10. Bounded Linear Functionals in $L^2$

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

**Definition 78** We call subsequence of a sequence $(x_n)_{n \geq 1}$, any sequence of the form $(x_{\phi(n)})_{n \geq 1}$ where $\phi : \mathbb{N}^* \to \mathbb{N}^*$ is a strictly increasing map.

**Exercise 1.** Let $(E, d)$ be a metric space, with metric topology $T$. Let $(x_n)_{n \geq 1}$ be a sequence in $E$. For all $n \geq 1$, let $F_n$ be the closure of the set $\{x_k : k \geq n\}$.

1. Show that for all $x \in E$, $x_n \xrightarrow{T} x$ is equivalent to:
   $$\forall \epsilon > 0, \exists n_0 \geq 1, \ n \geq n_0 \Rightarrow d(x_n, x) \leq \epsilon$$

2. Show that $(F_n)_{n \geq 1}$ is a decreasing sequence of closed sets in $E$.

3. Show that if $F_n \downarrow \emptyset$, then $(F_n^c)_{n \geq 1}$ is an open covering of $E$. 

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4. Show that if \((E, T)\) is compact then \(\cap_{n=1}^{+\infty} F_n \neq \emptyset\).

5. Show that if \((E, T)\) is compact, there exists \(x \in E\) such that for all \(n \geq 1\) and \(\epsilon > 0\), we have \(B(x, \epsilon) \cap \{x_k, k \geq n\} \neq \emptyset\).

6. By induction, construct a subsequence \((x_{n_p})_{p \geq 1}\) of \((x_n)_{n \geq 1}\) such that \(x_{n_p} \in B(x, 1/p)\) for all \(p \geq 1\).

7. Conclude that if \((E, T)\) is compact, any sequence \((x_n)_{n \geq 1}\) in \(E\) has a convergent subsequence.

**Exercise 2.** Let \((E, d)\) be a metric space, with metric topology \(T\). We assume that any sequence \((x_n)_{n \geq 1}\) in \(E\) has a convergent subsequence. Let \((V_i)_{i \in I}\) be an open covering of \(E\). For \(x \in E\), let:

\[
\hat{r}(x) \triangleq \sup\{r > 0 : B(x, r) \subseteq V_i, \text{ for some } i \in I\}
\]

1. Show that \(\forall x \in E, \exists i \in I, \exists r > 0, \text{ such that } B(x, r) \subseteq V_i\).
2. Show that $\forall x \in E, r(x) > 0$.

**Exercise 3.** Further to ex. (2), suppose $\inf_{x \in E} r(x) = 0$.

1. Show that for all $n \geq 1$, there is $x_n \in E$ such that $r(x_n) < 1/n$.

2. Extract a subsequence $(x_{n_k})_{k \geq 1}$ of $(x_n)_{n \geq 1}$ converging to some $x^* \in E$. Let $r^* > 0$ and $i \in I$ be such that $B(x^*, r^*) \subseteq V_i$. Show that we can find some $k_0 \geq 1$, such that $d(x^*, x_{n_{k_0}}) < r^*/2$ and $r(x_{n_{k_0}}) \leq r^*/4$.

3. Show that $d(x^*, x_{n_{k_0}}) < r^*/2$ implies that $B(x_{n_{k_0}}, r^*/2) \subseteq V_i$. Show that this contradicts $r(x_{n_{k_0}}) \leq r^*/4$, and conclude that $\inf_{x \in E} r(x) > 0$.

**Exercise 4.** Further to ex. (3), Let $r_0$ with $0 < r_0 < \inf_{x \in E} r(x)$. Suppose that $E$ cannot be covered by a finite number of open balls with radius $r_0$. 

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1. Show the existence of a sequence \((x_n)_{n \geq 1}\) in \(E\), such that for all \(n \geq 1\), \(x_{n+1} \notin B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)\).

2. Show that for all \(n > m\) we have \(d(x_n, x_m) \geq r_0\).

3. Show that \((x_n)_{n \geq 1}\) cannot have a convergent subsequence.

4. Conclude that there exists a finite subset \(\{x_1, \ldots, x_n\}\) of \(E\) such that \(E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)\).

5. Show that for all \(x \in E\), we have \(B(x, r_0) \subseteq V_i\) for some \(i \in I\).

6. Conclude that \((E, T)\) is compact.

7. Prove the following:

**Theorem 47** A metrizable topological space \((E, T)\) is compact, if and only if for every sequence \((x_n)_{n \geq 1}\) in \(E\), there exists a subsequence \((x_{n_k})_{k \geq 1}\) of \((x_n)_{n \geq 1}\) and some \(x \in E\), such that \(x_{n_k} \xrightarrow{T} x\).
**Exercise 5.** Let \(a, b \in \mathbb{R}, a < b\) and \((x_n)_{n \geq 1}\) be a sequence in \([a, b]\).

1. Show that \((x_n)_{n \geq 1}\) has a convergent subsequence.
2. Can we conclude that \([a, b]\) is a compact subset of \(\mathbb{R}\)?

**Exercise 6.** Let \(E = [-M, M] \times \ldots \times [-M, M] \subseteq \mathbb{R}^n\), where \(n \geq 1\) and \(M \in \mathbb{R}^+\). Let \(\mathcal{T}_{\mathbb{R}^n}\) be the usual product topology on \(\mathbb{R}^n\), and \(\mathcal{T}_E = (\mathcal{T}_{\mathbb{R}^n})|_E\) be the induced topology on \(E\).

1. Let \((x_p)_{p \geq 1}\) be a sequence in \(E\). Let \(x \in E\). Show that \(x_p \overset{\mathcal{T}_E}{\to} x\) is equivalent to \(x_p \overset{\mathcal{T}_{\mathbb{R}^n}}{\to} x\).
2. Propose a metric on \(\mathbb{R}^n\), inducing the topology \(\mathcal{T}_{\mathbb{R}^n}\).
3. Let \((x_p)_{p \geq 1}\) be a sequence in \(\mathbb{R}^n\). Let \(x \in \mathbb{R}^n\). Show that \(x_p \overset{\mathcal{T}_{\mathbb{R}^n}}{\to} x\) if and only if, \(x_p^i \overset{\mathcal{T}_{\mathbb{R}}}{\to} x^i\) for all \(i \in \mathbb{N}_n\).
EXERCISE 7. Further to ex. (6), suppose \((x_p)_{p \geq 1}\) is a sequence in \(E\).

1. Show the existence of a subsequence \((x_{\phi(p)})_{p \geq 1}\) of \((x_p)_{p \geq 1}\), such that \(x_{\phi(p)}^1 \xrightarrow{T_{[-M,M]}} x^1\) for some \(x^1 \in [-M, M]\).

2. Explain why the above convergence is equivalent to \(x_{\phi(p)}^1 \xrightarrow{TR} x^1\).

3. Suppose that \(1 \leq k \leq n - 1\) and \((y_p)_{p \geq 1} = (x_{\phi(p)})_{p \geq 1}\) is a subsequence of \((x_p)_{p \geq 1}\) such that:

\[
\forall j = 1, \ldots, k \ , \ x_j^{\phi(p)} \xrightarrow{TR} x^j \text{ for some } x^j \in [-M, M]
\]

Show the existence of a subsequence \((y_{\psi(p)})_{p \geq 1}\) of \((y_p)_{p \geq 1}\) such that \(y_1^{k+1} \xrightarrow{TR} x^{k+1}\) for some \(x^{k+1} \in [-M, M]\).

4. Show that \(\phi \circ \psi : \mathbb{N}^* \to \mathbb{N}^*\) is strictly increasing.
5. Show that \((x_{\phi \psi(p)})_{p \geq 1}\) is a subsequence of \((x_p)_{p \geq 1}\) such that:
\[
\forall j = 1, \ldots, k + 1, \quad x_j^j \overset{T_E}{\longrightarrow} x_j \in [-M, M]
\]

6. Show the existence of a subsequence \((x_{\phi(p)})_{p \geq 1}\) of \((x_p)_{p \geq 1}\), and \(x \in E\), such that \(x_{\phi(p)} \overset{T_E}{\longrightarrow} x\)

7. Show that \((E, T_E)\) is a compact topological space.

**Exercise 8.** Let \(A\) be a closed subset of \(\mathbb{R}^n\), \(n \geq 1\), which is bounded with respect to the usual metric of \(\mathbb{R}^n\).

1. Show that \(A \subseteq E = [-M, M] \times \ldots \times [-M, M]\), for some \(M \in \mathbb{R}^+\).

2. Show from \(E \setminus A = E \cap A^c\) that \(A\) is closed in \(E\).

3. Show \((A, (T_{\mathbb{R}^n})|_A)\) is a compact topological space.
4. Conversely, let $A$ is a compact subset of $\mathbb{R}^n$. Show that $A$ is closed and bounded.

**Theorem 48** A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded with respect to its usual metric.

**Exercise 9.** Let $n \geq 1$. Consider the map:

$$\phi : \begin{cases} 
\mathbb{C}^n &\rightarrow & \mathbb{R}^{2n} \\
(a_1 + ib_1, \ldots, a_n + ib_n) &\rightarrow & (a_1, b_1, \ldots, a_n, b_n)
\end{cases}$$

1. Recall the expressions of the usual metrics $d_{\mathbb{C}^n}$ and $d_{\mathbb{R}^{2n}}$ of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ respectively.

2. Show that for all $z, z' \in \mathbb{C}^n$, $d_{\mathbb{C}^n}(z, z') = d_{\mathbb{R}^{2n}}(\phi(z), \phi(z'))$.

3. Show that $\phi$ is a homeomorphism from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$.

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4. Show that a subset $K$ of $C^n$ is compact, if and only if $\phi(K)$ is a compact subset of $\mathbb{R}^{2n}$.

5. Show that $K$ is closed, if and only if $\phi(K)$ is closed.

6. Show that $K$ is bounded, if and only if $\phi(K)$ is bounded.

7. Show that a subset $K$ of $C^n$ is compact, if and only if it is closed and bounded with respect to its usual metric.

**Definition 79** Let $(E,d)$ be a metric space. A sequence $(x_n)_{n\geq 1}$ in $E$ is said to be a **Cauchy sequence** with respect to the metric $d$, if and only if for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:

$$n, m \geq n_0 \Rightarrow d(x_n, x_m) \leq \epsilon$$

**Definition 80** We say that a metric space $(E,d)$ is **complete**, if and only if for any Cauchy sequence $(x_n)_{n\geq 1}$ in $E$, there exists $x \in E$ such that $(x_n)_{n\geq 1}$ converges to $x$. 
Exercise 10.

1. Explain why strictly speaking, given $p \in [1, +\infty[$, definition (77) of Cauchy sequences in $L^p_C(\Omega, \mathcal{F}, \mu)$ is not covered by definition (79).

2. Explain why $L^p_C(\Omega, \mathcal{F}, \mu)$ is not a complete metric space, despite theorem (46) and definition (80).

Exercise 11. Let $(z_k)_{k \geq 1}$ be a Cauchy sequence in $\mathbb{C}^n$, $n \geq 1$, with respect to the usual metric $d(z, z') = \|z - z'\|$, where:

$$
\|z\| \triangleq \sqrt{\sum_{i=1}^{n} |z_i|^2}
$$

1. Show that the sequence $(z_k)_{k \geq 1}$ is bounded, i.e. that there exists $M \in \mathbb{R}^+$ such that $\|z_k\| \leq M$, for all $k \geq 1$.
2. Define $B = \{ z \in \mathbb{C}^n, \|z\| \leq M \}$. Show that $\delta(B) < +\infty$, and that $B$ is closed in $\mathbb{C}^n$.

3. Show the existence of a subsequence $(z_{k_p})_{p \geq 1}$ of $(z_k)_{k \geq 1}$ such that $z_{k_p} \xrightarrow{C^n} z$ for some $z \in B$.

4. Show that for all $\epsilon > 0$, there exists $p_0 \geq 1$ and $n_0 \geq 1$ such that $d(z, z_{k_{p_0}}) \leq \epsilon/2$ and:
   \[ k \geq n_0 \Rightarrow d(z_k, z_{k_{p_0}}) \leq \epsilon/2 \]

5. Show that $z_k \xrightarrow{C^n} z$.

6. Conclude that $\mathbb{C}^n$ is complete with respect to its usual metric.

7. For which theorem of Tutorial 9 was the completeness of $\mathbb{C}$ used?

**Exercise 12.** Let $(x_k)_{k \geq 1}$ be a sequence in $\mathbb{R}^n$ such that $x_k \xrightarrow{C^n} z$, for some $z \in \mathbb{C}^n$.  

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1. Show that $z \in \mathbb{R}^n$.

2. Show that $\mathbb{R}^n$ is complete with respect to its usual metric.

**Theorem 49** \( C^n \) and \( \mathbb{R}^n \) are complete w.r. to their usual metrics.

**Exercise 13.** Let \((E, d)\) be a metric space, with metric topology \( \mathcal{T} \). Let \( F \subseteq E \), and \( \bar{F} \) denote the closure of \( F \).

1. Explain why, for all \( x \in \bar{F} \) and \( n \geq 1 \), we have \( F \cap B(x, 1/n) \neq \emptyset \).

2. Show that for all \( x \in \bar{F} \), there exists a sequence \((x_n)_{n \geq 1}\) in \( F \), such that \( x_n \xrightarrow{\mathcal{T}} x \).

3. Show conversely that if there is a sequence \((x_n)_{n \geq 1}\) in \( F \) with \( x_n \xrightarrow{\mathcal{T}} x \), then \( x \in \bar{F} \).
4. Show that $F$ is closed if and only if for all sequence $(x_n)_{n \geq 1}$ in $F$ such that $x_n \xrightarrow{T} x$ for some $x \in E$, we have $x \in F$.

5. Explain why $(F, T_{|F})$ is metrizable.

6. Show that if $F$ is complete with respect to the metric $d_{|F \times F}$, then $F$ is closed in $E$.

7. Let $d_{\mathbb{R}}$ be a metric on $\mathbb{R}$, inducing the usual topology $T_{\mathbb{R}}$. Show that $d' = (d_{\mathbb{R}})_{|\mathbb{R} \times \mathbb{R}}$ is a metric on $\mathbb{R}$, inducing the topology $T_{\mathbb{R}}$.

8. Find a metric on $[-1, 1]$ which induces its usual topology.

9. Show that $\{-1, 1\}$ is not open in $[-1, 1]$.

10. Show that $\{-\infty, +\infty\}$ is not open in $\mathbb{R}$.

11. Show that $\mathbb{R}$ is not closed in $\overline{\mathbb{R}}$.

12. Let $d_{\mathbb{R}}$ be the usual metric of $\mathbb{R}$. Show that $d' = (d_{\mathbb{R}})_{|\mathbb{R} \times \mathbb{R}}$ and $d_{\mathbb{R}}$ induce the same topology on $\mathbb{R}$, but that however, $\mathbb{R}$
is complete with respect to $d_{\mathbb{R}}$, whereas it cannot be complete with respect to $d'$.

**Definition 81** Let $H$ be a $K$-vector space, where $K = \mathbb{R}$ or $\mathbb{C}$. We call **inner-product** on $H$, any map $\langle \cdot, \cdot \rangle : H \times H \to K$ with the following properties:

1. $\forall x, y \in H, \; \langle x, y \rangle = \langle y, x \rangle$
2. $\forall x, y, z \in H, \; \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$
3. $\forall x, y \in H, \forall \alpha \in K, \; \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
4. $\forall x \in H, \; \langle x, x \rangle \geq 0$
5. $\forall x \in H, \; (\langle x, x \rangle = 0 \iff x = 0)$

where for all $z \in \mathbb{C}$, $\bar{z}$ denotes the complex conjugate of $z$. For all $x \in H$, we call **norm** of $x$, denoted $\|x\|$, the number defined by:

$$\|x\| \triangleq \sqrt{\langle x, x \rangle}$$

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**Exercise 14.** Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $K$-vector space $\mathcal{H}$.

1. Show that for all $y \in \mathcal{H}$, the map $x \rightarrow \langle x, y \rangle$ is linear.

2. Show that for all $x \in \mathcal{H}$, the map $y \rightarrow \langle x, y \rangle$ is linear if $K = \mathbb{R}$, and conjugate-linear if $K = \mathbb{C}$.

**Exercise 15.** Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $K$-vector space $\mathcal{H}$.

Let $x, y \in \mathcal{H}$. Let $A = \|x\|^2$, $B = |\langle x, y \rangle|$ and $C = \|y\|^2$. Let $\alpha \in K$ be such that $|\alpha| = 1$ and:

$$B = \alpha \langle x, y \rangle$$

1. Show that $A, B, C \in \mathbb{R}^+$.

2. For all $t \in \mathbb{R}$, show that $\langle x - t\alpha y, x - t\alpha y \rangle = A - 2tB + t^2C$.

3. Show that if $C = 0$ then $B^2 \leq AC$. 

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4. Suppose that $C \neq 0$. Show that $P(t) = A - 2tB + t^2C$ has a minimal value which is in $\mathbb{R}^+$, and conclude that $B^2 \leq AC$.

5. Conclude with the following:

**Theorem 50 (Cauchy-Schwarz’s inequality:second)** Let $\mathcal{H}$ be a $K$-vector space, where $K = \mathbb{R}$ or $\mathbb{C}$, and $\langle \cdot, \cdot \rangle$ be an inner-product on $\mathcal{H}$. Then, for all $x, y \in \mathcal{H}$, we have:

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

**Exercise 16.** For all $f, g \in L^2_\mathbb{C}(\Omega, \mathcal{F}, \mu)$, we define:

$$\langle f, g \rangle \triangleq \int_{\Omega} f \overline{g} \, d\mu$$

1. Use the first Cauchy-Schwarz inequality (42) to prove that for all $f, g \in L^2_\mathbb{C}(\Omega, \mathcal{F}, \mu)$, we have $fg \in L^1_\mathbb{C}(\Omega, \mathcal{F}, \mu)$. Conclude that $\langle f, g \rangle$ is a well-defined complex number.
2. Show that for all \( f \in L^2_\mathcal{C}(\Omega, \mathcal{F}, \mu) \), we have \( \|f\|_2 = \sqrt{\langle f, f \rangle} \).

3. Make another use of the first Cauchy-Schwarz inequality to show that for all \( f, g \in L^2_\mathcal{C}(\Omega, \mathcal{F}, \mu) \), we have:
\[
|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2
\]

4. Go through definition (81), and indicate which of the properties (i) – (v) fails to be satisfied by \( \langle \cdot, \cdot \rangle \). Conclude that \( \langle \cdot, \cdot \rangle \) is not an inner-product on \( L^2_\mathcal{C}(\Omega, \mathcal{F}, \mu) \), and therefore that inequality (*) is not a particular case of the second Cauchy-Schwarz inequality (50).

5. Let \( f, g \in L^2_\mathcal{C}(\Omega, \mathcal{F}, \mu) \). By considering \( \int (|f|+|t|g|^2) d\mu \) for \( t \in \mathbb{R} \), imitate the proof of the second Cauchy-Schwarz inequality to show that:
\[
\int_\Omega |fg| d\mu \leq \left( \int_\Omega |f|^2 d\mu \right)^{\frac{1}{2}} \left( \int_\Omega |g|^2 d\mu \right)^{\frac{1}{2}}
\]
6. Let \( f, g : (\Omega, \mathcal{F}) \to [0, +\infty] \) non-negative and measurable. Show that if \( \int f^2 \, d\mu \) and \( \int g^2 \, d\mu \) are finite, then \( f \) and \( g \) are \( \mu \)-almost surely equal to elements of \( L^2_{\mu}(\Omega, \mathcal{F}, \mu) \). Deduce from 5. a new proof of the first Cauchy-Schwarz inequality:
\[
\int_\Omega f g \, d\mu \leq \left( \int_\Omega f^2 \, d\mu \right)^{\frac{1}{2}} \left( \int_\Omega g^2 \, d\mu \right)^{\frac{1}{2}}
\]

**Exercise 17.** Let \( \langle \cdot, \cdot \rangle \) be an inner-product on a \( \mathbb{K} \)-vector space \( \mathcal{H} \).

1. Show that for all \( x, y \in \mathcal{H} \), we have:
\[
\|x + y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle
\]
2. Using the second Cauchy-Schwarz inequality (50), show that:
\[
\|x + y\| \leq \|x\| + \|y\|
\]
3. Show that \( d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\| \) defines a metric on \( \mathcal{H} \).
**Definition 82** Let $H$ be a $K$-vector space, where $K = \mathbb{R}$ or $\mathbb{C}$, and $\langle \cdot, \cdot \rangle$ be an inner-product on $H$. We call **norm topology** on $H$, denoted $T_{\langle \cdot, \cdot \rangle}$, the metric topology associated with $d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\|$. 

**Definition 83** We call **Hilbert space** over $K$ where $K = \mathbb{R}$ or $\mathbb{C}$, any ordered pair $(H, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is an inner-product on a $K$-vector space $H$, which is complete w.r. to $d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\|$. 

**Exercise 18.** Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$ and let $M$ be a closed linear subspace of $H$, (closed with respect to the norm topology $T_{\langle \cdot, \cdot \rangle}$). Define $[\cdot, \cdot] = \langle \cdot, \cdot \rangle|_{M \times M}$.

1. Show that $[\cdot, \cdot]$ is an inner-product on the $K$-vector space $M$.
2. With obvious notations, show that $d_{[\cdot, \cdot]} = (d_{\langle \cdot, \cdot \rangle})|_{M \times M}$.
3. Deduce that $T_{[\cdot, \cdot]} = (T_{\langle \cdot, \cdot \rangle})|_{M}$. 

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**Exercise 19.** Further to ex. (18), Let \((x_n)_{n \geq 1}\) be a Cauchy sequence in \(M\), with respect to the metric \(d_{[,]}\).

1. Show that \((x_n)_{n \geq 1}\) is a Cauchy sequence in \(H\).
2. Explain why there exists \(x \in H\) such that \(x_n \xrightarrow{T_{[,]}^*}} x\).
3. Explain why \(x \in M\).
4. Explain why we also have \(x_n \xrightarrow{T_{[,]}^*}} x\).
5. Explain why \((M, \langle \cdot , \cdot \rangle|_{M \times M})\) is a Hilbert space over \(K\).

**Exercise 20.** For all \(z, z' \in \mathbb{C}^n, n \geq 1\), we define:

\[
\langle z, z' \rangle \triangleq \sum_{i=1}^{n} z_i z_i'
\]

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1. Show that $\langle \cdot, \cdot \rangle$ is an inner-product on $\mathbb{C}^n$.
2. Show that the metric $d_{\langle \cdot, \cdot \rangle}$ is equal to the usual metric of $\mathbb{C}^n$.
3. Conclude that $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space over $\mathbb{C}$.
4. Show that $\mathbb{R}^n$ is a closed subset of $\mathbb{C}^n$.
5. Show however that $\mathbb{R}^n$ is not a linear subspace of $\mathbb{C}^n$.
6. Show that $(\mathbb{R}^n, \langle \cdot, \cdot \rangle|_{\mathbb{R}^n \times \mathbb{R}^n})$ is a Hilbert space over $\mathbb{R}$.

**Definition 84** We call usual inner-product in $K^n$, where $K = \mathbb{R}$ or $\mathbb{C}$, the inner-product denoted $\langle \cdot, \cdot \rangle$ and defined by:

$$\forall x, y \in K^n, \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$
**Theorem 51**  \( \mathbb{C}^n \) and \( \mathbb{R}^n \) together with their usual inner-products, are Hilbert spaces over \( \mathbb{C} \) and \( \mathbb{R} \) respectively.

**Definition 85** Let \( \mathcal{H} \) be a \( K \)-vector space, where \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( \mathcal{C} \subseteq \mathcal{H} \). We say that \( \mathcal{C} \) is a **convex subset** of \( \mathcal{H} \), if and only if, for all \( x, y \in \mathcal{C} \) and \( t \in [0, 1] \), we have \( tx + (1 - t)y \in \mathcal{C} \).

**Exercise 21.** Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) be a Hilbert space over \( K \). Let \( \mathcal{C} \subseteq \mathcal{H} \) be a non-empty closed convex subset of \( \mathcal{H} \). Let \( x_0 \in \mathcal{H} \). Define:

\[ \delta_{\text{min}} = \inf \{ \|x - x_0\| : x \in \mathcal{C} \} \]

1. Show the existence of a sequence \( (x_n)_{n \geq 1} \) in \( \mathcal{C} \) such that \( \|x_n - x_0\| \to \delta_{\text{min}} \).

2. Show that for all \( x, y \in \mathcal{H} \), we have:

\[ \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4 \left\| \frac{x + y}{2} \right\|^2 \]
3. Explain why for all $n, m \geq 1$, we have:
\[ \delta_{\text{min}} \leq \left\| \frac{x_n + x_m}{2} - x_0 \right\| \]

4. Show that for all $n, m \geq 1$, we have:
\[ \|x_n - x_m\|^2 \leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\delta^2_{\text{min}} \]

5. Show the existence of some $x^* \in \mathcal{H}$, such that $x_n \xrightarrow{\mathcal{H}} x^*$.

6. Explain why $x^* \in C$

7. Show that for all $x, y \in \mathcal{H}$, we have $|\|x\| - \|y\| | \leq \|x - y\|$.

8. Show that $\|x_n - x_0\| \rightarrow \|x^* - x_0\|$.

9. Conclude that we have found $x^* \in C$ such that:
\[ \|x^* - x_0\| = \inf\{\|x - x_0\| : x \in C\} \]
10. Let $y^*$ be another element of $C$ with such property. Show that:

$$\|x^* - y^*\|^2 \leq 2\|x^* - x_0\|^2 + 2\|y^* - x_0\|^2 - 4\delta_{\min}^2$$

11. Conclude that $x^* = y^*$.

**Theorem 52** Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$, where $K = \mathbb{R}$ or $\mathbb{C}$. Let $C$ be a non-empty, closed and convex subset of $H$. For all $x_0 \in H$, there exists a unique $x^* \in C$ such that:

$$\|x^* - x_0\| = \inf \{\|x - x_0\| : x \in C\}$$

**Definition 86** Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$, where $K = \mathbb{R}$ or $\mathbb{C}$. Let $G \subseteq H$. We call **orthogonal** of $G$, the subset of $H$ denoted $G^\perp$ and defined by:

$$G^\perp \triangleq \{ x \in H : \langle x, y \rangle = 0, \forall y \in G \}$$

**Exercise 22.** Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$ and $G \subseteq H$. 

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1. Show that $\mathcal{G}^\perp$ is a linear subspace of $\mathcal{H}$, even if $\mathcal{G}$ isn’t.

2. Show that $\phi_y : \mathcal{H} \to K$ defined by $\phi_y(x) = \langle x, y \rangle$ is continuous.

3. Show that $\mathcal{G}^\perp = \cap_{y \in \mathcal{G}} \phi_y^{-1}(\{0\})$.

4. Show that $\mathcal{G}^\perp$ is a closed subset of $\mathcal{H}$, even if $\mathcal{G}$ isn’t.

5. Show that $\emptyset^\perp = \{0\}^\perp = \mathcal{H}$.

6. Show that $\mathcal{H}^\perp = \{0\}$.

**Exercise 23.** Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, and $x_0 \in \mathcal{H}$.

1. Explain why there exists $x^* \in \mathcal{M}$ such that:

   $$\|x^* - x_0\| = \inf \{ \|x - x_0\| : x \in \mathcal{M} \}$$

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2. Define $y^* = x_0 - x^* \in \mathcal{H}$. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$:

$$\|y^*\|^2 \leq \|y^* - \alpha y\|^2$$

3. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$, we have:

$$0 \leq -\alpha \langle y, y^* \rangle - \overline{\alpha} \langle y, y^* \rangle + |\alpha|^2 \|y\|^2$$

4. For all $y \in \mathcal{M} \setminus \{0\}$, taking $\alpha = \overline{\langle y, y^* \rangle} / \|y\|^2$, show that:

$$0 \leq -\|\langle y, y^* \rangle\|^2 / \|y\|^2$$

5. Conclude that $x^* \in \mathcal{M}$, $y^* \in \mathcal{M}^\perp$ and $x_0 = x^* + y^*$.

6. Show that $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$

7. Show that $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$ with $x_0 = x^* + y^*$, are unique.
**Theorem 53** Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a Hilbert space over \(K\), where \(K = \mathbb{R}\) or \(\mathbb{C}\). Let \(M\) be a closed linear subspace of \(\mathcal{H}\). Then, for all \(x_0 \in \mathcal{H}\), there is a unique decomposition:

\[ x_0 = x^* + y^* \]

where \(x^* \in M\) and \(y^* \in M^\perp\).

**Definition 87** Let \(\mathcal{H}\) be a \(K\)-vector space, where \(K = \mathbb{R}\) or \(\mathbb{C}\). We call linear functional, any map \(\lambda : \mathcal{H} \to K\), such that for all \(x, y \in \mathcal{H}\) and \(\alpha \in K\):

\[ \lambda(x + \alpha y) = \lambda(x) + \alpha \lambda(y) \]

**Exercise 24.** Let \(\lambda\) be a linear functional on a \(K\)-Hilbert\(^1\) space \(\mathcal{H}\).

1. Suppose that \(\lambda\) is continuous at some point \(x_0 \in \mathcal{H}\). Show the existence of \(\eta > 0\) such that:

\[ \forall x \in \mathcal{H}, \|x - x_0\| \leq \eta \Rightarrow |\lambda(x) - \lambda(x_0)| \leq 1 \]

\(^1\)Norm vector spaces are introduced later in these tutorials.
Tutorial 10: Bounded Linear Functionals in $L^2$

Show that for all $x \in H$ with $x \neq 0$, we have $|\lambda(\eta x/\|x\|)| \leq 1$.

2. Show that if $\lambda$ is continuous at $x_0$, there exists $M \in \mathbb{R}^+$, with:
\[
\forall x \in H, \ |\lambda(x)| \leq M\|x\| \tag{2}
\]

3. Show conversely that if (2) holds, $\lambda$ is continuous everywhere.

**Definition 88** Let $(\mathcal{H},\langle \cdot,\cdot \rangle)$ be a Hilbert space over $K = \mathbb{R}$ or $\mathbb{C}$. Let $\lambda$ be a linear functional on $\mathcal{H}$. Then, the following are equivalent:

(i) $\lambda : (\mathcal{H}, T_{\langle \cdot, \cdot \rangle}) \to (K, T_K)$ is continuous

(ii) $\exists M \in \mathbb{R}^+ , \forall x \in \mathcal{H} , \ |\lambda(x)| \leq M\|x\|

In which case, we say that $\lambda$ is a bounded linear functional.

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$^2$Norm vector spaces are introduced later in these tutorials.
Exercise 25. Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a Hilbert space over \(K\). Let \(\lambda\) be a bounded linear functional on \(\mathcal{H}\), such that \(\lambda(x) \neq 0\) for some \(x \in \mathcal{H}\), and define \(M = \lambda^{-1}(\{0\})\).

1. Show the existence of \(x_0 \in \mathcal{H}\), such that \(x_0 \not\in M\).

2. Show the existence of \(x^* \in M\) and \(y^* \in M^\perp\) with \(x_0 = x^* + y^*\).

3. Deduce the existence of some \(z \in M^\perp\) such that \(\|z\| = 1\).

4. Show that for all \(\alpha \in K \setminus \{0\}\) and \(x \in \mathcal{H}\), we have:
   \[
   \frac{\lambda(x)}{\alpha} \langle z, \alpha z \rangle = \lambda(x)
   \]

5. Show that in order to have:
   \[
   \forall x \in \mathcal{H}, \; \lambda(x) = \langle x, \alpha z \rangle
   \]

   it is sufficient to choose \(\alpha \in K \setminus \{0\}\) such that:

   \[
   \forall x \in \mathcal{H}, \; \frac{\lambda(x) z}{\alpha} - x \in M
   \]
6. Show the existence of \( y \in \mathcal{H} \) such that:
\[
\forall x \in \mathcal{H}, \lambda(x) = \langle x, y \rangle
\]

7. Show the uniqueness of such \( y \in \mathcal{H} \).

**Theorem 54** Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a Hilbert space over \( K \), where \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( \lambda \) be a bounded linear functional on \( \mathcal{H} \). Then, there exists a unique \( y \in \mathcal{H} \) such that: \( \forall x \in \mathcal{H}, \lambda(x) = \langle x, y \rangle \).

**Definition 89** Let \( K = \mathbb{R} \) or \( \mathbb{C} \). We call \( K \)-vector space, any set \( \mathcal{H} \), together with operators \( \oplus \) and \( \otimes \) for which there exists an element \( 0_{\mathcal{H}} \in \mathcal{H} \) such that for all \( x, y, z \in \mathcal{H} \) and \( \alpha, \beta \in K \), we have:

(i) \( 0_{\mathcal{H}} \oplus x = x \)
(ii) \( \exists (-x) \in \mathcal{H}, (-x) \oplus x = 0_{\mathcal{H}} \)
(iii) \( x \oplus (y \oplus z) = (x \oplus y) \oplus z \)
(iv) \( x \oplus y = y \oplus x \)
(v) \( 1 \otimes x = x \)
(vi) \( \alpha \otimes (\beta \otimes x) = (\alpha \beta) \otimes x \)
(vii) \( (\alpha + \beta) \otimes x = (\alpha \otimes x) \oplus (\beta \otimes x) \)
(viii) \( \alpha \otimes (x \oplus y) = (\alpha \otimes x) \oplus (\alpha \otimes y) \)

**Exercise 26.** For all \( f \in L^2_K(\Omega, \mathcal{F}, \mu) \), define:

\[ \mathcal{H} \triangleq \{ [f] : f \in L^2_K(\Omega, \mathcal{F}, \mu) \} \]

where \([f] = \{ g \in L^2_K(\Omega, \mathcal{F}, \mu) : g = f, \mu\text{-a.s.} \} \). Let \( 0_{\mathcal{H}} = [0] \), and for all \([f], [g] \in \mathcal{H}\), and \( \alpha \in K \), we define:

\[ [f] \oplus [g] \triangleq [f + g] \]
\[ \alpha \otimes [f] \triangleq [\alpha f] \]

We assume \( f, f', g \) and \( g' \) are elements of \( L^2_K(\Omega, \mathcal{F}, \mu) \).
Tutorial 10: Bounded Linear Functionals in \( L^2 \)

1. Show that for \( f = g \) \( \mu \)-a.s. is equivalent to \([f] = [g]\).
2. Show that if \([f] = [f']\) and \([g] = [g']\), then \([f + g] = [f' + g']\).
3. Conclude that \( \oplus \) is well-defined.
4. Show that \( \otimes \) is also well-defined.
5. Show that \((\mathcal{H}, \oplus, \otimes)\) is a \( \mathbb{K} \)-vector space.

**Exercise 27.** Further to ex. (26), we define for all \([f],[g] \in \mathcal{H}:\)

\[ \langle [f],[g] \rangle_{\mathcal{H}} \triangleq \int_{\Omega} f \bar{g} \, d\mu \]

1. Show that \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) is well-defined.
2. Show that \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) is an inner-product on \( \mathcal{H} \).
3. Show that \((\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})\) is a Hilbert space over \( \mathbb{K} \).
4. Why is \( \langle f, g \rangle \triangleq \int_{\Omega} f \bar{g} d\mu \) not an inner-product on \( L_{K}^{2}(\Omega, \mathcal{F}, \mu) \)?

**Exercise 28.** Further to ex. (27), Let \( \lambda : L_{K}^{2}(\Omega, \mathcal{F}, \mu) \to K \) be a continuous linear functional. Define \( \Lambda : \mathcal{H} \to K \) by \( \Lambda([f]) = \lambda(f) \).

1. Show the existence of \( M \in \mathbb{R}^{+} \) such that:
   \[
   \forall f \in L_{K}^{2}(\Omega, \mathcal{F}, \mu), \quad |\lambda(f)| \leq M \|f\|_{2}
   \]

2. Show that if \([f] = [g]\) then \( \lambda(f) = \lambda(g) \).

3. Show that \( \Lambda \) is a well defined bounded linear functional on \( \mathcal{H} \).

4. Conclude with the following:

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As defined in these tutorials, \( L_{K}^{2}(\Omega, \mathcal{F}, \mu) \) is not a Hilbert space (not even a norm vector space). However, both \( L_{K}^{2}(\Omega, \mathcal{F}, \mu) \) and \( K \) have natural topologies and it is therefore meaningful to speak of continuous linear functional. Note however that we are slightly outside the framework of definition (88).

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**Theorem 55**  Let $\lambda : L^2_K(\Omega, \mathcal{F}, \mu) \to K$ be a continuous linear functional, where $K = \mathbb{R}$ or $\mathbb{C}$. There exists $g \in L^2_K(\Omega, \mathcal{F}, \mu)$ such that:

$$\forall f \in L^2_K(\Omega, \mathcal{F}, \mu), \quad \lambda(f) = \int_{\Omega} f \bar{g} d\mu$$