\[
\int_{\Omega} (f \circ \phi)|J(\phi)|dx_{\Omega} = \int_{\Omega'} f dx_{\Omega'}
\]

**Exercise 31.** Let \( f : \mathbb{R}^2 \to [0, +\infty] \), with \( f(x, y) = \exp(-(x^2 + y^2)/2) \).

1. Show that:
\[
\int_{\mathbb{R}^2} f(x, y)dxdy = \left( \int_{-\infty}^{+\infty} e^{-u^2/2} du \right)^2
\]

2. Define:
\[
\Delta_1 \triangleq \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}
\]
\[
\Delta_2 \triangleq \{(x, y) \in \mathbb{R}^2 : x < 0, y > 0\}
\]
and let \( \Delta_3 \) and \( \Delta_4 \) be the other two open quarters of \( \mathbb{R}^2 \). Show:
\[
\int_{\mathbb{R}^2} f(x, y)dxdy = \int_{\Delta_1 \cup \ldots \cup \Delta_4} f(x, y)dxdy
\]
3. Let \( Q: \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( Q(x, y) = (-x, y) \). Show that:
\[
\int_{\Delta_1} f(x, y)dxdy = \int_{\Delta_2} f \circ Q^{-1}(x, y)dxdy
\]

4. Show that:
\[
\int_{\mathbb{R}^2} f(x, y)dxdy = 4\int_{\Delta_1} f(x, y)dxdy
\]

5. Let \( D_1 = ]0, +\infty[ \times ]0, \pi/2[ \subseteq \mathbb{R}^2 \), and define \( \phi: D_1 \to \Delta_1 \) by:
\[
\forall (r, \theta) \in D_1, \ \phi(r, \theta) = (r \cos \theta, r \sin \theta)
\]
Show that \( \phi \) is a bijection and that \( \psi = \phi^{-1} \) is given by:
\[
\forall (x, y) \in \Delta_1, \ \psi(x, y) = \left( \sqrt{x^2 + y^2}, \arctan(y/x) \right)
\]

6. Show that \( \phi \) is a \( C^1 \)-diffeomorphism, with:
\[
\forall (r, \theta) \in D_1, \ d\phi(r, \theta) = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}
\]
and:
\[
\forall (x, y) \in \Delta_1, \ d\psi(x, y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{pmatrix}
\]

7. Show that \( J(\phi)(r, \theta) = r \), for all \((r, \theta) \in D_1\).

8. Show that \( J(\psi)(x, y) = \frac{1}{(\sqrt{x^2+y^2})} \), for all \((x, y) \in \Delta_1\).

9. Show that:
\[
\int_{\Delta_1} f(x, y)dxdy = \frac{\pi}{2}
\]

10. Prove the following:

**Theorem 123** We have:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-u^2/2} du = 1
\]

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