20. Gaussian Measures

$\mathcal{M}_n(\mathbb{R})$ is the set of all $n \times n$-matrices with real entries, $n \geq 1$.

**Definition 141** A matrix $M \in \mathcal{M}_n(\mathbb{R})$ is said to be symmetric, if and only if $M = M^t$. $M$ is orthogonal, if and only if $M$ is non-singular and $M^{-1} = M^t$. If $M$ is symmetric, we say that $M$ is non-negative, if and only if:
\[ \forall u \in \mathbb{R}^n, \langle u, Mu \rangle \geq 0 \]

**Theorem 131** Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$, $n \geq 1$, be a symmetric and non-negative real matrix. There exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ and $P \in \mathcal{M}_n(\mathbb{R})$ orthogonal matrix, such that:
\[
\Sigma = P \begin{pmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{pmatrix} P^t
\]

In particular, there exists $A \in \mathcal{M}_n(\mathbb{R})$ such that $\Sigma = A.A^t$. 

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As a rare exception, theorem (131) is given without proof.

**Exercise 1.** Given $n \geq 1$ and $M \in \mathcal{M}_n(\mathbb{R})$, show that we have:
\[
\forall u, v \in \mathbb{R}^n, \quad \langle u, Mv \rangle = \langle M^t u, v \rangle
\]

**Exercise 2.** Let $n \geq 1$ and $m \in \mathbb{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative matrix. Let $\mu_1$ be the probability measure on $\mathbb{R}$:
\[
\forall B \in \mathcal{B}(\mathbb{R}) , \quad \mu_1(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx
\]

Let $\mu = \mu_1 \otimes \ldots \otimes \mu_1$ be the product measure on $\mathbb{R}^n$. Let $A \in \mathcal{M}_n(\mathbb{R})$ be such that $\Sigma = AA^t$. We define the map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by:
\[
\forall x \in \mathbb{R}^n, \quad \phi(x) \triangleq Ax + m
\]

1. Show that $\mu$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.
2. Explain why the image measure $P = \phi(\mu)$ is well-defined.
3. Show that $P$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. 

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4. Show that for all $u \in \mathbb{R}^n$:

$$FP(u) = \int_{\mathbb{R}^n} e^{i(u, \phi(x))} d\mu(x)$$

5. Let $v = A^t u$. Show that for all $u \in \mathbb{R}^n$:

$$FP(u) = e^{i(u, m) - \|v\|^2/2}$$

6. Show the following:

**Theorem 132** Let $n \geq 1$ and $m \in \mathbb{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. There exists a unique complex measure on $\mathbb{R}^n$, denoted $N_n(m, \Sigma)$, with Fourier transform:

$$\mathcal{F}N_n(m, \Sigma)(u) \triangleq \int_{\mathbb{R}^n} e^{i(u, x)} dN_n(m, \Sigma)(x) = e^{i\langle u, m \rangle} - \frac{1}{2} \langle u, \Sigma u \rangle$$

for all $u \in \mathbb{R}^n$. Furthermore, $N_n(m, \Sigma)$ is a probability measure.
**Definition 142** Let $n \geq 1$ and $m \in \mathbb{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. The probability measure $N_n(m, \Sigma)$ on $\mathbb{R}^n$ defined in theorem (132) is called the $n$-dimensional gaussian measure or normal distribution, with mean $m \in \mathbb{R}^n$ and covariance matrix $\Sigma$.

**Exercise 3.** Let $n \geq 1$ and $m \in \mathbb{R}^n$. Show that $N_n(m, 0) = \delta_m$.

**Exercise 4.** Let $m \in \mathbb{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. Let $A \in \mathcal{M}_n(\mathbb{R})$ be such that $\Sigma = AA^t$. A map $p : \mathbb{R}^n \to \mathbb{C}$ is said to be a polynomial, if and only if, it is a finite linear complex combination of maps $x \to x^\alpha$, $^1$ for $\alpha \in \mathbb{N}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbb{R})$, we have:

   $$N_1(0, 1)(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2}dx$$

---

$^1$See definition (140).
2. Show that:
\[ \int_{-\infty}^{+\infty} |x| dN_1(0,1)(x) < +\infty \]

3. Show that for all integer \( k \geq 1 \):
\[
\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x^{k+1} e^{-x^2/2} \, dx = \frac{k}{\sqrt{2\pi}} \int_0^{+\infty} x^{k-1} e^{-x^2/2} \, dx
\]

4. Show that for all integer \( k \geq 0 \):
\[ \int_{-\infty}^{+\infty} |x|^k dN_1(0,1)(x) < +\infty \]

5. Show that for all \( \alpha \in \mathbb{N}^n \):
\[ \int_{\mathbb{R}^n} |x|\alpha dN_1(0,1) \otimes \ldots \otimes N_1(0,1)(x) < +\infty \]
6. Let $p : \mathbb{R}^n \to \mathbb{C}$ be a polynomial. Show that:

$$\int_{\mathbb{R}^n} |p(x)| dN_1(0,1) \otimes \cdots \otimes N_1(0,1)(x) < +\infty$$

7. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\phi(x) = Ax + m$. Explain why the image measure $\phi(N_1(0,1) \otimes \cdots \otimes N_1(0,1))$ is well-defined.

8. Show that $\phi(N_1(0,1) \otimes \cdots \otimes N_1(0,1)) = N_n(m, \Sigma)$.

9. Show if $\beta \in \mathbb{N}^n$ and $|\beta| = 1$, then $x \to \phi(x)^\beta$ is a polynomial.

10. Show that if $\alpha' \in \mathbb{N}^n$ and $|\alpha'| = k+1$, then $\phi(x)^{\alpha'} = \phi(x)^\alpha \phi(x)^\beta$ for some $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| = k$ and $|\beta| = 1$.

11. Show that the product of two polynomials is a polynomial.

12. Show that for all $\alpha \in \mathbb{N}^n$, $x \to \phi(x)^\alpha$ is a polynomial.

13. Show that for all $\alpha \in \mathbb{N}^n$:

$$\int_{\mathbb{R}^n} |\phi(x)^\alpha| dN_1(0,1) \otimes \cdots \otimes N_1(0,1)(x) < +\infty$$
14. Show the following:

**Theorem 133** Let $n \geq 1$ and $m \in \mathbb{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. Then, for all $\alpha \in \mathbb{N}^n$, the map $x \to x^\alpha$ is integrable with respect to the gaussian measure $N_n(m, \Sigma)$:

$$\int_{\mathbb{R}^n} |x^\alpha| dN_n(m, \Sigma)(x) < +\infty$$

**Exercise 5.** Let $m \in \mathbb{R}^n$. Let $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. Let $j, k \in \mathbb{N}_n$. Let $\phi$ be the fourier transform of the gaussian measure $N_n(m, \Sigma)$, i.e.:

$$\forall u \in \mathbb{R}^n, \quad \phi(u) \triangleq e^{i(u, m) - \frac{1}{2}(u, \Sigma u)}$$

1. Show that:

$$\int_{\mathbb{R}^n} x_j dN_n(m, \Sigma)(x) = i^{-1} \frac{\partial \phi}{\partial u_j}(0)$$
2. Show that:
\[ \int_{\mathbb{R}^n} x_j dN_n(m, \Sigma)(x) = m_j \]

3. Show that:
\[ \int_{\mathbb{R}^n} x_j x_k dN_n(m, \Sigma)(x) = i^{-2} \frac{\partial^2 \phi}{\partial u_j \partial u_k}(0) \]

4. Show that:
\[ \int_{\mathbb{R}^n} x_j x_k dN_n(m, \Sigma)(x) = \sigma_{jk} + m_j m_k \]

5. Show that:
\[ \int_{\mathbb{R}^n} (x_j - m_j)(x_k - m_k) dN_n(m, \Sigma)(x) = \sigma_{jk} \]
Theorem 134  Let $n \geq 1$ and $m \in \mathbb{R}^n$. Let $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. Let $N_n(m, \Sigma)$ be the gaussian measure with mean $m$ and covariance matrix $\Sigma$. Then, for all $j, k \in \mathbb{N}_n$, we have:

$$\int_{\mathbb{R}^n} x_j dN_n(m, \Sigma)(x) = m_j$$

and:

$$\int_{\mathbb{R}^n} (x_j - m_j)(x_k - m_k) dN_n(m, \Sigma)(x) = \sigma_{jk}$$

Definition 143  Let $n \geq 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be a measurable map. We say that $X$ is an $n$-dimensional gaussian or normal vector, if and only if its distribution is a gaussian measure, i.e., $X(P) = N_n(m, \Sigma)$ for some $m \in \mathbb{R}^n$ and $\Sigma \in \mathcal{M}_n(\mathbb{R})$ symmetric and non-negative real matrix.

Exercise 6. Show the following:

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Theorem 135. Let \( n \geq 1 \). Let \((\Omega, \mathcal{F}, P)\) be a probability space. Let \( X : (\Omega, \mathcal{F}) \to \mathbb{R}^n \) be a measurable map. Then \( X \) is a gaussian vector, if and only if there exist \( m \in \mathbb{R}^n \) and \( \Sigma \in M_n(\mathbb{R}) \) symmetric and non-negative real matrix, such that:

\[
\forall u \in \mathbb{R}^n, \quad E[e^{i\langle u, X \rangle}] = e^{i\langle u, m \rangle - \frac{1}{2} \langle u, \Sigma u \rangle}
\]

where \( \langle \cdot, \cdot \rangle \) is the usual inner-product on \( \mathbb{R}^n \).

Definition 144. Let \( X : (\Omega, \mathcal{F}) \to \mathbb{R} \) (or \( \mathbb{C} \)) be a random variable on a probability space \((\Omega, \mathcal{F}, P)\). We say that \( X \) is integrable, if and only if we have \( E[|X|] < +\infty \). We say that \( X \) is square-integrable, if and only if we have \( E[|X|^2] < +\infty \).

Exercise 7. Further to definition (144), suppose \( X \) is \( \mathbb{C} \)-valued.

1. Show \( X \) is integrable if and only if \( X \in L_1^\mathbb{C}(\Omega, \mathcal{F}, P) \).
2. Show \( X \) is square-integrable, if and only if \( X \in L_2^\mathbb{C}(\Omega, \mathcal{F}, P) \).
Exercise 8. Further to definition (144), suppose $X$ is $\mathbb{R}$-valued.

1. Show that $X$ is integrable, if and only if $X$ is $P$-almost surely equal to an element of $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, P)$.

2. Show that $X$ is square-integrable, if and only if $X$ is $P$-almost surely equal to an element of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, P)$.

Exercise 9. Let $X, Y : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two square-integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$.

1. Show that both $X$ and $Y$ are integrable.

2. Show that $XY$ is integrable.

3. Show that $(X - E[X])(Y - E[Y])$ is a well-defined and integrable.
Definition 145 Let \( X, Y : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) be two square-integrable random variables on a probability space \((\Omega, \mathcal{F}, P)\). We define the covariance between \( X \) and \( Y \), denoted \( \text{cov}(X, Y) \), as:

\[
\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])]
\]

We say that \( X \) and \( Y \) are uncorrelated if and only if \( \text{cov}(X, Y) = 0 \). If \( X = Y \), \( \text{cov}(X, Y) \) is called the variance of \( X \), denoted \( \text{var}(X) \).

Exercise 10. Let \( X, Y \) be two square integrable, real random variable on a probability space \((\Omega, \mathcal{F}, P)\).

2. Show that \( \text{var}(X) = E[X^2] - E[X]^2 \).
3. Show that \( \text{var}(X + Y) = \text{var}(X) + 2\text{cov}(X, Y) + \text{var}(Y) \)
4. Show that \( X \) and \( Y \) are uncorrelated, if and only if:
   \[
   \text{var}(X + Y) = \text{var}(X) + \text{var}(Y)
   \]
Exercise 11. Let $X$ be an $n$-dimensional normal vector on some probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$, where $m \in \mathbb{R}^n$ and $\Sigma = (\sigma_{ij}) \in M_n(\mathbb{R})$ is a symmetric and non-negative real matrix.

1. Show that each coordinate $X_j : (\Omega, \mathcal{F}) \to \mathbb{R}$ is measurable.
2. Show that $E[X_j] < +\infty$ for all $j \in \mathbb{N}$.
3. Show that for all $j = 1, \ldots, n$, we have $E[X_j] = m_j$.
4. Show that for all $j, k = 1, \ldots, n$, we have $\text{cov}(X_j, X_k) = \sigma_{jk}$.

Theorem 136. Let $X$ be an $n$-dimensional normal vector on a probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$, where $m \in \mathbb{R}^n$ and $\Sigma = (\sigma_{ij}) \in M_n(\mathbb{R})$ is a symmetric and non-negative real matrix. Then, for all $j, k \in \mathbb{N}$, we have:

$$E[X_j] = m_j$$

and:

$$\text{cov}(X_j, X_k) = \sigma_{jk}$$

where $(\sigma_{ij}) = \Sigma$. 

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Exercise 12. Show the following:

**Theorem 137** Let $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a real random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then, $X$ is a normal random variable, if and only if it is square integrable, and:

$$
\forall u \in \mathbb{R}, \quad E[e^{iuX}] = e^{iuE[X]} - \frac{1}{2}u^2 \var(X)
$$

Exercise 13. Let $X$ be an $n$-dimensional normal vector on a probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$. Let $A \in \mathcal{M}_{d,n}(\mathbb{R})$ be an $d \times n$ real matrix, $(n, d \geq 1)$. Let $b \in \mathbb{R}^d$ and $Y = AX + b$.

1. Show that $Y : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is measurable.
2. Show that the law of $Y$ is $N_d(AM + b, A\Sigma A^t)$
3. Conclude that $Y$ is an $\mathbb{R}^d$-valued normal random vector.
**Theorem 138** Let $X$ be an $n$-dimensional normal vector with law $N_n(m, \Sigma)$ on a probability space $(\Omega, \mathcal{F}, P)$, ($n \geq 1$). Let $d \geq 1$ and $A \in \mathcal{M}_{d,n}(\mathbb{R})$ be an $d \times n$ real matrix. Let $b \in \mathbb{R}^d$. Then, $Y = AX + b$ is an $d$-dimensional normal vector, with law:

$$Y(P) = N_d(\text{Am} + b, A\Sigma A^t)$$

**Exercise 14.** Let $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be a measurable map, where $(\Omega, \mathcal{F}, P)$ is a probability space. Show that if $X$ is a gaussian vector, then for all $u \in \mathbb{R}^n$, $\langle u, X \rangle$ is a normal random variable.

**Exercise 15.** Let $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be a measurable map, where $(\Omega, \mathcal{F}, P)$ is a probability space. We assume that for all $u \in \mathbb{R}^n$, $\langle u, X \rangle$ is a normal random variable.

1. Show that for all $j = 1, \ldots, n$, $X_j$ is integrable.

2. Show that for all $j = 1, \ldots, n$, $X_j$ is square integrable.

3. Explain why given $j, k = 1, \ldots, n$, $\text{cov}(X_j, X_k)$ is well-defined.
4. Let $m \in \mathbb{R}^n$ be defined by $m_j = E[X_j]$, and $u \in \mathbb{R}^n$. Show:
   $$E[\langle u, X \rangle] = \langle u, m \rangle$$

5. Let $\Sigma = (\text{cov}(X_i, X_j))$. Show that for all $u \in \mathbb{R}^n$, we have:
   $$\text{var}(\langle u, X \rangle) = \langle u, \Sigma u \rangle$$

6. Show that $\Sigma$ is a symmetric and non-negative $n \times n$ real matrix.

7. Show that for all $u \in \mathbb{R}^n$:
   $$E[e^{i\langle u, X \rangle}] = e^{iE[\langle u, X \rangle] - \frac{1}{2}\text{var}(\langle u, X \rangle)}$$

8. Show that for all $u \in \mathbb{R}^n$:
   $$E[e^{i\langle u, X \rangle}] = e^{i\langle u, m \rangle - \frac{1}{2}\langle u, \Sigma u \rangle}$$

9. Show that $X$ is a normal vector.

10. Show the following:

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Theorem 139  Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be a measurable map on a probability space $(\Omega, \mathcal{F}, P)$. Then, $X$ is an $n$-dimensional normal vector, if and only if, any linear combination of its coordinates is itself normal, or in other words $\langle u, X \rangle$ is normal for all $u \in \mathbb{R}^n$.

Exercise 16. Let $(\Omega, \mathcal{F}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and $\mu$ be the probability on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by $\mu = \frac{1}{2}(\delta_0 + \delta_1)$. Let $P = N_1(0, 1) \otimes \mu$, and $X, Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be the canonical projections defined by $X(x, y) = x$ and $Y(x, y) = y$.

1. Show that $P$ is a probability measure on $(\Omega, \mathcal{F})$.
2. Explain why $X$ and $Y$ are measurable.
3. Show that $X$ has the distribution $N_1(0, 1)$.
4. Show that $P(\{Y = 0\}) = P(\{Y = 1\}) = \frac{1}{2}$.
5. Show that $P^{(X,Y)} = P$. 

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6. Show for all $\phi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to \mathbb{C}$ measurable and bounded:
\[
E[\phi(X, Y)] = \frac{1}{2}(E[\phi(X, 0)] + E[\phi(X, 1)])
\]

7. Let $X_1 = X$ and $X_2$ be defined as:
\[
X_2 \triangleq X1_{\{Y=0\}} - X1_{\{Y=1\}}
\]
Show that $E[e^{iuX_2}] = e^{-u^2/2}$ for all $u \in \mathbb{R}$.

8. Show that $X_1(P) = X_2(P) = N_1(0, 1)$.

9. Explain why $cov(X_1, X_2)$ is well-defined.

10. Show that $X_1$ and $X_2$ are uncorrelated.

11. Let $Z = \frac{1}{2}(X_1 + X_2)$. Show that:
\[
\forall u \in \mathbb{R} : E[e^{iuZ}] = \frac{1}{2}(1 + e^{-u^2/2})
\]
12. Show that $Z$ cannot be gaussian.

13. Conclude that although $X_1, X_2$ are normally distributed, (and even uncorrelated), $(X_1, X_2)$ is not a gaussian vector.

**Exercise 17.** Let $n \geq 1$ and $m \in \mathbb{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. Let $A \in \mathcal{M}_n(\mathbb{R})$ be such that $\Sigma = A A^t$. We assume that $\Sigma$ is non-singular. We define $p_{m,\Sigma} : \mathbb{R}^n \to \mathbb{R}^+$ by:

$$
\forall x \in \mathbb{R}^n, \quad p_{m,\Sigma}(x) \triangleq \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det(\Sigma)}} e^{-\frac{1}{2}(x - m, \Sigma^{-1}(x - m))}
$$

1. Explain why $\det(\Sigma) > 0$.

2. Explain why $\sqrt{\det(\Sigma)} = |\det(A)|$.

3. Explain why $A$ is non-singular.
4. Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by:

$$\forall x \in \mathbb{R}^n, \; \phi(x) \triangleq A^{-1}(x - m)$$

Show that for all $x \in \mathbb{R}^n$, $\langle x - m, \Sigma^{-1}(x - m) \rangle = ||\phi(x)||^2$.

5. Show that $\phi$ is a $C^1$-diffeomorphism.

6. Show that $\phi(dx) = |\det(A)|dx$.

7. Show that:

$$\int_{\mathbb{R}^n} p_{m, \Sigma}(x)dx = 1$$

8. Let $\mu = \int p_{m, \Sigma}dx$. Show that:

$$\forall u \in \mathbb{R}^n, \; \mathcal{F}_\mu(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i(u,Ax + m) - \frac{1}{2} \|x\|^2} dx$$

9. Show that the fourier transform of $\mu$ is therefore given by:

$$\forall u \in \mathbb{R}^n, \; \mathcal{F}_\mu(u) = e^{i(u,m) - \frac{1}{2} (u, \Sigma u)}$$
10. Show that $\mu = N_n(m, \Sigma)$.

11. Show that $N_n(m, \Sigma) \ll dx$, i.e. that $N_n(m, \Sigma)$ is absolutely continuous w.r. to the Lebesgue measure on $\mathbb{R}^n$.

**Exercise 18.** Let $n \geq 1$ and $m \in \mathbb{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and non-negative real matrix. We assume that $\Sigma$ is singular. Let $u \in \mathbb{R}^n$ be such that $\Sigma u = 0$ and $u \neq 0$. We define:

$$B \triangleq \{ x \in \mathbb{R}^n, \langle u, x \rangle = \langle u, m \rangle \}$$

Given $a \in \mathbb{R}^n$, let $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ be the translation of vector $a$.

1. Show $B = \tau_m^{-1}(u^\perp)$, where $u^\perp$ is the orthogonal of $u$ in $\mathbb{R}^n$.
2. Show that $B \in \mathcal{B}(\mathbb{R}^n)$.
3. Explain why $dx(u^\perp) = 0$. Is it important to have $u \neq 0$?
4. Show that $dx(B) = 0$. 

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5. Show that \( \phi : \mathbb{R}^n \to \mathbb{R} \) defined by \( \phi(x) = \langle u, x \rangle \), is measurable.

6. Explain why \( \phi(N_n(m, \Sigma)) \) is a well-defined probability on \( \mathbb{R} \).

7. Show that for all \( \alpha \in \mathbb{R} \), we have:
   \[
   \mathcal{F}\phi(N_n(m, \Sigma))(\alpha) = \int_{\mathbb{R}^n} e^{i\alpha \langle u, x \rangle} dN_n(m, \Sigma)(x)
   \]

8. Show that \( \phi(N_n(m, \Sigma)) \) is the dirac distribution on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \)
centered on \( \langle u, m \rangle \), i.e. \( \phi(N_n(m, \Sigma)) = \delta_{\langle u, m \rangle} \).

9. Show that \( N_n(m, \Sigma)(B) = 1 \).

10. Conclude that \( N_n(m, \Sigma) \) cannot be absolutely continuous with
    respect to the Lebesgue measure on \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \).

11. Show the following:
Theorem 140  Let \( n \geq 1 \) and \( m \in \mathbb{R}^n \). Let \( \Sigma \in \mathcal{M}_n(\mathbb{R}) \) be a symmetric and non-negative real matrix. Then, the gaussian measure \( N_n(m, \Sigma) \) is absolutely continuous with respect to the Lebesgue measure on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\), if and only if \( \Sigma \) is non-singular, in which case for all \( B \in \mathcal{B}(\mathbb{R}^n) \), we have:

\[
N_n(m, \Sigma)(B) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det(\Sigma)}} \int_B e^{-\frac{1}{2}(x-m, \Sigma^{-1}(x-m))} \, dx
\]