20. Gaussian Measures

$\mathcal{M}_n(\mathbf{R})$ is the set of all $n \times n$-matrices with real entries, $n \geq 1$.

Definition 141 *A matrix $M \in \mathcal{M}_n(\mathbf{R})$ is said to be* **symmetric**, *if and only if $M = M^t$. M is* **orthogonal**, *if and only if M is non-singular and $M^{-1} = M^t$. If M is symmetric, we say that M is* **non-negative**, *if and only if:*

$$\forall u \in \mathbf{R}^n \ , \ \langle u, Mu \rangle \geq 0$$

Theorem 131 *Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$, $n \geq 1$, be a symmetric and non-negative real matrix. There exist $\lambda_1, \ldots, \lambda_n \in \mathbf{R}^+$ and $P \in \mathcal{M}_n(\mathbf{R})$ orthogonal matrix, such that:*

$$\Sigma = P. \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} . P^t$$

In particular, there exists $A \in \mathcal{M}_n(\mathbf{R})$ such that $\Sigma = A.A^t$.

As a rare exception, theorem (131) is given without proof.

Exercise 1. Given $n \geq 1$ and $M \in \mathcal{M}_n(\mathbf{R})$, show that we have:

$$\forall u, v \in \mathbf{R}^n \ , \ \langle u, Mv \rangle = \langle M^t u, v \rangle$$

Exercise 2. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative matrix. Let μ_1 be the probability measure on $\mathbf{R}$:

$$\forall B \in \mathcal{B}(\mathbf{R}) \ , \ \mu_1(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

Let $\mu = \mu_1 \otimes \ldots \otimes \mu_1$ be the product measure on $\mathbf{R}^n$. Let $A \in \mathcal{M}_n(\mathbf{R})$ be such that $\Sigma = A.A^t$. We define the map $\phi : \mathbf{R}^n \to \mathbf{R}^n$ by:

$$\forall x \in \mathbf{R}^n \ , \ \phi(x) \stackrel{\triangle}{=} Ax + m$$

1. Show that μ is a probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

2. Explain why the image measure $P = \phi(\mu)$ is well-defined.

3. Show that P is a probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

4. Show that for all $u \in \mathbf{R}^n$:

$$\mathcal{F}P(u) = \int_{\mathbf{R}^n} e^{i\langle u,\phi(x)\rangle} d\mu(x)$$

5. Let $v = A^t u$. Show that for all $u \in \mathbf{R}^n$:

$$\mathcal{F}P(u) = e^{i\langle u,m\rangle - \|v\|^2/2}$$

6. Show the following:

Theorem 132 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. There exists a unique complex measure on $\mathbf{R}^n$, denoted $N_n(m, \Sigma)$, with fourier transform:*

$$\mathcal{F}N_n(m,\Sigma)(u) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} e^{i\langle u,x\rangle} dN_n(m,\Sigma)(x) = e^{i\langle u,m\rangle - \frac{1}{2}\langle u,\Sigma u\rangle}$$

for all $u \in \mathbf{R}^n$. Furthermore, $N_n(m, \Sigma)$ is a probability measure.

Definition 142 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. The probability measure $N_n(m, \Sigma)$ on $\mathbf{R}^n$ defined in theorem (132) is called the n-dimensional* **gaussian measure** *or* **normal distribution***, with mean $m \in \mathbf{R}^n$ and covariance matrix Σ.*

EXERCISE 3. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Show that $N_n(m, 0) = \delta_m$.

EXERCISE 4. Let $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $A \in \mathcal{M}_n(\mathbf{R})$ be such that $\Sigma = A.A^t$. A map $p : \mathbf{R}^n \to \mathbf{C}$ is said to be a *polynomial*, if and only if, it is a finite linear complex combination of maps $x \to x^\alpha$,[1] for $\alpha \in \mathbf{N}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R})$, we have:

$$N_1(0,1)(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

[1] See definition (140).

2. Show that:

$$\int_{-\infty}^{+\infty} |x| dN_1(0,1)(x) < +\infty$$

3. Show that for all integer $k \geq 1$:

$$\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x^{k+1} e^{-x^2/2} dx = \frac{k}{\sqrt{2\pi}} \int_0^{+\infty} x^{k-1} e^{-x^2/2} dx$$

4. Show that for all integer $k \geq 0$:

$$\int_{-\infty}^{+\infty} |x|^k dN_1(0,1)(x) < +\infty$$

5. Show that for all $\alpha \in \mathbf{N}^n$:

$$\int_{\mathbf{R}^n} |x^\alpha| dN_1(0,1) \otimes \ldots \otimes N_1(0,1)(x) < +\infty$$

6. Let $p : \mathbf{R}^n \to \mathbf{C}$ be a polynomial. Show that:

$$\int_{\mathbf{R}^n} |p(x)| dN_1(0,1) \otimes \ldots \otimes N_1(0,1)(x) < +\infty$$

7. Let $\phi : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $\phi(x) = Ax + m$. Explain why the image measure $\phi(N_1(0,1) \otimes \ldots \otimes N_1(0,1))$ is well-defined.

8. Show that $\phi(N_1(0,1) \otimes \ldots \otimes N_1(0,1)) = N_n(m, \Sigma)$.

9. Show if $\beta \in \mathbf{N}^n$ and $|\beta| = 1$, then $x \to \phi(x)^\beta$ is a polynomial.

10. Show that if $\alpha' \in \mathbf{N}^n$ and $|\alpha'| = k+1$, then $\phi(x)^{\alpha'} = \phi(x)^\alpha \phi(x)^\beta$ for some $\alpha, \beta \in \mathbf{N}^n$ such that $|\alpha| = k$ and $|\beta| = 1$.

11. Show that the product of two polynomials is a polynomial.

12. Show that for all $\alpha \in \mathbf{N}^n$, $x \to \phi(x)^\alpha$ is a polynomial.

13. Show that for all $\alpha \in \mathbf{N}^n$:

$$\int_{\mathbf{R}^n} |\phi(x)^\alpha| dN_1(0,1) \otimes \ldots \otimes N_1(0,1)(x) < +\infty$$

14. Show the following:

Theorem 133 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Then, for all $\alpha \in \mathbf{N}^n$, the map $x \to x^\alpha$ is integrable with respect to the gaussian measure $N_n(m, \Sigma)$:*

$$\int_{\mathbf{R}^n} |x^\alpha| dN_n(m, \Sigma)(x) < +\infty$$

EXERCISE 5. Let $m \in \mathbf{R}^n$. Let $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $j, k \in \mathbf{N}_n$. Let ϕ be the fourier transform of the gaussian measure $N_n(m, \Sigma)$, i.e.:

$$\forall u \in \mathbf{R}^n \ , \ \phi(u) \stackrel{\triangle}{=} e^{i\langle u,m\rangle - \frac{1}{2}\langle u,\Sigma u\rangle}$$

1. Show that:

$$\int_{\mathbf{R}^n} x_j dN_n(m, \Sigma)(x) = i^{-1} \frac{\partial \phi}{\partial u_j}(0)$$

2. Show that:

$$\int_{\mathbf{R}^n} x_j dN_n(m,\Sigma)(x) = m_j$$

3. Show that:

$$\int_{\mathbf{R}^n} x_j x_k dN_n(m,\Sigma)(x) = i^{-2}\frac{\partial^2\phi}{\partial u_j \partial u_k}(0)$$

4. Show that:

$$\int_{\mathbf{R}^n} x_j x_k dN_n(m,\Sigma)(x) = \sigma_{jk} + m_j m_k$$

5. Show that:

$$\int_{\mathbf{R}^n} (x_j - m_j)(x_k - m_k) dN_n(m,\Sigma)(x) = \sigma_{jk}$$

Theorem 134 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $N_n(m, \Sigma)$ be the gaussian measure with mean m and covariance matrix Σ. Then, for all $j, k \in \mathbf{N}_n$, we have:*

$$\int_{\mathbf{R}^n} x_j dN_n(m, \Sigma)(x) = m_j$$

and:

$$\int_{\mathbf{R}^n} (x_j - m_j)(x_k - m_k) dN_n(m, \Sigma)(x) = \sigma_{jk}$$

Definition 143 *Let $n \geq 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ be a measurable map. We say that X is an n-dimensional* **gaussian** *or* **normal vector***, if and only if its distribution is a gaussian measure, i.e. $X(P) = N_n(m, \Sigma)$ for some $m \in \mathbf{R}^n$ and $\Sigma \in \mathcal{M}_n(\mathbf{R})$ symmetric and non-negative real matrix.*

Exercise 6. Show the following:

Theorem 135 *Let $n \geq 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : (\Omega, \mathcal{F}) \to \mathbf{R}^n$ be a measurable map. Then X is a gaussian vector, if and only if there exist $m \in \mathbf{R}^n$ and $\Sigma \in \mathcal{M}_n(\mathbf{R})$ symmetric and non-negative real matrix, such that:*

$$\forall u \in \mathbf{R}^n \ , \ E[e^{i\langle u, X\rangle}] = e^{i\langle u, m\rangle - \frac{1}{2}\langle u, \Sigma u\rangle}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner-product on $\mathbf{R}^n$.

Definition 144 *Let $X : (\Omega, \mathcal{F}) \to \bar{\mathbf{R}}$ (or $\mathbf{C}$) be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. We say that X is* **integrable***, if and only if we have $E[|X|] < +\infty$. We say that X is* **square-integrable***, if and only if we have $E[|X|^2] < +\infty$.*

Exercise 7. Further to definition (144), suppose X is $\mathbf{C}$-valued.

1. Show X is integrable if and only if $X \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, P)$.

2. Show X is square-integrable, if and only if $X \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, P)$.

Exercise 8. Further to definition (144), suppose X is $\bar{\mathbf{R}}$-valued.

1. Show that X is integrable, if and only if X is P-almost surely equal to an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$.

2. Show that X is square-integrable, if and only if X is P-almost surely equal to an element of $L^2_{\mathbf{R}}(\Omega, \mathcal{F}, P)$.

Exercise 9. Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be two square-integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$.

1. Show that both X and Y are integrable.

2. Show that XY is integrable

3. Show that $(X - E[X])(Y - E[Y])$ is a well-defined and integrable.

Definition 145 *Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be two square-integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$. We define the* **covariance** *between X and Y, denoted $cov(X, Y)$, as:*

$$cov(X,Y) \stackrel{\triangle}{=} E[(X - E[X])(Y - E[Y])]$$

We say that X and Y are **uncorrelated** *if and only if $cov(X, Y) = 0$. If $X = Y$, $cov(X, Y)$ is called the* **variance** *of X, denoted $var(X)$.*

Exercise 10. Let X, Y be two square integrable, real random variable on a probability space $(\Omega, \mathcal{F}, P)$.

1. Show that $cov(X, Y) = E[XY] - E[X]E[Y]$.
2. Show that $var(X) = E[X^2] - E[X]^2$.
3. Show that $var(X + Y) = var(X) + 2cov(X, Y) + var(Y)$
4. Show that X and Y are uncorrelated, if and only if:

$$var(X + Y) = var(X) + var(Y)$$

Exercise 11. Let X be an n-dimensional normal vector on some probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$, where $m \in \mathbf{R}^n$ and $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbf{R})$ is a symmetric and non-negative real matrix.

1. Show that each coordinate $X_j : (\Omega, \mathcal{F}) \to \mathbf{R}$ is measurable.
2. Show that $E[|X^\alpha|] < +\infty$ for all $\alpha \in \mathbf{N}^n$.
3. Show that for all $j = 1, \ldots, n$, we have $E[X_j] = m_j$.
4. Show that for all $j, k = 1, \ldots, n$, we have $cov(X_j, X_k) = \sigma_{jk}$.

Theorem 136 *Let X be an n-dimensional normal vector on a probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$. Then, for all $\alpha \in \mathbf{N}^n$, X^α is integrable. Moreover, for all $j, k \in \mathbf{N}_n$, we have:*

$$E[X_j] = m_j$$

and:

$$cov(X_j, X_k) = \sigma_{jk}$$

where $(\sigma_{ij}) = \Sigma$.

Exercise 12. Show the following:

Theorem 137 *Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be a real random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then, X is a normal random variable, if and only if it is square integrable, and:*

$$\forall u \in \mathbf{R} \ , \ E[e^{iuX}] = e^{iuE[X] - \frac{1}{2}u^2 var(X)}$$

Exercise 13. Let X be an n-dimensional normal vector on a probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$. Let $A \in \mathcal{M}_{d,n}(\mathbf{R})$ be an $d \times n$ real matrix, $(n, d \geq 1)$. Let $b \in \mathbf{R}^d$ and $Y = AX + b$.

1. Show that $Y : (\Omega, \mathcal{F}) \to (\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ is measurable.
2. Show that the law of Y is $N_d(Am + b, A.\Sigma.A^t)$
3. Conclude that Y is an $\mathbf{R}^d$-valued normal random vector.

Theorem 138 *Let X be an n-dimensional normal vector with law $N_n(m,\Sigma)$ on a probability space $(\Omega,\mathcal{F},P)$, $(n \geq 1)$. Let $d \geq 1$ and $A \in \mathcal{M}_{d,n}(\mathbf{R})$ be an $d \times n$ real matrix. Let $b \in \mathbf{R}^d$. Then, $Y = AX + b$ is an d-dimensional normal vector, with law:*

$$Y(P) = N_d(Am + b, A.\Sigma.A^t)$$

EXERCISE 14. Let $X : (\Omega,\mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ be a measurable map, where $(\Omega,\mathcal{F},P)$ is a probability space. Show that if X is a gaussian vector, then for all $u \in \mathbf{R}^n$, $\langle u, X\rangle$ is a normal random variable.

EXERCISE 15. Let $X : (\Omega,\mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ be a measurable map, where $(\Omega,\mathcal{F},P)$ is a probability space. We assume that for all $u \in \mathbf{R}^n$, $\langle u, X\rangle$ is a normal random variable.

1. Show that for all $j = 1,\ldots,n$, X_j is integrable.
2. Show that for all $j = 1,\ldots,n$, X_j is square integrable.
3. Explain why given $j,k = 1,\ldots,n$, $cov(X_j, X_k)$ is well-defined.

4. Let $m \in \mathbf{R}^n$ be defined by $m_j = E[X_j]$, and $u \in \mathbf{R}^n$. Show:
$$E[\langle u, X\rangle] = \langle u, m\rangle$$

5. Let $\Sigma = (cov(X_i, X_j))$. Show that for all $u \in \mathbf{R}^n$, we have:
$$var(\langle u, X\rangle) = \langle u, \Sigma u\rangle$$

6. Show that Σ is a symmetric and non-negative $n \times n$ real matrix.

7. Show that for all $u \in \mathbf{R}^n$:
$$E[e^{i\langle u,X\rangle}] = e^{iE[\langle u,X\rangle]-\frac{1}{2}var(\langle u,X\rangle)}$$

8. Show that for all $u \in \mathbf{R}^n$:
$$E[e^{i\langle u,X\rangle}] = e^{i\langle u,m\rangle-\frac{1}{2}\langle u,\Sigma u\rangle}$$

9. Show that X is a normal vector.

10. Show the following:

Theorem 139 *Let* $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ *be a measurable map on a probability space* $(\Omega, \mathcal{F}, P)$*. Then,* X *is an* n*-dimensional normal vector, if and only if, any linear combination of its coordinates is itself normal, or in other words* $\langle u, X\rangle$ *is normal for all* $u \in \mathbf{R}^n$*.*

Exercise 16. Let $(\Omega, \mathcal{F}) = (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2))$ and μ be the probability on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ defined by $\mu = \frac{1}{2}(\delta_0 + \delta_1)$. Let $P = N_1(0, 1) \otimes \mu$, and $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be the canonical projections defined by $X(x, y) = x$ and $Y(x, y) = y$.

1. Show that P is a probability measure on $(\Omega, \mathcal{F})$.
2. Explain why X and Y are measurable.
3. Show that X has the distribution $N_1(0, 1)$.
4. Show that $P(\{Y = 0\}) = P(\{Y = 1\}) = \frac{1}{2}$.
5. Show that $P^{(X,Y)} = P$.

6. Show for all $\phi : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \to \mathbf{C}$ measurable and bounded:

$$E[\phi(X,Y)] = \frac{1}{2}(E[\phi(X,0)] + E[\phi(X,1)])$$

7. Let $X_1 = X$ and X_2 be defined as:

$$X_2 \stackrel{\triangle}{=} X1_{\{Y=0\}} - X1_{\{Y=1\}}$$

Show that $E[e^{iuX_2}] = e^{-u^2/2}$ for all $u \in \mathbf{R}$.

8. Show that $X_1(P) = X_2(P) = N_1(0,1)$.

9. Explain why $cov(X_1, X_2)$ is well-defined.

10. Show that X_1 and X_2 are uncorrelated.

11. Let $Z = \frac{1}{2}(X_1 + X_2)$. Show that:

$$\forall u \in \mathbf{R} \; , \; E[e^{iuZ}] = \frac{1}{2}(1 + e^{-u^2/2})$$

12. Show that Z cannot be gaussian.

13. Conclude that although X_1, X_2 are normally distributed, (and even uncorrelated), (X_1, X_2) is not a gaussian vector.

Exercise 17. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $A \in \mathcal{M}_n(\mathbf{R})$ be such that $\Sigma = A.A^t$. We assume that Σ is non-singular. We define $p_{m,\Sigma} : \mathbf{R}^n \to \mathbf{R}^+$ by:

$$\forall x \in \mathbf{R}^n \ , \ p_{m,\Sigma}(x) \stackrel{\triangle}{=} \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}\langle x-m, \Sigma^{-1}(x-m)\rangle}$$

1. Explain why $\det(\Sigma) > 0$.

2. Explain why $\sqrt{\det(\Sigma)} = |\det(A)|$.

3. Explain why A is non-singular.

4. Let $\phi : \mathbf{R}^n \to \mathbf{R}^n$ be defined by:

$$\forall x \in \mathbf{R}^n \ , \ \phi(x) \stackrel{\triangle}{=} A^{-1}(x-m)$$

Show that for all $x \in \mathbf{R}^n$, $\langle x-m, \Sigma^{-1}(x-m)\rangle = \|\phi(x)\|^2$.

5. Show that ϕ is a C^1-diffeomorphism.

6. Show that $\phi(dx) = |\det(A)|dx$.

7. Show that:

$$\int_{\mathbf{R}^n} p_{m,\Sigma}(x)dx = 1$$

8. Let $\mu = \int p_{m,\Sigma}dx$. Show that:

$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}\mu(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{i\langle u, Ax+m\rangle - \|x\|^2/2} dx$$

9. Show that the fourier transform of μ is therefore given by:

$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}\mu(u) = e^{i\langle u,m\rangle - \frac{1}{2}\langle u, \Sigma u\rangle}$$

10. Show that $\mu = N_n(m, \Sigma)$.

11. Show that $N_n(m, \Sigma) << dx$, i.e. that $N_n(m, \Sigma)$ is absolutely continuous w.r. to the Lebesgue measure on $\mathbf{R}^n$.

Exercise 18. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. We assume that Σ is singular. Let $u \in \mathbf{R}^n$ be such that $\Sigma u = 0$ and $u \neq 0$. We define:

$$B \stackrel{\triangle}{=} \{x \in \mathbf{R}^n \ , \ \langle u, x \rangle = \langle u, m \rangle\}$$

Given $a \in \mathbf{R}^n$, let $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ be the translation of vector a.

1. Show $B = \tau_{-m}^{-1}(u^\perp)$, where $u^\perp$ is the orthogonal of u in $\mathbf{R}^n$.

2. Show that $B \in \mathcal{B}(\mathbf{R}^n)$.

3. Explain why $dx(u^\perp) = 0$. Is it important to have $u \neq 0$?

4. Show that $dx(B) = 0$.

5. Show that $\phi : \mathbf{R}^n \to \mathbf{R}$ defined by $\phi(x) = \langle u, x \rangle$, is measurable.

6. Explain why $\phi(N_n(m, \Sigma))$ is a well-defined probability on $\mathbf{R}$.

7. Show that for all $\alpha \in \mathbf{R}$, we have:

$$\mathcal{F}\phi(N_n(m, \Sigma))(\alpha) = \int_{\mathbf{R}^n} e^{i\alpha\langle u,x\rangle} dN_n(m, \Sigma)(x)$$

8. Show that $\phi(N_n(m, \Sigma))$ is the dirac distribution on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ centered on $\langle u, m \rangle$, i.e. $\phi(N_n(m, \Sigma)) = \delta_{\langle u,m \rangle}$.

9. Show that $N_n(m, \Sigma)(B) = 1$.

10. Conclude that $N_n(m, \Sigma)$ cannot be absolutely continuous with respect to the Lebesgue measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

11. Show the following:

Theorem 140 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Then, the gaussian measure $N_n(m, \Sigma)$ is absolutely continuous with respect to the Lebesgue measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, if and only if Σ is non-singular, in which case for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:*

$$N_n(m, \Sigma)(B) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\Sigma)}} \int_B e^{-\frac{1}{2}\langle x-m, \Sigma^{-1}(x-m)\rangle} dx$$