19. Fourier Transform

Exercise 1. Let \( a, b \in \mathbb{R}, a < b \). Let \( f : [a, b] \to \mathbb{C} \) be a map such that \( f'(t) \) exists for all \( t \in [a, b] \). We assume that:

\[
\int_a^b |f'(t)| \, dt < +\infty
\]

1. Show that \( f' : ([a, b], \mathcal{B}([a, b])) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) is measurable.

2. Show that:

\[
f(b) - f(a) = \int_a^b f'(t) \, dt
\]

Exercise 2. We define the maps \( \psi : \mathbb{R}^2 \to \mathbb{C} \) and \( \phi : \mathbb{R} \to \mathbb{C} \):

\[
\forall (u, x) \in \mathbb{R}^2, \; \psi(u, x) \triangleq e^{iu x - x^2 / 2}
\]

\[
\forall u \in \mathbb{R}, \; \phi(u) \triangleq \int_{-\infty}^{+\infty} \psi(u, x) \, dx
\]
1. Show that for all $u \in \mathbb{R}$, the map $x \to \psi(u,x)$ is measurable.

2. Show that for all $u \in \mathbb{R}$, we have:

$$\int_{-\infty}^{+\infty} |\psi(u,x)|dx = \sqrt{2\pi} < +\infty$$

and conclude that $\phi$ is well defined.

3. Let $u \in \mathbb{R}$ and $(u_n)_{n \geq 1}$ be a sequence in $\mathbb{R}$ converging to $u$.

Show that $\phi(u_n) \to \phi(u)$ and conclude that $\phi$ is continuous.

4. Show that:

$$\int_{0}^{+\infty} xe^{-x^2/2}dx = 1$$

5. Show that for all $u \in \mathbb{R}$, we have:

$$\int_{-\infty}^{+\infty} \left| \frac{\partial \psi}{\partial u}(u,x) \right| dx = 2 < +\infty$$
6. Let $a, b \in \mathbb{R}$, $a < b$. Show that:
\[ e^{ib} - e^{ia} = \int_{a}^{b} ie^{ix} dx \]

7. Let $a, b \in \mathbb{R}$, $a < b$. Show that:
\[ |e^{ib} - e^{ia}| \leq |b - a| \]

8. Let $a, b \in \mathbb{R}$, $a \neq b$. Show that for all $x \in \mathbb{R}$:
\[ \left| \frac{\psi(b, x) - \psi(a, x)}{b - a} \right| \leq |x|e^{-x^2/2} \]

9. Let $u \in \mathbb{R}$ and $(u_n)_{n \geq 1}$ be a sequence in $\mathbb{R}$ converging to $u$, with $u_n \neq u$ for all $n$. Show that:
\[ \lim_{n \to +\infty} \frac{\phi(u_n) - \phi(u)}{u_n - u} = \int_{-\infty}^{+\infty} \frac{\partial \psi}{\partial u}(u, x) dx \]
10. Show that \( \phi \) is differentiable with:
\[
\forall u \in \mathbb{R} \ , \ \phi'(u) = \int_{-\infty}^{+\infty} \frac{\partial \phi}{\partial u}(u, x) dx
\]

11. Show that \( \phi \) is of class \( C^1 \).

12. Show that for all \((u, x) \in \mathbb{R}^2\), we have:
\[
\frac{\partial \psi}{\partial u}(u, x) = -u\psi(u, x) - i\frac{\partial \psi}{\partial x}(u, x)
\]

13. Show that for all \( u \in \mathbb{R} \):
\[
\int_{-\infty}^{+\infty} \left| \frac{\partial \psi}{\partial x}(u, x) \right| dx < +\infty
\]

14. Let \( a, b \in \mathbb{R}, \ a < b \). Show that for all \( u \in \mathbb{R} \):
\[
\psi(u, b) - \psi(u, a) = \int_a^b \frac{\partial \psi}{\partial x}(u, x) dx
\]
15. Show that for all $u \in \mathbb{R}$:
\[
\int_{-\infty}^{+\infty} \frac{\partial \psi}{\partial x}(u, x) dx = 0
\]

16. Show that for all $u \in \mathbb{R}$:
\[
\phi'(u) = -u\phi(u)
\]

**Exercise 3.** Let $\mathcal{S}$ be the set of functions defined by:
\[
\mathcal{S} \triangleq \{ h : h \in C^1(\mathbb{R}, \mathbb{R}) \ , \ \forall u \in \mathbb{R} \ , \ h'(u) = -uh(u) \}
\]

1. Let $\phi$ be as in ex. (2). Show that $Re(\phi)$ and $Im(\phi)$ lie in $\mathcal{S}$.

2. Given $h \in \mathcal{S}$, we define $g : \mathbb{R} \rightarrow \mathbb{R}$, by:
\[
\forall u \in \mathbb{R} \ , \ g(u) \triangleq h(u)e^{u^2/2}
\]
Show that $g$ is of class $C^1$ with $g' = 0$. 

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3. Let $a, b \in \mathbb{R}$, $a < b$. Show the existence of $c \in [a, b]$, such that:
\[ g(b) - g(a) = g'(c)(b - a) \]

4. Conclude that for all $h \in \mathcal{S}$, we have:
\[ \forall u \in \mathbb{R}, \ h(u) = h(0)e^{-u^2/2} \]

5. Prove the following:

**Theorem 124**  For all $u \in \mathbb{R}$, we have:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iux-x^2/2}dx = e^{-u^2/2}
\]
Definition 135 Let $\mu_1, \ldots, \mu_p$ be complex measures on $\mathbb{R}^n$, where $n, p \geq 1$. We call convolution of $\mu_1, \ldots, \mu_p$, denoted $\mu_1 \ast \cdots \ast \mu_p$, the image measure of the product measure $\mu_1 \otimes \cdots \otimes \mu_p$ by the measurable map $S : (\mathbb{R}^n)^p \rightarrow \mathbb{R}^n$ defined by:

$S(x_1, \ldots, x_p) \overset{\Delta}{=} x_1 + \cdots + x_p$

In other words, $\mu_1 \ast \cdots \ast \mu_p$ is the complex measure on $\mathbb{R}^n$, defined by:

$\mu_1 \ast \cdots \ast \mu_p \overset{\Delta}{=} S(\mu_1 \otimes \cdots \otimes \mu_p)$

Recall that the product $\mu_1 \otimes \cdots \otimes \mu_p$ is defined in theorem (66).

Exercise 4. Let $\mu, \nu$ be complex measures on $\mathbb{R}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$:

$\mu \ast \nu(B) = \int_{\mathbb{R}^n \times \mathbb{R}^n} 1_B(x + y) d\mu \otimes \nu(x, y)$
2. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$:

$$\mu \ast \nu(B) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} 1_B(x + y)d\mu(x) \right) d\nu(y)$$

3. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$:

$$\mu \ast \nu(B) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} 1_B(x + y)d\nu(x) \right) d\mu(y)$$

4. Show that $\mu \ast \nu = \nu \ast \mu$.

5. Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be bounded and measurable. Show that:

$$\int_{\mathbb{R}^n} f d\mu \ast \nu = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x + y)d\mu \otimes \nu(x, y)$$
**Exercise 5.** Let $\mu, \nu$ be complex measures on $\mathbb{R}^n$. Given $B \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define $B - x = \{ y \in \mathbb{R}^n : y + x \in B \}$.

1. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $B - x \in \mathcal{B}(\mathbb{R}^n)$.

2. Show $x \rightarrow \mu(B - x)$ is measurable and bounded, for $B \in \mathcal{B}(\mathbb{R}^n)$.

3. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$:
   \[ \mu * \nu(B) = \int_{\mathbb{R}^n} \mu(B - x) d\nu(x) \]

4. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$:
   \[ \mu * \nu(B) = \int_{\mathbb{R}^n} \nu(B - x) d\mu(x) \]
Exercise 6. Let $\mu_1, \mu_2, \mu_3$ be complex measures on $\mathbb{R}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$:

$$
\mu_1 * (\mu_2 * \mu_3)(B) = \int_{\mathbb{R}^n \times \mathbb{R}^n} 1_B(x + y) d\mu_1 \otimes (\mu_2 * \mu_3)(x, y)
$$

2. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$
\int_{\mathbb{R}^n} 1_B(x + y) d\mu_2 * \mu_3(y) = \int_{\mathbb{R}^n \times \mathbb{R}^n} 1_B(x + y + z) d\mu_2 \otimes \mu_3(y, z)
$$

3. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$:

$$
\mu_1 * (\mu_2 * \mu_3)(B) = \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} 1_B(x + y + z) d\mu_1 \otimes \mu_2 \otimes \mu_3(x, y, z)
$$

4. Show that $\mu_1 * (\mu_2 * \mu_3) = \mu_1 * \mu_2 * \mu_3 = (\mu_1 * \mu_2) * \mu_3$
Definition 136 \( \text{Let } n \geq 1 \text{ and } a \in \mathbb{R}^n. \text{ We define } \delta_a : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}^+ : \)
\[ \forall B \in \mathcal{B}(\mathbb{R}^n), \, \delta_a(B) \triangleq 1_B(a) \]
\( \delta_a \) \( \text{is called the Dirac probability measure on } \mathbb{R}^n, \text{ centered in } a. \)

Exercise 7. \( \text{Let } n \geq 1 \text{ and } a \in \mathbb{R}^n. \)

1. Show that \( \delta_a \) is indeed a probability measure on \( \mathbb{R}^n. \)

2. Show for all \( f : \mathbb{R}^n \to [0, +\infty] \) non-negative and measurable:
\[ \int_{\mathbb{R}^n} f \delta_a = f(a) \]

3. Show if \( f : \mathbb{R}^n \to \mathbb{C} \) is measurable, \( f \in L^1_{\mathbb{C}}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \delta_a) \) and:
\[ \int_{\mathbb{R}^n} f \delta_a = f(a) \]
4. Show that for any complex measure $\mu$ on $\mathbb{R}^n$:

$$\mu * \delta_0 = \delta_0 * \mu = \mu$$

5. Let $\tau_a(x) = a + x$ define the translation of vector $a$ in $\mathbb{R}^n$. Show that for any complex measure $\mu$ on $\mathbb{R}^n$:

$$\mu * \delta_a = \delta_a * \mu = \tau_a(\mu)$$

**Exercise 8.** Let $f, g : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space. Let $u = \text{Re}(f)$, $v = \text{Im}(f)$, $u' = \text{Re}(g)$ and $v' = \text{Im}(g)$.

1. Show that $u, v, u', v' : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are all measurable.

2. Show that $u + u'$, $v + v'$, $uu' - vv'$ and $uv' + u'v$ are measurable.

3. Show that $f + g, fg : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ are measurable.

4. Show that $\mathbb{C} = \mathbb{R}^2$ has a countable base.
5. Show that \( \mathcal{B}(\mathbb{C} \times \mathbb{C}) = \mathcal{B}(\mathbb{C}) \otimes \mathcal{B}(\mathbb{C}) \).

6. Show that \((z, z') \mapsto z + z'\) and \((z, z') \mapsto zz'\) are continuous.

7. Show that \( \omega \mapsto (f(\omega), g(\omega)) \) is measurable w.r. to \( \mathcal{B}(\mathbb{C}) \otimes \mathcal{B}(\mathbb{C}) \).

8. Conclude once more that \( f + g \) and \( fg \) are measurable.

**Exercise 9.** Let \( n \geq 1 \) and \( \mu, \nu \) be complex measures on \( \mathbb{R}^n \). We assume that \( \nu << dx \), i.e. that \( \nu \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^n \).

1. Show there is \( f \in L^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx) \), such that \( \nu = \int f dx \).

2. Show that for all \( B \in \mathcal{B}(\mathbb{R}^n) \), we have:
   \[
   \mu \ast \nu(B) = \int_{\mathbb{R}^n} \nu(B - x) d\mu(x)
   \]
3. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$\nu(B - x) = \int_{\mathbb{R}^n} 1_B(y) f(y - x) dy$$

4. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$ the map:

$$(x, y) \mapsto 1_B(y) f(y - x)$$

lies in $L^1_C(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n), |\mu| \otimes dy)$.

5. Let $h \in L^1_C(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), |\mu|)$ with $|h| = 1$, $\mu = \int h d|\mu|$. Show:

$$(x, y) \mapsto 1_B(y) f(y - x) h(x)$$

also lies in $L^1_C(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n), |\mu| \otimes dy)$.

6. Show that for all $B \in \mathcal{B}(\mathbb{R}^n)$, we have:

$$\mu \ast \nu(B) = \int_B \left( \int_{\mathbb{R}^n} f(y - x) d\mu(x) \right) dy$$
7. Let $g$ be the map defined by $g(y) = \int_{\mathbb{R}^n} f(y - x) d\mu(x)$. Recall why $g$ is $dy$-almost surely well-defined, and $dy$-almost surely equal to an element of $L^1_c(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dy)$.

8. Show that $\mu \ast \nu = \int g \, dx$ and $\mu \ast \nu \ll dx$.

**Theorem 125** Let $\mu, \nu$ be two complex measures on $\mathbb{R}^n$, $n \geq 1$. If $\nu \ll dx$, i.e. $\nu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, with density $f \in L^1_c(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, then the convolution $\mu \ast \nu = \nu \ast \mu$ is itself absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, with density:

$$g(y) = \int_{\mathbb{R}^n} f(y - x) d\mu(x), \quad dy - a.s.$$ 

In other words, $\mu \ast \nu = \nu \ast \mu = \int g \, dx$.
Exercise 10. Let $f \in L^1_c(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let $\nu$ be the complex measure on $(\Omega, \mathcal{F})$ defined by $\nu = \int f d\mu$. Let $g : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ be a measurable map.

1. Show that $g \in L^1_c(\Omega, \mathcal{F}, \nu) \iff gf \in L^1_c(\Omega, \mathcal{F}, \mu)$.

2. Show that for all $g \in L^1_c(\Omega, \mathcal{F}, \nu)$:

$$\int g d\nu = \int gf d\mu$$

Exercise 11. Further to theorem (125), show that if $\mu = \int h dx$ for some $h \in L^1_c(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, then:

$$g(y) = \int_{\mathbb{R}^n} f(y - x)h(x)dx \ , \ dy - a.s.$$
Definition 137 Let $\mu$ be a complex measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \geq 1$. We call Fourier transform of $\mu$, the map $\mathcal{F}_\mu : \mathbb{R}^n \rightarrow \mathbb{C}$ defined by:

$$\forall u \in \mathbb{R}^n, \quad \mathcal{F}_\mu(u) \equiv \int_{\mathbb{R}^n} e^{i(u,x)} d\mu(x)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner-product in $\mathbb{R}^n$.

Exercise 12. Further to definition (137):

1. Show that $\mathcal{F}_\mu$ is well-defined.
2. Show that $\mathcal{F}_\mu \in C_b^1(\mathbb{R}^n)$, i.e $\mathcal{F}_\mu$ is continuous and bounded.
3. Show that for all $a, u \in \mathbb{R}^n$, we have $\mathcal{F}_\delta_a(u) = e^{i(u,a)}$.
4. Let $\mu$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by:

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad \mu(B) \equiv \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

Show that $\mathcal{F}_\mu(u) = e^{-u^2/2}$, for all $u \in \mathbb{R}$. 

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Exercise 13. Let $\mu_1, \ldots, \mu_p$ be complex measures on $\mathbb{R}^n$, $p \geq 2$.

1. Show that for all $u \in \mathbb{R}^n$, we have:
   $$\mathcal{F}(\mu_1 \ast \ldots \ast \mu_p)(u) = \int_{(\mathbb{R}^n)^p} e^{i(u,x_1 + \ldots + x_p)} \, d\mu_1 \otimes \ldots \otimes d\mu_p(x).$$

2. Show that if $p \geq 3$ then $\mu_1 \ast \ldots \ast \mu_p = (\mu_1 \ast \ldots \ast \mu_{p-1}) \ast \mu_p$.

3. Show that $\mathcal{F}(\mu_1 \ast \ldots \ast \mu_p) = \Pi_{j=1}^p \mathcal{F}\mu_j$.

Exercise 14. Let $n \geq 1$, $\sigma > 0$ and $g_\sigma : \mathbb{R}^n \to \mathbb{R}^+$ defined by:

$$\forall x \in \mathbb{R}^n, \quad g_\sigma(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\|x\|^2/2\sigma^2}.$$

1. Show that $\int_{\mathbb{R}^n} g_\sigma(x) \, dx = 1$.

2. Show that for all $u \in \mathbb{R}^n$, we have:
   $$\int_{\mathbb{R}^n} g_\sigma(x) e^{i(u,x)} \, dx = e^{-\sigma^2\|u\|^2/2}$$
3. Show that $P_\sigma = \int g_\sigma dx$ is a probability on $\mathbb{R}^n$, and:

$$\forall u \in \mathbb{R}^n, \quad \mathcal{F}P_\sigma(u) = e^{-\sigma^2\|u\|^2/2}$$

4. Show that for all $x \in \mathbb{R}^n$, we have:

$$g_\sigma(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x,u) - \sigma^2\|u\|^2/2} du$$

**EXERCISE 15.** Further to ex. (14), let $\mu$ be a complex measure on $\mathbb{R}^n$.

1. Show that $\mu \ast P_\sigma = \int \phi_\sigma dx$ where:

$$\phi_\sigma(x) = \int_{\mathbb{R}^n} g_\sigma(x - y) d\mu(y), \quad dx - a.s.$$  

2. Show that we also have:

$$\phi_\sigma(x) = \int_{\mathbb{R}^n} g_\sigma(y - x) d\mu(y), \quad dx - a.s.$$
3. Show that:
\[
\phi_\sigma(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{i(y-x,u) - \sigma^2 \|u\|^2/2} \, du \right) \, d\mu(y) \ , \ \text{d}x - \text{a.s.}
\]

4. Show that:
\[
\phi_\sigma(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(x,u) - \sigma^2 \|u\|^2/2} (\mathcal{F}\mu)(u) \, du
\]

5. Show that if \( \mu, \nu \) are two complex measures on \( \mathbb{R}^n \) such that \( \mathcal{F}\mu = \mathcal{F}\nu \), then for all \( \sigma > 0 \), we have \( \mu \ast P_\sigma = \nu \ast P_\sigma \).

**Definition 138** Let \((\Omega, T)\) be a topological space. Let \((\mu_k)_{k \geq 1}\) be a sequence of complex measures on \((\Omega, \mathcal{B}(\Omega))\). We say that the sequence \((\mu_k)_{k \geq 1}\) **narrowly converges**, or **weakly converges** to a complex measure \(\mu\) on \((\Omega, \mathcal{B}(\Omega))\), and we write \(\mu_k \rightharpoonup \mu\), if and only if:
\[
\forall f \in C^b_R(\Omega) \ , \ \lim_{k \rightarrow +\infty} \int f \, d\mu_k = \int f \, d\mu
\]
Exercise 16. Further to definition (138):

1. Show that \( \mu_k \rightarrow \mu \) narrowly, is equivalent to:
\[
\forall f \in C_c^b(\Omega), \quad \lim_{k \rightarrow +\infty} \int f d\mu_k = \int f d\mu.
\]

2. Show that if \((\Omega, T)\) is metrizable and \(\nu\) is a complex measure on \((\Omega, \mathcal{B}(\Omega))\) such that \(\mu_k \rightarrow \mu\) and \(\mu_k \rightarrow \nu\) narrowly, then \(\mu = \nu\).

Theorem 126  On a metrizable topological space, the narrow or weak limit when it exists, of any sequence of complex measures, is unique.

Exercise 17.

1. Show that on \((\mathbb{R}, \mathcal{B}(\mathbb{R})), \) we have \(\delta_{1/n} \rightarrow \delta_0\) narrowly.

2. Show there is \(B \in \mathcal{B}(\mathbb{R}), \) such that \(\delta_{1/n}(B) \nrightarrow \delta_0(B)\).

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Exercise 18. Let \( n \geq 1 \). Given \( \sigma > 0 \), let \( P_\sigma \) be the probability measure on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\) defined as in ex. (14). Let \((\sigma_k)_{k \geq 1}\) be a sequence in \( \mathbb{R}^+ \) such that \( \sigma_k > 0 \) and \( \sigma_k \to 0 \).

1. Show that for all \( f \in C^b_{\mathbb{R}}(\mathbb{R}^n) \), we have:
   \[
   \int_{\mathbb{R}^n} f(x) g_{\sigma_k}(x) \, dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\sigma_k x) e^{-\|x\|^2/2} \, dx
   \]

2. Show that for all \( f \in C^b_{\mathbb{R}}(\mathbb{R}^n) \), we have:
   \[
   \lim_{k \to +\infty} \int_{\mathbb{R}^n} f(x) g_{\sigma_k}(x) \, dx = f(0)
   \]

3. Show that \( P_{\sigma_k} \to \delta_0 \) narrowly.
Exercise 19. Let $\mu, \nu$ be two complex measures on $\mathbb{R}^n$. Let $(\nu_k)_{k \geq 1}$ be a sequence of complex measures on $\mathbb{R}^n$, which narrowly converges to $\nu$. Let $f \in C_b^0(\mathbb{R}^n)$, and $\phi : \mathbb{R}^n \to \mathbb{R}$ be defined by:

$$\forall y \in \mathbb{R}^n, \phi(y) \triangleq \int_{\mathbb{R}^n} f(x+y) d\mu(x)$$

1. Show that:

$$\int_{\mathbb{R}^n} f d\mu * \nu_k = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x+y) d\mu \otimes \nu_k(x,y)$$

2. Show that:

$$\int_{\mathbb{R}^n} f d\mu * \nu_k = \int_{\mathbb{R}^n} \phi d\nu_k$$

3. Show that $\phi \in C_b^0(\mathbb{R}^n)$.

4. Show that:

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} \phi d\nu_k = \int_{\mathbb{R}^n} \phi d\nu$$

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5. Show that:
\[
\lim_{k \to +\infty} \int_{\mathbb{R}^n} f \, d\mu_k \ast \nu_k = \int_{\mathbb{R}^n} f \, d\mu \ast \nu
\]

6. Show that \( \mu \ast \nu_k \to \mu \ast \nu \) narrowly.

**Theorem 127** Let \( \mu, \nu \) be two complex measures on \( \mathbb{R}^n \), \( n \geq 1 \). Let \( (\nu_k)_{k \geq 1} \) be a sequence of complex measures on \( \mathbb{R}^n \). Then:

\[\nu_k \to \nu \text{ narrowly} \Rightarrow \mu \ast \nu_k \to \mu \ast \nu \text{ narrowly}\]

**Exercise 20.** Let \( \mu, \nu \) be two complex measures on \( \mathbb{R}^n \), such that \( \mathcal{F}\mu = \mathcal{F}\nu \). For all \( \sigma > 0 \), let \( P_\sigma \) be the probability measure on \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \) as defined in ex. (14). Let \( (\sigma_k)_{k \geq 1} \) be a sequence in \( \mathbb{R}^+ \) such that \( \sigma_k > 0 \) and \( \sigma_k \to 0 \).

1. Show that \( \mu \ast P_{\sigma_k} = \nu \ast P_{\sigma_k} \), for all \( k \geq 1 \).

2. Show that \( \mu \ast P_{\sigma_k} \to \mu \ast \delta_0 \) narrowly.
3. Show that \((\mu * P_{\sigma_k})_{k \geq 1}\) narrowly converges to both \(\mu\) and \(\nu\).

4. Prove the following:

**Theorem 128** Let \(\mu, \nu\) be two complex measures on \(\mathbb{R}^n\). Then:
\[
\mathcal{F} \mu = \mathcal{F} \nu \implies \mu = \nu
\]
i.e. the Fourier transform is an injective mapping on \(M^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\).

**Definition 139** Let \((\Omega, \mathcal{F}, P)\) be a probability space. Given \(n \geq 1\), and a measurable map \(X : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\), the mapping \(\phi_X\) defined as:
\[
\forall u \in \mathbb{R}^n, \quad \phi_X(u) \triangleq E[e^{i(u,X)}]
\]
is called the **characteristic function** \(^1\) of the random variable \(X\).

\(^1\)Do not confuse with the characteristic function \(1_A\) of a set \(A\), definition (39).
Exercise 21. Further to definition (139):

1. Show that $\phi_X$ is well-defined, bounded and continuous.

2. Show that we have:
   \[
   \forall u \in \mathbb{R}^n, \quad \phi_X(u) = \int_{\mathbb{R}^n} e^{i(u,x)} dX(P)(x)
   \]

3. Show $\phi_X$ is the Fourier transform of the image measure $X(P)$.

4. Show the following:

**Theorem 129** Let $X, Y : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \geq 1$, be two random variables on a probability space $(\Omega, \mathcal{F}, P)$. If $X$ and $Y$ have the same characteristic functions, i.e.
   \[
   \forall u \in \mathbb{R}^n, \quad E[e^{i(u,X)}] = E[e^{i(u,Y)}]
   \]
then $X$ and $Y$ have the same distributions, i.e.
   \[
   \forall B \in \mathcal{B}(\mathbb{R}^n), \quad P(\{X \in B\}) = P(\{Y \in B\})
   \]
**Definition 140**  Let \( n \geq 1 \). Given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we define the modulus of \( \alpha \), denoted \( |\alpha| \), by \( |\alpha| = \alpha_1 + \ldots + \alpha_n \). Given \( x \in \mathbb{R}^n \) and \( \alpha \in \mathbb{N}^n \), we put:

\[
x^\alpha \triangleq x_1^{\alpha_1} \ldots x_n^{\alpha_n}
\]

where it is understood that \( x_j^{\alpha_j} = 1 \) whenever \( \alpha_j = 0 \). Given a map \( f : U \to \mathbb{C} \), where \( U \) is an open subset of \( \mathbb{R}^n \), we denote \( \partial^\alpha f \) the \( |\alpha| \)-th partial derivative, when it exists:

\[
\partial^\alpha f \triangleq \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}
\]

Note that \( \partial^\alpha f = f \), whenever \( |\alpha| = 0 \). Given \( k \geq 0 \), we say that \( f \) is of **class** \( C^k \), if and only if for all \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq k \), \( \partial^\alpha f \) exists and is continuous on \( U \).

**Exercise 22.** Explain why def. (140) is consistent with def. (130).
Exercise 23. Let $\mu$ be a complex measure on $\mathbb{R}^n$, and $\alpha \in \mathbb{N}^n$, with:

$$\int_{\mathbb{R}^n} |x^\alpha| |d\mu|(x) < +\infty$$  \hspace{1cm} (1)

Let $x^\alpha \mu$ the complex measure on $\mathbb{R}^n$ defined by $x^\alpha \mu = \int x^\alpha d\mu$.

1. Explain why the above integral (1) is well-defined.
2. Show that $x^\alpha \mu$ is a well-defined complex measure on $\mathbb{R}^n$.
3. Show that the total variation of $x^\alpha \mu$ is given by:

$$\forall B \in \mathcal{B}(\mathbb{R}^n), \quad |x^\alpha \mu|(B) = \int_B |x^\alpha| |d\mu|(x)$$

4. Show that the Fourier transform of $x^\alpha \mu$ is given by:

$$\forall u \in \mathbb{R}^n, \quad \mathcal{F}(x^\alpha \mu)(u) = \int_{\mathbb{R}^n} x^\alpha e^{iu \cdot x} d\mu(x)$$
Exercise 24. Let \( \mu \) be a complex measure on \( \mathbb{R}^n \). Let \( \beta \in \mathbb{N}^n \) with \( |\beta| = 1 \), and:

\[
\int_{\mathbb{R}^n} |x^\beta| d|\mu|(x) < +\infty
\]

Let \( x^\beta \mu \) be the complex measure on \( \mathbb{R}^n \) defined as in ex. (23).

1. Show that there is \( j \in \mathbb{N}_n \) with \( x^\beta = x_j \) for all \( x \in \mathbb{R}^n \).

2. Show that for all \( u \in \mathbb{R}^n \), \( \frac{\partial F\mu}{\partial u_j}(u) \) exists and that we have:

\[
\frac{\partial F\mu}{\partial u_j}(u) = i \int_{\mathbb{R}^n} x_j e^{i(u,x)} d\mu(x)
\]

3. Conclude that \( \partial F\mu \) exists and that we have:

\[
\partial F\mu = i F(x^\beta \mu)
\]

4. Explain why \( \partial F\mu \) is continuous.

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**Exercise 25.** Let $\mu$ be a complex measure on $\mathbb{R}^n$. Let $k \geq 0$ be an integer. We assume that for all $\alpha \in \mathbb{N}^n$, we have:

$$|\alpha| \leq k \Rightarrow \int_{\mathbb{R}^n} |x^{\alpha}| d|\mu|(x) < +\infty$$

(2)

In particular, if $|\alpha| \leq k$, the measure $x^{\alpha}\mu$ of ex. (23) is well-defined. We claim that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, $\partial^\alpha \mathcal{F}\mu$ exists, and:

$$\partial^\alpha \mathcal{F}\mu = i^{|\alpha|} \mathcal{F}(x^{\alpha}\mu)$$

1. Show that if $k = 0$, then the property is obviously true. We assume the property is true for some $k \geq 0$, and that the above integrability condition (2) holds for $k + 1$.

2. Let $\alpha' \in \mathbb{N}^n$ be such that $|\alpha'| \leq k + 1$. Explain why if $|\alpha'| \leq k$, then $\partial^{\alpha'} \mathcal{F}\mu$ exists, with:

$$\partial^{\alpha'} \mathcal{F}\mu = i^{|\alpha'|} \mathcal{F}(x^{\alpha'}\mu)$$

3. We assume that $|\alpha'| = k + 1$. Show the existence of $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha + \beta = \alpha'$, $|\alpha| = k$ and $|\beta| = 1$.  

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4. Explain why $\partial^\alpha F\mu$ exists, and:
$$\partial^\alpha F\mu = i^{\|\alpha\|}F(x^\alpha \mu)$$

5. Show that:
$$\int_{\mathbb{R}^n} |x^\beta|d|x^\alpha \mu|< +\infty$$

6. Show that $\partial^{\beta} F(x^\alpha \mu)$ exists, with:
$$\partial^{\beta} F(x^\alpha \mu) = iF(x^{\beta}(x^\alpha \mu))$$

7. Show that $\partial^{\beta}(\partial^\alpha F\mu)$ exists, with:
$$\partial^{\beta}(\partial^\alpha F\mu) = i^{\|\alpha\|+1}F(x^{\beta}(x^\alpha \mu))$$

8. Show that $x^{\beta}(x^\alpha \mu) = x^\alpha \mu$.

9. Conclude that the property is true for $k + 1$.

10. Show the following:

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Theorem 130  Let $\mu$ be a complex measure on $\mathbb{R}^n$, $n \geq 1$. Let $k \geq 0$ be an integer such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, we have:

$$\int_{\mathbb{R}^n} |x^\alpha| d|\mu|(x) < +\infty$$

Then, the Fourier transform $\mathcal{F}_\mu$ is of class $C^k$ on $\mathbb{R}^n$, and for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, we have:

$$\forall u \in \mathbb{R}^n, \ \partial^\alpha \mathcal{F}_\mu(u) = i^{|\alpha|} \int_{\mathbb{R}^n} x^\alpha e^{i(u,x)} d\mu(x)$$

In particular:

$$\int_{\mathbb{R}^n} x^\alpha d\mu(x) = i^{-|\alpha|} \partial^\alpha \mathcal{F}_\mu(0)$$