11. Complex Measures

In the following, \((\Omega, \mathcal{F})\) denotes an arbitrary measurable space.

**Definition 90** Let \((a_n)_{n \geq 1}\) be a sequence of complex numbers. We say that \((a_n)_{n \geq 1}\) has the **permutation property** if and only if, for all bijections \(\sigma : \mathbb{N}^* \rightarrow \mathbb{N}^*\), the series \(\sum_{k=1}^{+\infty} a_{\sigma(k)}\) converges in \(\mathbb{C}\).

**Exercise 1.** Let \((a_n)_{n \geq 1}\) be a sequence of complex numbers.

1. Show that if \((a_n)_{n \geq 1}\) has the permutation property, then the same is true of \((\text{Re}(a_n))_{n \geq 1}\) and \((\text{Im}(a_n))_{n \geq 1}\).

2. Suppose \(a_n \in \mathbb{R}\) for all \(n \geq 1\). Show that if \(\sum_{k=1}^{+\infty} a_k\) converges:

\[
\sum_{k=1}^{+\infty} |a_k| = +\infty \Rightarrow \sum_{k=1}^{+\infty} a_k^+ = \sum_{k=1}^{+\infty} a_k^- = +\infty
\]

\(^{1}\) which excludes \(\pm \infty\) as limit.
EXERCISE 2. Let \((a_n)_{n \geq 1}\) be a sequence in \(\mathbb{R}\), such that the series 
\[
\sum_{k=1}^{+\infty} a_k \text{ converges, and } \sum_{k=1}^{+\infty} |a_k| = +\infty.
\]
Let \(A > 0\). We define:
\[
N^+ \triangleq \{k \geq 1 : a_k \geq 0\} , \quad N^- \triangleq \{k \geq 1 : a_k < 0\}
\]

1. Show that \(N^+\) and \(N^-\) are infinite.

2. Let \(\phi^+ : \mathbb{N}^* \to N^+\) and \(\phi^- : \mathbb{N}^* \to N^-\) be two bijections. Show the existence of \(k_1 \geq 1\) such that:
\[
\sum_{k=1}^{k_1} a_{\phi^+(k)} \geq A
\]

3. Show the existence of an increasing sequence \((k_p)_{p \geq 1}\) such that:
\[
\sum_{k=k_{p-1}+1}^{k_p} a_{\phi^+(k)} \geq A
\]
for all \( p \geq 1 \), where \( k_0 = 0 \).

4. Consider the permutation \( \sigma : \mathbb{N}^* \to \mathbb{N}^* \) defined informally by:

\[
(\phi^-(1), \phi^+(1), \ldots, \phi^+(k_1), \phi^-(2), \phi^+(k_1 + 1), \ldots, \phi^+(k_2), \ldots)
\]

representing \((\sigma(1), \sigma(2), \ldots)\). More specifically, define \( k_0^* = 0 \) and \( k_p^* = k_p + p \) for all \( p \geq 1 \). For all \( n \in \mathbb{N}^* \) and \( p \geq 1 \) with:

\[
k_{p-1}^* < n \leq k_p^*
\]

we define:

\[
\sigma(n) = \begin{cases} 
\phi^-(p) & \text{if } n = k_{p-1}^* + 1 \\
\phi^+(n - p) & \text{if } n > k_{p-1}^* + 1
\end{cases}
\]

Show that \( \sigma : \mathbb{N}^* \to \mathbb{N}^* \) is indeed a bijection.

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\(^2\)Given an integer \( n \geq 1 \), there exists a unique \( p \geq 1 \) such that (1) holds.
5. Show that if $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges, there is $N \geq 1$, such that:

$$n \geq N, \ p \geq 1 \ \Rightarrow \ \left| \sum_{k=n+1}^{n+p} a_{\sigma(k)} \right| < A$$

6. Explain why $(a_n)_{n \geq 1}$ cannot have the permutation property.

7. Prove the following theorem:

**Theorem 56**  Let $(a_n)_{n \geq 1}$ be a sequence of complex numbers such that for all bijections $\sigma : N^* \rightarrow N^*$, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges. Then, the series $\sum_{k=1}^{+\infty} a_k$ converges absolutely, i.e.

$$\sum_{k=1}^{+\infty} |a_k| < +\infty$$
**Definition 91** Let $(\Omega, \mathcal{F})$ be a measurable space and $E \in \mathcal{F}$. We call **measurable partition** of $E$, any sequence $(E_n)_{n \geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $E = \biguplus_{n \geq 1} E_n$.

**Definition 92** We call **complex measure** on a measurable space $(\Omega, \mathcal{F})$ any map $\mu : \mathcal{F} \to \mathbb{C}$, such that for all $E \in \mathcal{F}$ and $(E_n)_{n \geq 1}$ measurable partition of $E$, the series $\sum_{n=1}^{+\infty} \mu(E_n)$ converges to $\mu(E)$. The set of all complex measures on $(\Omega, \mathcal{F})$ is denoted $M^1(\Omega, \mathcal{F})$.

**Definition 93** We call **signed measure** on a measurable space $(\Omega, \mathcal{F})$, any complex measure on $(\Omega, \mathcal{F})$ with values in $\mathbb{R}$.\(^3\)

**Exercise 3.**

1. Show that a measure on $(\Omega, \mathcal{F})$ may not be a complex measure.

2. Show that for all $\mu \in M^1(\Omega, \mathcal{F})$, $\mu(\emptyset) = 0$.

\(^3\)In these tutorials, signed measure may not have values in $\{-\infty, +\infty\}$.
3. Show that a finite measure on $(\Omega, \mathcal{F})$ is a complex measure with values in $\mathbb{R}^+$, and conversely.

4. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n \geq 1}$ be a measurable partition of $E$. Show that:
\[
\sum_{n=1}^{+\infty} |\mu(E_n)| < +\infty
\]

5. Let $\mu$ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_C(\Omega, \mathcal{F}, \mu)$. Define:
\[
\forall E \in \mathcal{F}, \quad \nu(E) \triangleq \int_E f d\mu
\]
Show that $\nu$ is a complex measure on $(\Omega, \mathcal{F})$. 

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Definition 94. Let \( \mu \) be a complex measure on a measurable space \((\Omega, \mathcal{F})\). We call total variation of \( \mu \), the map \(|\mu| : \mathcal{F} \rightarrow [0, +\infty]\), defined by:

\[
\forall E \in \mathcal{F}, \quad |\mu|(E) \triangleq \sup \sum_{n=1}^{\infty} |\mu(E_n)|
\]

where the 'sup' is taken over all measurable partitions \((E_n)_{n \geq 1}\) of \(E\).

Exercise 4. Let \( \mu \) be a complex measure on \((\Omega, \mathcal{F})\).

1. Show that for all \( E \in \mathcal{F} \), \(|\mu(E)| \leq |\mu|(E)\).
2. Show that \(|\mu|(\emptyset) = 0\).

Exercise 5. Let \( \mu \) be a complex measure on \((\Omega, \mathcal{F})\). Let \(E \in \mathcal{F}\) and \((E_n)_{n \geq 1}\) be a measurable partition of \(E\).

1. Show that there exists \((t_n)_{n \geq 1}\) in \(\mathbb{R}\), with \(t_n < |\mu|(E_n)\) for all \(n\).
2. Show that for all $n \geq 1$, there exists a measurable partition $(E_n^p)_{p \geq 1}$ of $E_n$ such that:

$$t_n < \sum_{p=1}^{+\infty} |\mu(E_n^p)|$$

3. Show that $(E_n^p)_{n,p \geq 1}$ is a measurable partition of $E$.

4. Show that for all $N \geq 1$, we have $\sum_{n=1}^{N} t_n \leq |\mu|(E)$.

5. Show that for all $N \geq 1$, we have:

$$\sum_{n=1}^{N} |\mu|(E_n) \leq |\mu|(E)$$

6. Suppose that $(A_p)_{p \geq 1}$ is another arbitrary measurable partition
of $E$. Show that for all $p \geq 1$:

$$|\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu(A_p \cap E_n)|$$

7. Show that for all $n \geq 1$:

$$\sum_{p=1}^{+\infty} |\mu(A_p \cap E_n)| \leq |\mu|(E_n)$$

8. Show that:

$$\sum_{p=1}^{+\infty} |\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu|(E_n)$$

9. Show that $|\mu| : \mathcal{F} \rightarrow [0, +\infty]$ is a measure on $(\Omega, \mathcal{F})$.
Exercise 6. Let $a, b \in \mathbb{R}, a < b$. Let $F \in C^1([a, b]; \mathbb{R})$, and define:

$$
\forall x \in [a, b], \quad H(x) \triangleq \int_a^x F'(t) dt
$$

1. Show that $H \in C^1([a, b]; \mathbb{R})$ and $H' = F'$.

2. Show that:

$$
F(b) - F(a) = \int_a^b F'(t) dt
$$

3. Show that:

$$
\frac{1}{2\pi} \int_{-\pi/2}^{+\pi/2} \cos \theta d\theta = \frac{1}{\pi}
$$

4. Let $u \in \mathbb{R}^n$ and $\tau_u : \mathbb{R}^n \to \mathbb{R}^n$ be the translation $\tau_u(x) = x + u$. Show that the Lebesgue measure $dx$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is invariant by translation $\tau_u$, i.e. $dx(\{ \tau_u \in B \}) = dx(B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$. 

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5. Show that for all $f \in L^1_C(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, and $u \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} f(x + u)dx = \int_{\mathbb{R}^n} f(x)dx$$

6. Show that for all $\alpha \in \mathbb{R}$, we have:

$$\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta)d\theta = \int_{-\pi - \alpha}^{+\pi - \alpha} \cos^+ \theta d\theta$$

7. Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $k \leq \alpha/2\pi < k + 1$. Show:

$$-\pi - \alpha \leq -2k\pi - \pi < \pi - \alpha \leq -2k\pi + \pi$$

8. Show that:

$$\int_{-\pi - \alpha}^{-2k\pi - \pi} \cos^+ \theta d\theta = \int_{\pi - \alpha}^{-2k\pi + \pi} \cos^+ \theta d\theta$$
9. Show that:
\[
\int_{-\pi - \alpha}^{\pi - \alpha} \cos^+ \theta d\theta = \int_{-2k\pi - \pi}^{-2k\pi + \pi} \cos^+ \theta d\theta = \int_{-\pi}^{\pi} \cos^+ \theta d\theta
\]

10. Show that for all \( \alpha \in \mathbb{R} \):
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^+ (\alpha - \theta) d\theta = \frac{1}{\pi}
\]

**Exercise 7.** Let \( z_1, \ldots, z_N \) be \( N \) complex numbers. Let \( \alpha_k \in \mathbb{R} \) be such that \( z_k = |z_k| e^{i\alpha_k} \), for all \( k = 1, \ldots, N \). For all \( \theta \in [-\pi, +\pi] \), we define \( S(\theta) = \{k = 1, \ldots, N : \cos(\alpha_k - \theta) > 0\} \).

1. Show that for all \( \theta \in [-\pi, +\pi] \), we have:
\[
\left| \sum_{k \in S(\theta)} z_k \right| = \left| \sum_{k \in S(\theta)} z_k e^{-i\theta} \right| \geq \sum_{k \in S(\theta)} |z_k| \cos(\alpha_k - \theta)
\]

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2. Define \( \phi : [-\pi, +\pi] \to \mathbb{R} \) by \( \phi(\theta) = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta) \).

Show the existence of \( \theta_0 \in [-\pi, +\pi] \) such that:

\[
\phi(\theta_0) = \sup_{\theta \in [-\pi, +\pi]} \phi(\theta)
\]

3. Show that:

\[
\frac{1}{2\pi} \int_{-\pi}^{+\pi} \phi(\theta) d\theta = \frac{1}{\pi} \sum_{k=1}^{N} |z_k|
\]

4. Conclude that:

\[
\frac{1}{\pi} \sum_{k=1}^{N} |z_k| \leq \left| \sum_{k \in S(\theta_0)} z_k \right|
\]

**Exercise 8.** Let \( \mu \in M^1(\Omega, \mathcal{F}) \). Suppose that \( |\mu|(E) = +\infty \) for some \( E \in \mathcal{F} \). Define \( t = \pi(1 + |\mu(E)|) \in \mathbb{R}^+ \).
1. Show that there is a measurable partition \((E_n)_{n \geq 1}\) of \(E\), with:
   \[
   t < \sum_{n=1}^{+\infty} |\mu(E_n)|
   \]

2. Show the existence of \(N \geq 1\) such that:
   \[
   t < \sum_{n=1}^{N} |\mu(E_n)|
   \]

3. Show the existence of \(S \subseteq \{1, \ldots, N\}\) such that:
   \[
   \sum_{n=1}^{N} |\mu(E_n)| \leq \pi \left| \sum_{n \in S} \mu(E_n) \right|
   \]

4. Show that \(|\mu(A)| > t/\pi\), where \(A = \cup_{n \in S} E_n\).

5. Let \(B = E \setminus A\). Show that \(|\mu(B)| \geq |\mu(A)| - |\mu(E)|\).
6. Show that \( E = A \cup B \) with \( |\mu(A)| > 1 \) and \( |\mu(B)| > 1 \).

7. Show that \( |\mu|(A) = +\infty \) or \( |\mu|(B) = +\infty \).

**Exercise 9.** Let \( \mu \in M^1(\Omega, \mathcal{F}) \). Suppose that \( |\mu|(\Omega) = +\infty \).

1. Show the existence of \( A_1, B_1 \in \mathcal{F} \), such that \( \Omega = A_1 \cup B_1 \), \( |\mu(A_1)| > 1 \) and \( |\mu|(B_1) = +\infty \).

2. Show the existence of a sequence \( (A_n)_{n \geq 1} \) of pairwise disjoint elements of \( \mathcal{F} \), such that \( |\mu(A_n)| > 1 \) for all \( n \geq 1 \).

3. Show that the series \( \sum_{n=1}^{\infty} \mu(A_n) \) does not converge to \( \mu(A) \) where \( A = \bigcup_{n=1}^{+\infty} A_n \).

4. Conclude that \( |\mu|(\Omega) < +\infty \).
**Theorem 57** Let $\mu$ be a complex measure on a measurable space $(\Omega, \mathcal{F})$. Then, its total variation $|\mu|$ is a finite measure on $(\Omega, \mathcal{F})$.

**Exercise 10.** Show that $\mathcal{M}_1(\Omega, \mathcal{F})$ is a $\mathbb{C}$-vector space, with:

\[
(\lambda + \mu)(E) \triangleq \lambda(E) + \mu(E) \\
(\alpha \lambda)(E) \triangleq \alpha \cdot \lambda(E)
\]

where $\lambda, \mu \in \mathcal{M}_1(\Omega, \mathcal{F})$, $\alpha \in \mathbb{C}$, and $E \in \mathcal{F}$.

**Definition 95** Let $\mathcal{H}$ be a $\mathbb{K}$-vector space, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We call norm on $\mathcal{H}$, any map $N : \mathcal{H} \rightarrow \mathbb{R}^+$, with the following properties:

(i) $\forall x \in \mathcal{H}, (N(x) = 0 \iff x = 0)$

(ii) $\forall x \in \mathcal{H}, \forall \alpha \in \mathbb{K}, N(\alpha x) = |\alpha|N(x)$

(iii) $\forall x, y \in \mathcal{H}, N(x + y) \leq N(x) + N(y)$
Exercise 11.

1. Explain why \( \| . \|_p \) may not be a norm on \( L^p_k(\Omega, \mathcal{F}, \mu) \).

2. Show that \( \| . \| = \sqrt{\langle ., . \rangle} \) is a norm, when \( \langle ., . \rangle \) is an inner-product.

3. Show that \( \| \mu \| \overset{\triangle}{=} |\mu|(\Omega) \) defines a norm on \( M^1(\Omega, \mathcal{F}) \).

Exercise 12. Let \( \mu \in M^1(\Omega, \mathcal{F}) \) be a signed measure. Show that:

\[
\mu^+ \overset{\triangle}{=} \frac{1}{2}(|\mu| + \mu) \\
\mu^- \overset{\triangle}{=} \frac{1}{2}(|\mu| - \mu)
\]

are finite measures such that:

\[
\mu = \mu^+ - \mu^- , \quad |\mu| = \mu^+ + \mu^-
\]

Exercise 13. Let \( \mu \in M^1(\Omega, \mathcal{F}) \) and \( l : \mathbb{R}^2 \to \mathbb{R} \) be a linear map.
1. Show that $l$ is continuous.

2. Show that $l \circ \mu$ is a signed measure on $(\Omega, \mathcal{F})$.  

3. Show that all $\mu \in M^1(\Omega, \mathcal{F})$ can be decomposed as:

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are finite measures.

\[ \text{Note: } l \circ \mu \text{ refers strictly speaking to } l(\text{Re}(\mu), \text{Im}(\mu)).]