1. Dynkin systems

**Definition 1** A **Dynkin system** on a set $\Omega$ is a subset $\mathcal{D}$ of the power set $\mathcal{P}(\Omega)$, with the following properties:

(i) $\Omega \in \mathcal{D}$

(ii) $A, B \in \mathcal{D}, A \subseteq B \Rightarrow B \setminus A \in \mathcal{D}$

(iii) $A_n \in \mathcal{D}, A_n \subseteq A_{n+1}, n \geq 1 \Rightarrow \bigcup_{n=1}^{+\infty} A_n \in \mathcal{D}$

**Definition 2** A **$\sigma$-algebra** on a set $\Omega$ is a subset $\mathcal{F}$ of the power set $\mathcal{P}(\Omega)$ with the following properties:

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \triangleq \Omega \setminus A \in \mathcal{F}$

(iii) $A_n \in \mathcal{F}, n \geq 1 \Rightarrow \bigcup_{n=1}^{+\infty} A_n \in \mathcal{F}$
**Exercise 1.** Let \( \mathcal{F} \) be a \( \sigma \)-algebra on \( \Omega \). Show that \( \emptyset \in \mathcal{F} \), that if \( A, B \in \mathcal{F} \) then \( A \cup B \in \mathcal{F} \) and also \( A \cap B \in \mathcal{F} \). Recall that \( B \setminus A = B \cap A^c \) and conclude that \( \mathcal{F} \) is also a Dynkin system on \( \Omega \).

**Exercise 2.** Let \((\mathcal{D}_i)_{i \in I}\) be an arbitrary family of Dynkin systems on \( \Omega \), with \( I \neq \emptyset \). Show that \( \mathcal{D} \triangleq \bigcap_{i \in I} \mathcal{D}_i \) is also a Dynkin system on \( \Omega \).

**Exercise 3.** Let \((\mathcal{F}_i)_{i \in I}\) be an arbitrary family of \( \sigma \)-algebras on \( \Omega \), with \( I \neq \emptyset \). Show that \( \mathcal{F} \triangleq \bigcap_{i \in I} \mathcal{F}_i \) is also a \( \sigma \)-algebra on \( \Omega \).

**Exercise 4.** Let \( A \) be a subset of the power set \( \mathcal{P}(\Omega) \). Define:

\[
D(A) \triangleq \{ \mathcal{D} \text{ Dynkin system on } \Omega : A \subseteq \mathcal{D} \}
\]

Show that \( \mathcal{P}(\Omega) \) is a Dynkin system on \( \Omega \), and that \( D(A) \) is not empty. Define:

\[
\mathcal{D}(A) \triangleq \bigcap_{\mathcal{D} \in D(A)} \mathcal{D}
\]
Show that $\mathcal{D}(\mathcal{A})$ is a Dynkin system on $\Omega$ such that $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$, and that it is the smallest Dynkin system on $\Omega$ with such property, (i.e. if $\mathcal{D}$ is a Dynkin system on $\Omega$ with $\mathcal{A} \subseteq \mathcal{D}$, then $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}$).

**Definition 3** Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call Dynkin system generated by $\mathcal{A}$, the Dynkin system on $\Omega$, denoted $\mathcal{D}(\mathcal{A})$, equal to the intersection of all Dynkin systems on $\Omega$, which contain $\mathcal{A}$.

**Exercise 5.** Do exactly as before, but replacing Dynkin systems by $\sigma$-algebras.

**Definition 4** Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call $\sigma$-algebra generated by $\mathcal{A}$, the $\sigma$-algebra on $\Omega$, denoted $\mathcal{\sigma}(\mathcal{A})$, equal to the intersection of all $\sigma$-algebras on $\Omega$, which contain $\mathcal{A}$.

**Definition 5** A subset $\mathcal{A}$ of the power set $\mathcal{P}(\Omega)$ is called a $\pi$-system on $\Omega$, if and only if it is closed under finite intersection, i.e. if it has the property:

$$A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$$
Exercise 6. Let $\mathcal{A}$ be a $\pi$-system on $\Omega$. For all $A \in \mathcal{D}(\mathcal{A})$, we define:

$$\Gamma(A) \triangleq \{ B \in \mathcal{D}(\mathcal{A}) : A \cap B \in \mathcal{D}(\mathcal{A}) \}$$

1. If $A \in \mathcal{A}$, show that $A \subseteq \Gamma(A)$
2. Show that for all $A \in \mathcal{D}(\mathcal{A})$, $\Gamma(A)$ is a Dynkin system on $\Omega$.
3. Show that if $A \in \mathcal{A}$, then $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(A)$.
4. Show that if $B \in \mathcal{D}(\mathcal{A})$, then $\mathcal{A} \subseteq \Gamma(B)$.
5. Show that for all $B \in \mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(B)$.
6. Conclude that $\mathcal{D}(\mathcal{A})$ is also a $\pi$-system on $\Omega$.

Exercise 7. Let $\mathcal{D}$ be a Dynkin system on $\Omega$ which is also a $\pi$-system.

1. Show that if $A, B \in \mathcal{D}$ then $A \cup B \in \mathcal{D}$. 

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2. Let $A_n \in \mathcal{D}, n \geq 1$. Consider $B_n \triangleq \bigcup_{i=1}^{n} A_i$. Show that $\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n$.

3. Show that $\mathcal{D}$ is a $\sigma$-algebra on $\Omega$.

**Exercise 8.** Let $\mathcal{A}$ be a $\pi$-system on $\Omega$. Explain why $\mathcal{D}(\mathcal{A})$ is a $\sigma$-algebra on $\Omega$, and $\sigma(\mathcal{A})$ is a Dynkin system on $\Omega$. Conclude that $\mathcal{D}(\mathcal{A}) = \sigma(\mathcal{A})$. Prove the theorem:

**Theorem 1 (Dynkin system)** Let $\mathcal{C}$ be a collection of subsets of $\Omega$ which is closed under pairwise intersection. If $\mathcal{D}$ is a Dynkin system containing $\mathcal{C}$ then $\mathcal{D}$ also contains the $\sigma$-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$.