3. Stieltjes-Lebesgue Measure

Definition 12 Let $A \subseteq \mathcal{P}(\Omega)$ and $\mu : A \rightarrow [0, +\infty]$ be a map. We say that $\mu$ is **finitely additive** if and only if, given $n \geq 1$:

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A = \bigcup_{i=1}^{n} A_i \Rightarrow \mu(A) = \sum_{i=1}^{n} \mu(A_i)$$

We say that $\mu$ is **finitely sub-additive** if and only if, given $n \geq 1$:

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^{n} A_i \Rightarrow \mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

Exercise 1. Let $S \triangleq \{ [a, b] , \ a, b \in \mathbb{R} \}$ be the set of all intervals $[a, b]$, defined as $[a, b] = \{ x \in \mathbb{R}, a < x \leq b \}$.

1. Show that $[a, b] \cap [c, d] = [a \land c, b \land d]$
2. Show that $[a, b] \setminus [c, d] = [a, b \land c] \cup [a \lor d, b]$
3. Show that $c \leq d \Rightarrow b \land c \leq a \lor d$.

4. Show that $\mathcal{S}$ is a semi-ring on $\mathbb{R}$.

**Exercise 2.** Suppose $\mathcal{S}$ is a semi-ring in $\Omega$ and $\mu : \mathcal{S} \to [0, +\infty]$ is finitely additive. Show that $\mu$ can be extended to a finitely additive map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$, with $\bar{\mu}|_\mathcal{S} = \mu$.

**Exercise 3.** Everything being as before, Let $A \in \mathcal{R}(\mathcal{S})$, $A_i \in \mathcal{R}(\mathcal{S})$, $A \subseteq \bigcup_{i=1}^n A_i$ where $n \geq 1$. Define $B_1 = A_1 \cap A$ and for $i = 1, \ldots, n-1$:

$B_{i+1} \triangleq (A_{i+1} \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_i \cap A))$

1. Show that $B_1, \ldots, B_n$ are pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$ such that $A = \bigcup_{i=1}^n B_i$.

2. Show that for all $i = 1, \ldots, n$, we have $\bar{\mu}(B_i) \leq \bar{\mu}(A_i)$.

3. Show that $\bar{\mu}$ is finitely sub-additive.
4. Show that $\mu$ is finitely sub-additive.

**Exercise 4.** Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-decreasing map. Let $\mathcal{S}$ be the semi-ring on $\mathbb{R}$, $\mathcal{S} = \{[a, b], a, b \in \mathbb{R}\}$. Define the map $\mu : \mathcal{S} \to [0, +\infty]$ by $\mu(\emptyset) = 0$, and:

$$\forall a \leq b, \mu([a, b]) \triangleq F(b) - F(a) \quad (1)$$

Let $a < b$ and $a_i < b_i$ for $i = 1, \ldots, n$ and $n \geq 1$, with:

$$[a, b] = \bigcup_{i=1}^{n} [a_i, b_i]$$

1. Show that there is $i_1 \in \{1, \ldots, n\}$ such that $a_{i_1} = a$.

2. Show that $[b_{i_1}, b] = \bigcup_{j \in \{1, \ldots, n\} \setminus \{i_1\}} [a_j, b_j]$.

3. Show the existence of a permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ such that $a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n} = b$. 

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4. Show that $\mu$ is finitely additive and finitely sub-additive.

**Exercise 5.** $\mu$ being defined as before, suppose $a < b$ and $a_n < b_n$ for $n \geq 1$ with:

$$ |a, b| = \bigcup_{n=1}^{+\infty} [a_n, b_n] $$

Given $N \geq 1$, let $(i_1, \ldots, i_N)$ be a permutation of $\{1, \ldots, N\}$ with:

$$ a \leq a_{i_1} < b_{i_1} \leq a_{i_2} < \ldots < b_{i_N} \leq b $$

1. Show that $\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$.

2. Show that $\sum_{n=1}^{+\infty} \mu([a_n, b_n]) \leq \mu([a, b])$

3. Given $\epsilon > 0$, show that there is $\eta \in [0, b - a]$ such that:

$$ 0 \leq F(a + \eta) - F(a) \leq \epsilon $$

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4. For \( n \geq 1 \), show that there is \( \eta_n > 0 \) such that:
\[
0 \leq F(b_n + \eta_n) - F(b_n) \leq \frac{\epsilon}{2^n}
\]

5. Show that \([a + \eta, b] \subseteq \bigcup_{n=1}^{+\infty} [a_n, b_n + \eta_n]\).

6. Explain why there exist \( p \geq 1 \) and integers \( n_1, \ldots, n_p \) such that:
\[
[a + \eta, b] \subseteq \bigcup_{k=1}^{p} [a_{n_k}, b_{n_k} + \eta_{n_k}]
\]

7. Show that \( F(b) - F(a) \leq 2\epsilon + \sum_{n=1}^{+\infty} F(b_n) - F(a_n) \)

8. Show that \( \mu : \mathcal{S} \to [0, +\infty] \) is a measure.
**Definition 13**  A **topology** on $\Omega$ is a subset $\mathcal{T}$ of the power set $\mathcal{P}(\Omega)$, with the following properties:

(i) $\Omega, \emptyset \in \mathcal{T}$
(ii) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$
(iii) $A_i \in \mathcal{T}, \forall i \in I \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$

Property (iii) of definition (13) can be translated as: for any family $(A_i)_{i \in I}$ of elements of $\mathcal{T}$, the union $\bigcup_{i \in I} A_i$ is still an element of $\mathcal{T}$.

Hence, a topology on $\Omega$, is a set of subsets of $\Omega$ containing $\Omega$ and the empty set, which is closed under finite intersection and arbitrary union.

**Definition 14**  A **topological space** is an ordered pair $(\Omega, \mathcal{T})$, where $\Omega$ is a set and $\mathcal{T}$ is a topology on $\Omega$.
**Definition 15** Let \((Ω, T)\) be a topological space. We say that \(A \subseteq Ω\) is an open set in \(Ω\), if and only if it is an element of the topology \(T\). We say that \(A \subseteq Ω\) is a closed set in \(Ω\), if and only if its complement \(A^c\) is an open set in \(Ω\).

**Definition 16** Let \((Ω, T)\) be a topological space. We define the Borel \(σ\)-algebra on \(Ω\), denoted \(B(Ω)\), as the \(σ\)-algebra on \(Ω\), generated by the topology \(T\). In other words, \(B(Ω) = σ(T)\).

**Definition 17** We define the usual topology on \(R\), denoted \(T_\mathbb{R}\), as the set of all \(U \subseteq R\) such that:

\[
\forall x \in U, \ \exists \epsilon > 0, \ |x - \epsilon, x + \epsilon| \subseteq U
\]

**Exercise 6.** Show that \(T_\mathbb{R}\) is indeed a topology on \(R\).

**Exercise 7.** Consider the semi-ring \(S = \{[a, b] \ , \ a, b \in \mathbb{R}\}\). Let \(T_\mathbb{R}\) be the usual topology on \(R\), and \(B(\mathbb{R})\) be the Borel \(σ\)-algebra on \(R\).

1. Let \(a \leq b\). Show that \([a, b] = \cap_{n=1}^{+\infty}[a, b + 1/n]\).
2. Show that $\sigma(S) \subseteq B(\mathbb{R})$.

3. Let $U$ be an open subset of $\mathbb{R}$. Show that for all $x \in U$, there exist $a_x, b_x \in \mathbb{Q}$ such that $x \in [a_x, b_x] \subseteq U$.

4. Show that $U = \bigcup_{x \in U} [a_x, b_x]$.

5. Show that the set $I \triangleq \{ [a_x, b_x] \mid x \in U \}$ is countable.

6. Show that $U$ can be written $U = \bigcup_{i \in I} A_i$ with $A_i \in S$.

7. Show that $\sigma(S) = B(\mathbb{R})$.

**Theorem 6**  Let $S$ be the semi-ring $S = \{ [a, b] \mid a, b \in \mathbb{R} \}$. Then, the Borel $\sigma$-algebra $B(\mathbb{R})$ on $\mathbb{R}$, is generated by $S$, i.e. $B(\mathbb{R}) = \sigma(S)$.

**Definition 18** A measurable space is an ordered pair $(\Omega, F)$ where $\Omega$ is a set and $F$ is a $\sigma$-algebra on $\Omega$. 

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**Definition 19**  A **measure space** is a triple $(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F})$ is a measurable space and $\mu : \mathcal{F} \to [0, +\infty]$ is a measure on $\mathcal{F}$.

**Exercise 8.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_n \subseteq A_{n+1}$ for all $n \geq 1$, and let $A = \bigcup_{n=1}^{+\infty} A_n$ (we write $A_n \uparrow A$). Define $B_1 = A_1$ and for all $n \geq 1$, $B_{n+1} = A_{n+1} \setminus A_n$.

1. Show that $(B_n)$ is a sequence of pairwise disjoint elements of $\mathcal{F}$ such that $A = \bigcup_{n=1}^{+\infty} B_n$.
2. Given $N \geq 1$ show that $A_N = \bigcup_{n=1}^{N} B_n$.
3. Show that $\mu(A_N) \to \mu(A)$ as $N \to +\infty$
4. Show that $\mu(A_n) \leq \mu(A_{n+1})$ for all $n \geq 1$. 

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**Theorem 7**: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then if $(A_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \uparrow A$, we have $\mu(A_n) \uparrow \mu(A)$.

**Exercise 9.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_{n+1} \subseteq A_n$ for all $n \geq 1$, and let $A = \cap_{n=1}^{+\infty} A_n$ (we write $A_n \downarrow A$). We assume that $\mu(A_1) < +\infty$.

1. Define $B_n \triangleq A_1 \setminus A_n$ and show that $B_n \in \mathcal{F}, B_n \uparrow A_1 \setminus A$.
2. Show that $\mu(B_n) \uparrow \mu(A_1 \setminus A)$
3. Show that $\mu(A_n) = \mu(A_1) - \mu(A_1 \setminus A_n)$
4. Show that $\mu(A) = \mu(A_1) - \mu(A_1 \setminus A)$
5. Why is $\mu(A_1) < +\infty$ important in deriving those equalities.
6. Show that $\mu(A_n) \to \mu(A)$ as $n \to +\infty$

\[ ^1 \text{i.e. the sequence } (\mu(A_n))_{n \geq 1} \text{ is non-decreasing and converges to } \mu(A). \]
7. Show that $\mu(A_{n+1}) \leq \mu(A_n)$ for all $n \geq 1$.

**Theorem 8**  Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then if $(A_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \downarrow A$ and $\mu(A_1) < +\infty$, we have $\mu(A_n) \downarrow \mu(A)$.

**Exercise 10**. Take $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(\mathbb{R})$. Suppose $\mu$ is a measure on $\mathcal{B}(\mathbb{R})$ such that $\mu([a, b]) = b - a$, for $a < b$. Take $A_n = ]n, +\infty[$.

1. Show that $A_n \downarrow \emptyset$.
2. Show that $\mu(A_n) = +\infty$, for all $n \geq 1$.
3. Conclude that $\mu(A_n) \downarrow \mu(\emptyset)$ fails to be true.

**Exercise 11**. Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-decreasing map. Show the existence of a measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, +\infty]$ such that:

$$\forall a, b \in \mathbb{R}, \ a \leq b, \ \mu([a, b]) = F(b) - F(a)$$  \hspace{1cm} (2)
Exercise 12. Let $\mu_1, \mu_2$ be two measures on $\mathcal{B}(\mathbb{R})$ with property (2). For $n \geq 1$, we define:

$$D_n \triangleq \{ B \in \mathcal{B}(\mathbb{R}) : \mu_1(B \cap [-n, n]) = \mu_2(B \cap [-n, n]) \}$$

1. Show that $D_n$ is a Dynkin system on $\mathbb{R}$.

2. Explain why $\mu_1([-n, n]) < +\infty$ and $\mu_2([-n, n]) < +\infty$ is needed when proving 1.

3. Show that $S \triangleq \{ [a, b] : a, b \in \mathbb{R} \} \subseteq D_n$.

4. Show that $\mathcal{B}(\mathbb{R}) \subseteq D_n$.

5. Show that $\mu_1 = \mu_2$.

6. Prove the following theorem.
Theorem 9  Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-decreasing map. There exists a unique measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, +\infty]$ such that:

$$\forall a, b \in \mathbb{R}, \ a \leq b \ , \ \mu([a, b]) = F(b) - F(a)$$

Definition 20  Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-decreasing map. We call Stieltjes measure on $\mathbb{R}$ associated with $F$, the unique measure on $\mathcal{B}(\mathbb{R})$, denoted $dF$, such that:

$$\forall a, b \in \mathbb{R}, \ a \leq b \ , \ dF([a, b]) = F(b) - F(a)$$

Definition 21  We call Lebesgue measure on $\mathbb{R}$, the unique measure on $\mathcal{B}(\mathbb{R})$, denoted $dx$, such that:

$$\forall a, b \in \mathbb{R}, \ a \leq b \ , \ dx([a, b]) = b - a$$

Exercise 13. Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-decreasing map. Let $x_0 \in \mathbb{R}$.

1. Show that the limit $F(x_0-) = \lim_{x \to x_0} F(x)$ exists and is an element of $\mathbb{R}$. 

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2. Show that $\{x_0\} = \cap_{n=1}^{\infty}[x_0 - 1/n, x_0]$. 
3. Show that $\{x_0\} \in \mathcal{B}(\mathbb{R})$ 
4. Show that $dF(\{x_0\}) = F(x_0) - F(x_0^-)$

Exercise 14. Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-decreasing map. Let $a \leq b$.

1. Show that $[a, b] \in \mathcal{B}(\mathbb{R})$ and $dF([a, b]) = F(b) - F(a)$ 
2. Show that $[a, b] \in \mathcal{B}(\mathbb{R})$ and $dF([a, b]) = F(b) - F(a^-)$ 
3. Show that $]a, b[ \in \mathcal{B}(\mathbb{R})$ and $dF([a, b]) = F(b-) - F(a)$ 
4. Show that $]a, b[ \in \mathcal{B}(\mathbb{R})$ and $dF([a, b]) = F(b-) - F(a^-)$
**Exercise 15.** Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$. Define:

$$\mathcal{A}_{|\Omega'} \triangleq \{ A \cap \Omega', A \in \mathcal{A} \}$$

1. Show that if $\mathcal{A}$ is a topology on $\Omega$, $\mathcal{A}_{|\Omega'}$ is a topology on $\Omega'$.
2. Show that if $\mathcal{A}$ is a $\sigma$-algebra on $\Omega$, $\mathcal{A}_{|\Omega'}$ is a $\sigma$-algebra on $\Omega'$.

**Definition 22.** Let $\Omega$ be a set, and $\Omega' \subseteq \Omega$. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. We call **trace** of $\mathcal{A}$ on $\Omega'$, the subset $\mathcal{A}_{|\Omega'}$ of the power set $\mathcal{P}(\Omega')$ defined by:

$$\mathcal{A}_{|\Omega'} \triangleq \{ A \cap \Omega', A \in \mathcal{A} \}$$

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Definition 23  Let $(\Omega, T)$ be a topological space and $\Omega' \subseteq \Omega$. We call induced topology on $\Omega'$, denoted $T_{|\Omega'}$, the topology on $\Omega'$ defined by:

$$T_{|\Omega'} \triangleq \{ A \cap \Omega' : A \in T \}$$

In other words, the induced topology $T_{|\Omega'}$ is the trace of $T$ on $\Omega'$.

Exercise 16. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$, and $\mathcal{A}_{|\Omega'}$ be the trace of $\mathcal{A}$ on $\Omega'$. Define:

$$\Gamma \triangleq \{ A \in \sigma(\mathcal{A}) : A \cap \Omega' \in \sigma(\mathcal{A}_{|\Omega'}) \}$$

where $\sigma(\mathcal{A}_{|\Omega'})$ refers to the $\sigma$-algebra generated by $\mathcal{A}_{|\Omega'}$ on $\Omega'$.

1. Explain why the notation $\sigma(\mathcal{A}_{|\Omega'})$ by itself is ambiguous.
2. Show that $\mathcal{A} \subseteq \Gamma$.
3. Show that $\Gamma$ is a $\sigma$-algebra on $\Omega$.
4. Show that $\sigma(\mathcal{A}_{|\Omega'}) = \sigma(\mathcal{A})_{|\Omega'}$

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**Theorem 10**  Let $\Omega' \subseteq \Omega$ and $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Then, the trace on $\Omega'$ of the $\sigma$-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$, is equal to the $\sigma$-algebra on $\Omega'$ generated by the trace of $\mathcal{A}$ on $\Omega'$. In other words, $\sigma(\mathcal{A}) |_{\Omega'} = \sigma(\mathcal{A}_{|\Omega'})$.

**Exercise 17.** Let $(\Omega, T)$ be a topological space and $\Omega' \subseteq \Omega$ with its induced topology $T_{|\Omega'}$.

1. Show that $\mathcal{B}(\Omega)_{|\Omega'} = \mathcal{B}(\Omega')$.
2. Show that if $\Omega' \in \mathcal{B}(\Omega)$ then $\mathcal{B}(\Omega') \subseteq \mathcal{B}(\Omega)$.
3. Show that $\mathcal{B}(\mathbb{R}^+) = \{ A \cap \mathbb{R}^+ , A \in \mathcal{B}(\mathbb{R}) \}$.
4. Show that $\mathcal{B}(\mathbb{R}^+) \subseteq \mathcal{B}(\mathbb{R})$.

**Exercise 18.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \subseteq \Omega$
1. Show that \((\Omega', \mathcal{F}|_{\Omega'})\) is a measurable space.

2. If \(\Omega' \in \mathcal{F}\), show that \(\mathcal{F}|_{\Omega'} \subseteq \mathcal{F}\).

3. If \(\Omega' \in \mathcal{F}\), show that \((\Omega', \mathcal{F}|_{\Omega'}, \mu|_{\Omega'})\) is a measure space, where \(\mu|_{\Omega'}\) is defined as \(\mu|_{\Omega'} = \mu(\mathcal{F}|_{\Omega'})\).

**Exercise 19.** Let \(F : \mathbb{R}^+ \to \mathbb{R}\) be a right-continuous, non-decreasing map with \(F(0) \geq 0\). Define:

\[
\tilde{F}(x) \triangleq \begin{cases} 
0 & \text{if } x < 0 \\
F(x) & \text{if } x \geq 0
\end{cases}
\]

1. Show that \(\tilde{F} : \mathbb{R} \to \mathbb{R}\) is right-continuous and non-decreasing.

2. Show that \(\mu : \mathcal{B}(\mathbb{R}^+) \to [0, +\infty]\) defined by \(\mu = d\tilde{F}|_{\mathcal{B}(\mathbb{R}^+)}\), is a measure on \(\mathcal{B}(\mathbb{R}^+)\) with the properties:

\[
\begin{align*}
(i) & \quad \mu(\{0\}) = F(0) \\
(ii) & \quad \forall 0 \leq a \leq b, \quad \mu([a, b]) = F(b) - F(a)
\end{align*}
\]
**Exercise 20.** Define: $\mathcal{C} = \{\{0\}\} \cup \{[a, b], \ 0 \leq a \leq b\}$

1. Show that $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R}^+)$

2. Let $U$ be open in $\mathbb{R}^+$. Show that $U$ is of the form:

   $$U = \bigcup_{i \in I}(\mathbb{R}^+ \cap [a_i, b_i])$$

   where $I$ is a countable set and $a_i, b_i \in \mathbb{R}$ with $a_i \leq b_i$.

3. For all $i \in I$, show that $\mathbb{R}^+ \cap [a_i, b_i] \in \sigma(\mathcal{C})$.

4. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^+)$

**Exercise 21.** Let $\mu_1$ and $\mu_2$ be two measures on $\mathcal{B}(\mathbb{R}^+)$ with:

   \begin{align*}
   (i) \quad & \mu_1(\{0\}) = \mu_2(\{0\}) = F(0) \\
   (ii) \quad & \mu_1([a, b]) = \mu_2([a, b]) = F(b) - F(a)
   \end{align*}
for all $0 \leq a \leq b$. For $n \geq 1$, we define:

$$D_n = \{ B \in \mathcal{B}(\mathbb{R}^+) \mid \mu_1(B \cap [0, n]) = \mu_2(B \cap [0, n]) \}$$

1. Show that $D_n$ is a Dynkin system on $\mathbb{R}^+$ with $C \subseteq D_n$, where the set $C$ is defined as in exercise (20).

2. Explain why $\mu_1([0, n]) < +\infty$ and $\mu_2([0, n]) < +\infty$ is important when proving 1.

3. Show that $\mu_1 = \mu_2$.

4. Prove the following theorem.

**Theorem 11** Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. There exists a unique $\mu : \mathcal{B}(\mathbb{R}^+) \rightarrow [0, +\infty]$ measure on $\mathcal{B}(\mathbb{R}^+)$ such that:

(i) $\mu(\{0\}) = F(0)$

(ii) $\forall 0 \leq a \leq b, \mu([a, b]) = F(b) - F(a)$

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Definition 24  Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. We call Stieltjes measure on $\mathbb{R}^+$ associated with $F$, the unique measure on $\mathcal{B}(\mathbb{R}^+)$, denoted $dF$, such that:

(i) $dF(\{0\}) = F(0)$

(ii) $\forall 0 \leq a \leq b$, $dF([a,b]) = F(b) - F(a)$
Solutions to Exercises

Exercise 1.

1. \( x \in [a, b] \cap [c, d] \) is equivalent to \( a < x \leq b \) and \( c < x \leq d \). This is in turn equivalent to:

\[
\bigvee a \lor c \equiv \max(a,c) < x \leq \min(b,d) \equiv b \land d
\]

We have proved that:

\[
[a, b] \cap [c, d] = [a \lor c, b \land d]
\]

2. Suppose \( x \in [a, b] \setminus [c, d] \). Then, either \( x \leq c \) or \( d < x \). In the first case, \( x \in [a, b \land c] \). In the second, \( x \in [a \lor d, b] \). Conversely, if \( x \in [a, b \land c] \cup [a \lor d, b] \), then \( a < x \leq b \) is true. Moreover, \( x \leq c \) or \( d < x \). In any case, \( x \notin [c, d] \). So \( x \in [a, b] \setminus [c, d] \). We have proved that:

\[
[a, b] \setminus [c, d] = [a, b \land c] \cup [a \lor d, b]
\]
3. If \( c \leq d \), then in particular:

\[
b \land c \leq c \leq d \leq a \lor d
\]

4. \( S \) is a set of subsets of \( \mathbb{R} \) which obviously contains the empty set. From 1., it is also closed under finite intersection. Let \( [a, b] \) and \( [c, d] \) be two elements of \( S \). If \( c > d \), then \( ]c, d[ = \emptyset \) and we have \( [a, b] \setminus ]c, d[ = [a, b] \). If \( c \leq d \), then it follows from 3. that \( b \land c \leq a \lor d \). We conclude from 2. that:

\[
[a, b] \setminus [c, d] = [a, b \land c] \cup [a \lor d, b]
\]

In any case, \( [a, b] \setminus [c, d] \) can be written as a finite union of pair-wise disjoint elements of \( S \). We have proved that \( S \) is indeed a semi-ring on \( \mathbb{R} \), as defined in definition (6).

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**Exercise 1**
Exercise 2. The solution to this exercise is very similar to the proof of theorem (2) : a measure defined on a semi-ring can be extended to a measure defined on the ring generated by this semi-ring. In this case, we are dealing with a finitely additive map which is not exactly a measure, but the techniques involved are almost the same. We know from the previous tutorial that the ring $\mathcal{R}(\mathcal{S})$ generated by the semi-ring $\mathcal{S}$, is the set of all finite unions of pairwise disjoint elements of $\mathcal{S}$. It is tempting to define $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$, by:

$$\forall A = \bigcup_{i=1}^{n} A_i \in \mathcal{R}(\mathcal{S}) \ , \ \bar{\mu}(A) = \sum_{i=1}^{n} \mu(A_i) \quad (3)$$

However, such definition may not be valid, unless the sum involved in equation (3), is independent of the particular representation of $A \in \mathcal{R}(\mathcal{S})$ as a finite union of pairwise disjoint elements of $\mathcal{S}$. Suppose that $A = \bigcup_{j=1}^{p} B_j$ is another such representation of $A$. Then, for all $i = 1, \ldots, n$, we have:

$$A_i = A_i \cap A = \bigcup_{j=1}^{p} A_i \cap B_j$$
Solutions to Exercises

Since each $A_i \cap B_j$ is an element of $\mathcal{S}$, and $\mu$ is finitely additive, for all $i = 1, \ldots , n$, we have:

$$\mu(A_i) = \sum_{j=1}^{p} \mu(A_i \cap B_j)$$

and similarly for all $j = 1, \ldots , p$:

$$\mu(B_j) = \sum_{i=1}^{n} \mu(A_i \cap B_j)$$

from which we conclude that:

$$\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \sum_{j=1}^{p} \mu(A_i \cap B_j) = \sum_{j=1}^{p} \mu(B_j)$$

It follows that the map $\tilde{\mu}$ as defined by equation (3), is perfectly well defined. Let $A_1, \ldots , A_n$ be $n$ pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$, $n \geq 1$, each $A_i$ having the representation:

$$A_i = \biguplus_{k=1}^{p_i} A^k_i, \quad i = 1, \ldots , n$$

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as a finite union of pairwise disjoint elements of $\mathcal{S}$. Suppose moreover that $A = \biguplus_{i=1}^n A_i$ (which is an element of $\mathcal{R}(\mathcal{S})$ since a ring is closed under finite union). Then $A$ has a representation:

$$A = \bigcup_{i=1}^n \bigcup_{k=1}^{p_i} A_i^k$$

where the $A_i^k$’s are pairwise disjoint. From the very definition of $\bar{\mu}$:

$$\bar{\mu}(A) = \sum_{i=1}^n \sum_{k=1}^{p_i} \mu(A_i^k)$$

and furthermore for all $i = 1, \ldots, n$:

$$\bar{\mu}(A_i) = \sum_{k=1}^{p_i} \mu(A_i^k)$$

So we conclude that:

$$\bar{\mu}(A) = \sum_{i=1}^n \bar{\mu}(A_i)$$
Solutions to Exercises

We have proved that $\bar{\mu} : \mathcal{R}(\mathcal{S}) \rightarrow [0, +\infty]$ is a finitely additive map. Finally, if $A \in \mathcal{S}$, taking $n = 1$ and $A_1 = A$, $A = \cup_{i=1}^n A_i$ is a representation of $A$ as a finite union of pairwise disjoint elements of $\mathcal{S}$. By definition of $\bar{\mu}$, $\bar{\mu}(A) = \sum_{i=1}^n \mu(A_i) = \mu(A)$. Hence, we see that $\bar{\mu}|_\mathcal{S} = \mu$. We have proved the existence of a finitely additive map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \rightarrow [0, +\infty]$, such that $\bar{\mu}|_\mathcal{S} = \mu$.

Exercise 2
Exercise 3.

1. A ring being closed under finite union, intersection and difference, each $B_i$ is an element of $\mathcal{R}(S)$. Suppose $B_i \cap B_j \neq \emptyset$ for some $i, j = 1, \ldots, n$. Without loss of generality we can assume that $i \leq j$. Suppose that $i < j$ and let $x \in B_i \cap B_j$. From $x \in B_i$ we have $x \in A_i \cap A$. From $x \in B_j$, we have $x \notin (A_1 \cap A) \cup \ldots \cup (A_{j-1} \cap A)$. In particular $x \notin A_i \cap A$. This is a contradiction, and it follows that $i = j$. The $B_i$’s are therefore pairwise disjoint. For all $i = 1, \ldots, n$ we have $B_i \subseteq A_i \cap A \subseteq A$. Hence $\bigcup_{i=1}^{n} B_i \subseteq A$. Conversely, suppose $x \in A \subseteq \bigcup_{i=1}^{n} A_i$. There exists $i \in \{1, \ldots, n\}$ such that $x \in A_i$. Let $i$ be the smallest of such integer. If $i = 1$, then $x \in A_1 \cap A = B_1$. If $i > 1$, then $x \in A_i \cap A$ and $x \notin A_j \cap A$ for all $j < i$. So $x \in B_i$. In any case, $x \in B_i$. It follows that $A \subseteq \bigcup_{i=1}^{n} B_i$. We have proved that $B_1, \ldots, B_n$ are pairwise disjoint elements of $\mathcal{R}(S)$ with $A = \bigcup_{i=1}^{n} B_i$.

2. $\bar{\mu} : \mathcal{R}(S) \to [0, +\infty]$ being defined as in exercise (2), it is a
finitely additive map. We have $B_i \subseteq A_i \cap A \subseteq A_i$, for all $i = 1, \ldots, n$. It follows that $A_i = B_i \cup (A_i \setminus B_i)$, from which we conclude that:

$$\bar{\mu}(A_i) = \bar{\mu}(B_i) + \bar{\mu}(A_i \setminus B_i) \geq \bar{\mu}(B_i)$$

3. From $A = \bigcup_{i=1}^n B_i$ and $\bar{\mu}$ being finitely additive, we have:

$$\bar{\mu}(A) = \sum_{i=1}^n \bar{\mu}(B_i)$$

Using 2., we obtain:

$$\bar{\mu}(A) \leq \sum_{i=1}^n \bar{\mu}(A_i)$$

This is true for all $A \in \mathcal{R}(S)$ and $A_1, \ldots, A_n$ in $\mathcal{R}(S)$ such that $A \subseteq \bigcup_{i=1}^n A_i$. It follows from definition (12) that $\bar{\mu}$ is indeed a finitely sub-additive map.
4. Suppose $A \in \mathcal{S}$ and $A_1, \ldots, A_n \in \mathcal{S}$, $(n \geq 1)$, with $A \subseteq \bigcup_{i=1}^{n} A_i$. Since $\bar{\mu}|_{\mathcal{S}} = \mu$, and $\bar{\mu}$ is finitely sub-additive (from 3.), we have:

$$\mu(A) = \bar{\mu}(A) \leq \sum_{i=1}^{n} \bar{\mu}(A_i) = \sum_{i=1}^{n} \mu(A_i)$$

It follows from definition (12) that $\mu$ is indeed finitely sub-additive. The purpose of this exercise is to show that any finitely additive map defined on a semi-ring $\mathcal{S}$, is in fact also finitely sub-additive. Note that proving that $\bar{\mu}$ is finitely sub-additive is pretty straightforward. This is because $\bar{\mu}$ is defined on a ring, which is closed under various finite operations (union, intersection, difference). However, $\mu$ being defined on a semi-ring only, it is impossible to apply the same line of argument as the one used for $\bar{\mu}$. It is in fact necessary for us to initially extend $\mu$ from $\mathcal{S}$ to $\mathcal{R}(\mathcal{S})$, then prove the finite sub-additivity on $\mathcal{R}(\mathcal{S})$, and conclude with the finite sub-additivity of $\mu$ on $\mathcal{S}$.

Exercise 3

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Exercise 4.

1. Take $i_1$ such that $a_{i_1} = \min(a_1, \ldots, a_n)$. From $[a_{i_1}, b_{i_1}] \subseteq [a, b]$ and $a_{i_1} < b_{i_1}$, we see that $a \leq a_{i_1} < b_{i_1} \leq b$. Suppose that $a < a_{i_1}$, and let $x$ be such that $a < x < a_{i_1} \leq b$. Since $x \in [a, b]$, there is $j \in \{1, \ldots, n\}$ such that $x \in [a_j, b_j]$. By definition of $i_1$, we have $a_{i_1} \leq a_j < x$. This is a contradiction, and it follows that $a_{i_1} = a$. We have proved the existence of $i_1 \in \{1, \ldots, n\}$ such that $a_{i_1} = a$.

2. Suppose $x \in [a_i, b_i]$ for some $i \in \{1, \ldots, n\}$, $i \neq i_1$. Since $[a_i, b_i] \subseteq [a, b]$, $x \in [a, b]$ and $x \leq b$. Also, $a \leq a_i$. From 1., $a_{i_1} = a$. It follows that $a_{i_1} \leq a_i < x$. However, the $[a_i, b_i]$'s being pairwise disjoint and $i \neq i_1$, $x \not\in [a_{i_1}, b_{i_1}]$. Therefore $x > b_{i_1}$. We have proved that $x \in [b_{i_1}, b]$ and consequently:

$$
\bigcup_{i=1,i \neq i_1}^n [a_i, b_i] \subseteq [b_{i_1}, b]
$$
Conversely, let \( x \in [b_{i_1}, b] \subseteq [a, b] \). There exists \( i \in \{1, \ldots, n\} \) such that \( x \in [a_i, b_i] \). If \( i = i_1 \), then \( x \in [a_{i_1}, b_{i_1}] \) which contradicts \( b_{i_1} < x \). It follows that \( i \neq i_1 \) and:

\[
|b_{i_1}, b| \subseteq \bigcup_{i=1, i \neq i_1} |a_i, b_i|
\]

3. Using 1. and 2., starting from:

\[
|a, b| = \bigcup_{i=1}^n |a_i, b_i|
\]

we have \( i_1 \in \{1, \ldots, n\} \) such that \( a = a_{i_1} < b_{i_1} \) and:

\[
|b_{i_1}, b| = \bigcup_{i=1, i \neq i_1} |a_i, b_i|
\]

Going one step further, there exists \( i_2 \in \{1, \ldots, n\} \setminus \{i_1\} \) such
that \( b_{i_1} = a_{i_2} < b_{i_2} \) and:

\[
|b_{i_2}, b| = \bigcup_{i=1, i \neq i_1, i_2}^n |a_i, b_i|
\]

By induction, we define \( i_1, \ldots, i_n \) distinct integers in \( \{1, \ldots, n\} \), (hence a permutation on \( \{1, \ldots, n\} \)), such that:

\[
a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n}
\]

and \( |b_{i_n}, b| = \emptyset \). Since \( [a_{i_n}, b_{i_n}] \subseteq [a, b] \) and \( a_{i_n} < b_{i_n} \), we have \( b_{i_n} \leq b \). From \( |b_{i_n}, b| = \emptyset \), we conclude that \( b_{i_n} = b \).

4. Let \( (i_1, \ldots, i_n) \) be a permutation of \( \{1, \ldots, n\} \), such that:

\[
a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n} = b
\]

We have:

\[
F(b) - F(a) = \sum_{k=1}^{n} F(b_{i_k}) - F(a_{i_k})
\]
from which we see that:

\[ \mu([a, b]) = \sum_{k=1}^{n} \mu([a_{ik}, b_{ik}]) = \sum_{i=1}^{n} \mu([a_i, b_i]) \]

This is true for all \( a < b, \ n \geq 1 \) and \( a_i < b_i \) for \( i = 1, \ldots, n \), such that:

\[ [a, b] = \bigcup_{i=1}^{n} [a_i, b_i] \]

Suppose \( A \in \mathcal{S}, \ n \geq 1 \) and \( A_1, \ldots, A_n \in \mathcal{S} \), with \( A = \bigcup_{i=1}^{n} A_i \). If \( A = \emptyset \), then for all \( i = 1, \ldots, n \), we have \( A_i = \emptyset \). In particular, \( \mu(A) = \sum_{i=1}^{n} \mu(A_i) \) is obviously satisfied. If \( A \neq \emptyset \), then \( A \) is of the form \( A = [a, b] \) for some \( a < b \) in \( \mathbb{R} \). If we consider \( J = \{ i = 1, \ldots, n, A_i \neq \emptyset \} \), then \( J \neq \emptyset \), and for all \( i \in J, \ A_i \) is of the form \( A_i = [a_i, b_i] \) with \( a_i < b_i \). Moreover, \( A = \bigcup_{i \in J} A_i \) and it follows from our previous developments that \( \mu(A) = \sum_{i \in J} \mu(A_i) \). However, for all \( i = 1, \ldots, n \), if \( i \notin J \), then
\[ A_i = \emptyset \text{ and } \mu(A_i) = 0. \text{ Consequently:} \]

\[ \mu(A) = \sum_{i \in J} \mu(A_i) + \sum_{i \notin J} \mu(A_i) = \sum_{i=1}^{n} \mu(A_i) \]

We have proved that \( \mu : S \rightarrow [0, +\infty] \) as defined by (1) is finitely additive. From exercise (3), it is also finitely sub-additive.

Exercise 4
Exercise 5.

1. The sum $\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k})$ can be written as:

$$F(b_{i_N}) - F(a_{i_1}) + \sum_{k=1}^{N-1} F(b_{i_k}) - F(a_{i_k+1})$$

$F$ being non-decreasing, with $b_{i_N} \leq b$ and $a \leq a_{i_1}$, we have $F(b_{i_N}) \leq F(b)$ and $F(a) \leq F(a_{i_1})$. Moreover, since $b_{i_k} \leq a_{i_{k+1}}$ for all $k = 1, \ldots, N - 1$, we have $F(b_{i_k}) \leq F(a_{i_{k+1}})$. It follows that:

$$\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$$

2. Let $N \geq 1$, and $(i_1, \ldots, i_N)$ be a permutation of $\{1, \ldots, N\}$ such that $a_{i_1} \leq a_{i_2} \leq \ldots \leq a_{i_N}$. Since $[a_{i_1}, b_{i_1}] \subseteq [a, b]$ (and the fact that $a_{i_1} < b_{i_1}$), we have $a \leq a_{i_1} < b_{i_1}$. We also have $[a_{i_N}, b_{i_N}] \subseteq [a, b]$ with $a_{i_N} < b_{i_N}$. Hence, $a_{i_N} < b_{i_N} \leq b$. Let $k \in \{1, \ldots, N - 1\}$. Since the $[a_n, b_n]$'s are pairwise disjoint,
in particular, $|a_{i_k}, b_{i_k}| \cap |a_{i_{k+1}}, b_{i_{k+1}}| = \emptyset$. Let $\epsilon > 0$ be such that $a_{i_{k+1}} + \epsilon \in |a_{i_{k+1}}, b_{i_{k+1}}|$. Then $a_{i_k} \leq a_{i_{k+1}} < a_{i_{k+1}} + \epsilon$, and $a_{i_{k+1}} + \epsilon$ cannot be an element of $|a_{i_k}, b_{i_k}|$. Hence, $b_{i_k} < a_{i_{k+1}} + \epsilon$.

Taking the limit as $\epsilon \to 0$, we have $b_{i_k} \leq a_{i_{k+1}}$. It follows that the permutation $(i_1, \ldots, i_N)$ of $\{1, \ldots, N\}$ is such that:

$$a \leq a_{i_1} < b_{i_1} \leq a_{i_2} < \ldots < b_{i_N} \leq b$$

From 1., we obtain:

$$\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$$

and consequently:

$$\sum_{n=1}^{N} \mu([a_n, b_n]) = \sum_{k=1}^{N} \mu([a_{i_k}, b_{i_k}]) \leq \mu([a, b])$$

(4)

Taking the supremum over all $N \geq 1$ (or the limit as $N \to +\infty$)
in the left-hand side of (4), we obtain:

\[ \sum_{n=1}^{+\infty} \mu([a_n, b_n]) \leq \mu([a, b]) \]

3. \( F \) being right-continuous, it is right-continuous in \( a \in \mathbb{R} \). Given \( \epsilon > 0 \), there exists \( \eta' > 0 \) such that:

\[ \forall x \in [a, a + \eta'], \quad |F(x) - F(a)| \leq \epsilon \]

Take \( \eta = \min(b - a, \eta')/2 \). Then \( \eta \in ]0, b - a[ \), and we have \( a + \eta \in [a, a + \eta'] \). Therefore, \( |F(a + \eta) - F(a)| \leq \epsilon \), and \( F \) being non-decreasing, we finally have:

\[ 0 \leq F(a + \eta) - F(a) \leq \epsilon \]

4. Given \( n \geq 1 \), \( F \) is right-continuous in \( b_n \in \mathbb{R} \). Given \( \epsilon > 0 \) and \( \epsilon' = \epsilon/2^n \), there exists \( \eta'_n > 0 \) such that:

\[ \forall x \in [b_n, b_n + \eta'_n], \quad |F(x) - F(b_n)| \leq \epsilon' \]
Take $\eta_n = \eta'_n/2$. Then $b_n + \eta_n \in [b_n, b_n + \eta'_n]$, and we have $|F(b_n + \eta_n) - F(b_n)| \leq \epsilon/2^n$. $F'$ being non-decreasing, we finally have:

$$0 \leq F(b_n + \eta_n) - F(b_n) \leq \frac{\epsilon}{2^n}$$

5. Let $x \in [a + \eta, b]$. Then $x \in [a, b]$, and there exists $n \geq 1$ such that $x \in [a_n, b_n]$. In particular, $x \in [a_n, b_n + \eta_n]$. It follows that:

$$[a + \eta, b] \subseteq \bigcup_{n=1}^{+\infty} [a_n, b_n + \eta_n] \quad (5)$$

6. We see from (5) that the closed interval $[a + \eta, b]$ of $\mathbb{R}$, is covered by the family of open sets $([a_n, b_n + \eta_n])_{n \geq 1}$ in $\mathbb{R}$. Since $[a + \eta, b]$ is a compact subset of $\mathbb{R}^2$, we can extract a finite sub-covering

---

Note that the notion of compact subsets and the fact that any closed interval $[a, b]$ in $\mathbb{R}$ is indeed a compact subset of $\mathbb{R}$, has not been approached so far in these tutorials. This seems to contradict our promise that no results in these tutorials should be used without proof. In fact, Tutorial 8 will give you ample reminders on compactness. Just be a little patient.

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of \([a + \eta, b]\). In other words, there exist \(p \geq 1\), and integers \(n_1, \ldots, n_p\) such that:

\[
[a + \eta, b] \subseteq \bigcup_{k=1}^{p} [a_{n_k}, b_{n_k} + \eta_{n_k}]
\]

In particular:

\[
[a + \eta, b] \subseteq \bigcup_{k=1}^{p} [a_{n_k}, b_{n_k} + \eta_{n_k}]
\] (6)

7. From exercise (4), we know that \(\mu\) as defined in (1), is finitely

\[
\mu([a + \eta, b]) \leq \sum_{k=1}^{p} \mu([a_{n_k}, b_{n_k} + \eta_{n_k}])
\] (7)

Since \(a + \eta < b\) and \(a_n < b_n < b_n + \eta_n\) for all \(n \geq 1\), inequality (7)
can be written as:

\[ F(b) - F(a + \eta) \leq \sum_{k=1}^{p} F(b_{n_k} + \eta_{n_k}) - F(a_{n_k}) \]

Using 3. and 4., we obtain:

\[ F(b) - F(a) \leq \epsilon + \sum_{k=1}^{p} (F(b_{n_k}) - F(a_{n_k}) + \frac{\epsilon}{2^{n_k}}) \]

and since \( F \) is non-decreasing, we finally have:

\[ F(b) - F(a) \leq 2\epsilon + \sum_{n=1}^{+\infty} F(b_{n}) - F(a_{n}) \] (8)

8. Taking the limit as \( \epsilon \to 0 \) in (8), we obtain:

\[ F(b) - F(a) \leq \sum_{n=1}^{+\infty} F(b_{n}) - F(a_{n}) \]
Since $a < b$ and $a_n < b_n$ for all $n \geq 1$, we have:

$$\mu([a, b]) \leq \sum_{n=1}^{+\infty} \mu([a_n, b_n])$$

From 2., we conclude that:

$$\mu([a, b]) = \sum_{n=1}^{+\infty} \mu([a_n, b_n]) \quad (9)$$

It follows that if $A \in \mathcal{S}$ and $(A_n)_{n \geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{S}$, such that $A = \bigcup_{n=1}^{+\infty} A_n$, we have:

$$\mu(A) = \sum_{n=1}^{+\infty} \mu(A_n) \quad (10)$$

Indeed, if $A = \emptyset$, then all $A_n$’s are empty and (10) is obviously satisfied. If $A \neq \emptyset$, then $A = [a, b]$ for some $a < b$. Moreover, if we define $J = \{ n \geq 1, A_n \neq \emptyset \}$, then $A = \bigcup_{n \in J} A_n$, and the
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following holds,

\[ \mu(A) = \sum_{n \in J} \mu(A_n) \quad (11) \]

either as a consequence of (9), in the case when \( J \) is infinite, or as a consequence of \( \mu \) being finitely additive (exercise (4)), in the case when \( J \) is finite. In any case, (10) follows immediately from (11) and the fact that \( \mu(\emptyset) = 0 \). Having proved (10), we conclude that \( \mu : \mathcal{S} \to [0, +\infty] \) as defined in (1) is indeed a measure on the semi-ring \( \mathcal{S} \).

Exercise 5

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Exercise 6. Any statement of the form \( \forall x \in \emptyset \ldots \) is true. So \( \emptyset \in \mathcal{T}_R \), and it is clear that \( \mathbb{R} \in \mathcal{T}_R \). So (i) of definition (13) is satisfied for \( \mathcal{T}_R \). Let \( A, B \in \mathcal{T}_R \). Let \( x \in A \cap B \). Since \( x \in A \), from definition (17), there exists \( \epsilon_1 > 0 \) such that \( |x - \epsilon_1, x + \epsilon_1| \subseteq A \). Since \( x \in B \), there exists \( \epsilon_2 > 0 \) such that \( |x - \epsilon_2, x + \epsilon_2| \subseteq B \). It follows that if \( \epsilon \) is defined as \( \epsilon = \min(\epsilon_1, \epsilon_2) \), then \( |x - \epsilon, x + \epsilon| \subseteq A \cap B \). Hence \( A \cap B \in \mathcal{T}_R \), and (ii) of definition (13) is satisfied for \( \mathcal{T}_R \). Let \( (A_i)_{i \in I} \) be a family of elements of \( \mathcal{T}_R \). Let \( x \in \bigcup_{i \in I} A_i \). There exists \( i \in I \) such that \( x \in A_i \). Since by assumption \( A_i \in \mathcal{T}_R \), there exists \( \epsilon > 0 \) such that \( |x - \epsilon, x + \epsilon| \subseteq A_i \). In particular, \( |x - \epsilon, x + \epsilon| \subseteq \bigcup_{i \in I} A_i \). It follows that \( \bigcup_{i \in I} A_i \in \mathcal{T}_R \), and (iii) of definition (13) is satisfied for \( \mathcal{T}_R \). We have proved that \( \mathcal{T}_R \) is indeed a topology on \( \mathbb{R} \).

Exercise 6

---

3 Recall that \( \forall x \in \emptyset, H \) is equivalent to \( x \in \emptyset \Rightarrow H \), and \( G \Rightarrow H \) is equivalent to \( (G \text{ is false}) \text{ or (both } G \text{ and } H \text{ are true}) \).
Exercise 7.

1. For all $n \geq 1$, we have $]a, b[ \subseteq ]a, b + 1/n[$. Hence, we have $]a, b[ \subseteq \bigcap_{n=1}^{\infty} ]a, b + 1/n[$. Conversely, if $x \in \bigcap_{n=1}^{\infty} ]a, b + 1/n[$, then for all $n \geq 1$, we have $a < x < b + 1/n$. Taking the limit as $n \to +\infty$, we obtain $a < x \leq b$. It follows that $x \in ]a, b[$ and $\bigcap_{n=1}^{\infty} ]a, b + 1/n[ \subseteq ]a, b[$. Finally, $]a, b[ = \bigcap_{n=1}^{\infty} ]a, b + 1/n[$.

2. Let $a, b \in \mathbb{R}$, $a \leq b$. For all $n \geq 1$, the interval $]a, b + 1/n[$ is an open set in $\mathbb{R}$, (i.e. an element of $\mathcal{T}_\mathbb{R}$). Indeed, if $x \in ]a, b + 1/n[$, take $\epsilon = \min(b + 1/n - x, x - a)$, then $]x - \epsilon, x + \epsilon[ \subseteq ]a, b + 1/n[$. Since $\mathcal{T}_\mathbb{R} \subseteq \sigma(\mathcal{T}_\mathbb{R}) = \mathcal{B}(\mathbb{R})$, $]a, b + 1/n[$ is also a Borel set in $\mathbb{R}$, (i.e. an element of $\mathcal{B}(\mathbb{R})$). From 1., we have:

$$]a, b[ = \bigcap_{n=1}^{\infty} ]a, b + 1/n[ = \left( \bigcup_{n=1}^{\infty} ]a, b + 1/n[ \right)^c$$

$\mathcal{B}(\mathbb{R})$ being a $\sigma$-algebra, it is closed under complementation and countable union. It follows that $]a, b[ \in \mathcal{B}(\mathbb{R})$. This being true
for all \( a \leq b \), we have proved that \( S \subseteq B(\mathbb{R}) \). The \( \sigma \)-algebra \( \sigma(S) \) generated by \( S \) being the smallest \( \sigma \)-algebra on \( \mathbb{R} \) containing \( S \), we finally have \( \sigma(S) \subseteq B(\mathbb{R}) \).

3. Let \( U \in \mathcal{T}_\mathbb{R} \) and \( x \in U \). From definition (17), there exists \( \epsilon > 0 \) such that \( [x - \epsilon, x + \epsilon] \subseteq U \). \( \mathbb{Q} \) being the set of all rational numbers, it is dense in \( \mathbb{R} \): in other words, for all \( a < b \), \( \mathbb{Q} \cap ]a, b[ \) is a non-empty set\(^4\). In particular, there exist \( a_x \in \mathbb{Q} \cap ]x - \epsilon, x[ \) and \( b_x \in \mathbb{Q} \cap ]x, x + \epsilon[ \). We have \( x \in ]a_x, b_x[ \subseteq U \).

4. Since for all \( x \in U \), \([a_x, b_x] \subseteq U \), we have \( \cup_{x \in U} [a_x, b_x] \subseteq U \). If \( x \in U \), then \( x \in ]a_x, b_x[ \). So \( U \subseteq \cup_{x \in U} [a_x, b_x] \). We have proved that \( U = \cup_{x \in U} [a_x, b_x] \).

5. Let \( I = \{ [a_x, b_x], x \in U \} \). Since \( \mathbb{Q} \) is a countable set, the product \( \mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q} \) is also countable. There exists a one-to-one map \( \phi : \mathbb{Q}^2 \to \mathbb{N} \). Consider \( \psi : I \to \mathbb{N} \) defined by

\(^4\)This density property of \( \mathbb{Q} \) in \( \mathbb{R} \) is not proved anywhere in these tutorials. Please refer to any textbook containing a formal construction of the field \( \mathbb{R} \).
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ψ([a_x, b_x]) = φ(a_x, b_x). Then if ψ([a_{x'}, b_{x'}]) = ψ([a_x, b_x]), we have φ(a_{x'}, b_{x'}) = φ(a_x, b_x), and thus, (a_{x'}, b_{x'}) = (a_x, b_x).
Hence, [a_{x'}, b_{x'}] = [a_x, b_x]. It follows that the map ψ : I → \mathbb{N} is an injective map. We have proved that I is a countable set.

6. For all i ∈ I, i = [a_x, b_x] for some x ∈ U. So i ∈ S. Define A_i = i. Then A_i ∈ S for all i ∈ I, and we have:

\[ U = \bigcup_{x \in U} [a_x, b_x] = \bigcup_{i \in I} A_i \]

7. Since I is a countable set, and A_i ∈ S for all i ∈ I, we have U = \bigcup_{i \in I} A_i ∈ \sigma(S). This being true for all U ∈ \mathcal{T}_R, we have proved that \mathcal{T}_R \subseteq \sigma(S). The Borel \sigma-algebra \mathcal{B}(\mathbb{R}) generated by \mathcal{T}_R being the smallest \sigma-algebra on \mathbb{R} containing \mathcal{T}_R, we have \mathcal{B}(\mathbb{R}) \subseteq \sigma(S). From 2., we conclude that \mathcal{B}(\mathbb{R}) = \sigma(S). The purpose of this exercise is to show theorem (6).

Exercise 7

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Exercise 8.

1. A σ-algebra being closed under difference, \((B_n)_{n \geq 1}\) is indeed a sequence of elements of \(\mathcal{F}\). Suppose \(B_n \cap B_p \neq \emptyset\). Without loss of generality, we can assume that \(n \leq p\). Suppose \(n < p\) and let \(x \in B_n \cap B_p\). From \(x \in B_n\), we have \(x \in A_n\). From \(x \in B_p\), we have \(x \notin A_{p-1}\). However, \(A_n \subseteq A_{p-1}\). This is a contradiction, and it follows that \(n = p\). We have proved that the \(B_n\)'s are pairwise disjoint. Since \(B_n \subseteq A_n\) for all \(n \geq 1\), we have \(\bigcup_{n=1}^{\infty} B_n \subseteq A\). Conversely, let \(x \in A\). There exists \(n \geq 1\) such that \(x \in A_n\). Let \(n\) be the smallest integer such that \(x \in A_n\). Then if \(n = 1\), \(x \in B_1\). If \(n > 1\), then \(x \in A_n \setminus A_{n-1} = B_n\). In any case \(x \in B_n\) and \(A \subseteq \bigcup_{n=1}^{\infty} B_n\). We have proved that \((B_n)_{n \geq 1}\) is a sequence of pairwise disjoint elements of \(\mathcal{F}\), such that \(A = \bigcup_{n=1}^{\infty} B_n\).

2. Let \(N \geq 1\). For all \(n = 1, \ldots, N\), we have \(B_n \subseteq A_n \subseteq A_N\). So \(\bigcup_{n=1}^{N} B_n \subseteq A_N\). Conversely, let \(x \in A_N\). Let \(n\) be the smallest integer such that \(x \in A_n\). Then \(1 \leq n \leq N\). If \(n = 1\), then \(x \in B_1\).
Solutions to Exercises

$x \in B_1$. If $n > 1$, then $x \in A_n \setminus A_{n-1} = B_n$. In any case, $x \in B_n$ and $A_N \subseteq \bigcup_{n=1}^{N} B_n$. We have proved that $A_N = \bigcup_{n=1}^{N} B_n$.

3. $\mu$ being a measure on $\mathcal{F}$, from 1. we obtain:

$$\lim_{N \to +\infty} \sum_{n=1}^{N} \mu(B_n) \triangleq \sum_{n=1}^{+\infty} \mu(B_n) = \mu(A)$$

However, it follows from 2.

$$\sum_{n=1}^{N} \mu(B_n) = \mu(A_N)$$

Hence:

$$\lim_{N \to +\infty} \mu(A_N) = \mu(A)$$

4. Since $A_n \subseteq A_{n+1}$, we have $\mu(A_n) \leq \mu(A_{n+1})$ for all $n \geq 1$. The purpose of this exercise is to prove theorem (7).

Exercise 8
Exercise 9.

1. A σ-algebra being closed under difference, each $B_n$ is an element of $\mathcal{F}$. For all $n \geq 1$, we have:

   $$B_n = A_1 \cap A_n^c \subseteq A_1 \cap A_{n+1}^c = B_{n+1}$$

   Moreover:

   $$\bigcup_{n=1}^{+\infty} B_n = A_1 \cap \left( \bigcup_{n=1}^{+\infty} A_n^c \right) = A_1 \cap \left( \bigcap_{n=1}^{+\infty} A_n \right)^c = A_1 \setminus A$$

   We have proved that $B_n \uparrow A_1 \setminus A$.

2. $\mu(B_n) \uparrow \mu(A_1 \setminus A)$ is a direct application of theorem (7).

3. Since $A_n \subseteq A_1$, we have $A_1 = A_n \uplus (A_1 \setminus A_n)$. $\mu$ being a measure on $\mathcal{F}$, we obtain $\mu(A_1) = \mu(A_n) + \mu(A_1 \setminus A_n)$. Since $\mu(A_1) < +\infty$, all the terms involved in this equality are finite. Hence, it is legitimate to write:

   $$\mu(A_n) = \mu(A_1) - \mu(A_1 \setminus A_n)$$

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4. Since $A \subseteq A_1$, we have $A_1 = A \cup (A_1 \setminus A)$. $\mu$ being a measure on $\mathcal{F}$, we obtain $\mu(A_1) = \mu(A) + \mu(A_1 \setminus A)$. Since $\mu(A_1) < +\infty$, all the terms involved in this equality are finite. Hence, it is legitimate to write:

$$\mu(A) = \mu(A_1) - \mu(A_1 \setminus A)$$

5. Since for all $n \geq 1$, $A \subseteq A_n \subseteq A_1$, $\mu$ being a measure on $\mathcal{F}$, $\mu(A) \leq \mu(A_n) \leq \mu(A_1)$. Similarly, $A_1 \setminus A \subseteq A_1$ implies that $\mu(A_1 \setminus A) \leq \mu(A_1)$. Having $\mu(A_1) < +\infty$ ensures that all the terms involved in say $\mu(A_1) = \mu(A) + \mu(A_1 \setminus A)$ are finite, allowing to subtract $\mu(A_1 \setminus A)$ on both side of such equality. One common mistake to make is to get involved in algebra with non-finite terms, ending up with meaningless expressions of the form $+\infty - (+\infty)\ldots$
6. Using 2., 3., 4. and the fact that $\mu(A_1) < +\infty$:

$$\lim_{n \to +\infty} \mu(A_n) = \mu(A_1) - \lim_{n \to +\infty} \mu(B_n) = \mu(A_1) - \mu(A_1 \setminus A) = \mu(A)$$

7. For all $n \geq 1$, $A_{n+1} \subseteq A_n$, and therefore $\mu(A_{n+1}) \leq \mu(A_n)$. The purpose of this exercise is to prove theorem (8).

**Exercise 9**

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5 $\lim_{n \to +\infty}(+\infty - n) = +\infty$, whereas $+\infty - \lim_{n \to +\infty} n$ is meaningless…
Exercise 10.

1. For all $n \geq 1$, we have $A_{n+1} \subseteq A_n$, and:

$$\bigcap_{n=1}^{+\infty} A_n = \bigcap_{n=1}^{+\infty} n, +\infty[ = \emptyset$$

It follows that $A_n \downarrow \emptyset$.

2. Let $n \geq 1$. Given $p \geq n$, define $A_p^n = [n, p]$. Then $A_p^n \uparrow A_n$ as $p \to +\infty$, and from theorem (7), we have:

$$\mu(A_n) = \lim_{p \to +\infty} \mu(A_p^n) = \lim_{p \to +\infty} p - n = +\infty$$

3. Since $\mu(A_n) = +\infty$ for all $n \geq 1$, $\mu(A_n) \to +\infty$ as $n \to +\infty$. Since $\mu(\emptyset) = 0$, $\mu(A_n) \downarrow \mu(\emptyset)$ fails to be true. The purpose of this exercise is to serve as counter example to theorem (8), if the condition $\mu(A_1) < +\infty$ is relaxed. Indeed, $A_n \downarrow \emptyset$, yet we do not have $\mu(A_n) \downarrow \mu(\emptyset)$. Note however that to apply...

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theorem (8), \( \mu(A_1) < +\infty \) is not strictly speaking necessary: if a slightly weaker assumption is made that \( \mu(A_p) < +\infty \) for some \( p \geq 1 \), one can always apply theorem (8) to the sequence \( (A'_n)_{n \geq 1} = (A_{n+p-1})_{n \geq 1} \ldots \)

Exercise 10
Exercise 11. Let \( S \) be the semi-ring \( S = \{ [a, b], a, b \in \mathbb{R} \} \), and \( \mu : S \to [0, +\infty] \) be the map defined by equation (2). We know from exercise (5) that \( \mu \) is in fact a measure on \( S \). From theorem (5), \( \mu \) can be extended to a measure defined on the \( \sigma \)-algebra \( \sigma(S) \) generated by \( S \). In other words, there exists a measure \( \bar{\mu} : \sigma(S) \to [0, +\infty] \), such that \( \bar{\mu}|_S = \mu \). From theorem (6), we know that the \( \sigma \)-algebra \( \sigma(S) \) is in fact equal to the Borel \( \sigma \)-algebra \( B(\mathbb{R}) \) on \( \mathbb{R} \). Hence, we have found a measure \( \bar{\mu} : B(\mathbb{R}) \to [0, +\infty] \) such that \( \bar{\mu}|_S = \mu \). In particular, we have:

\[
\forall a, b \in \mathbb{R} \; , \; a \leq b \; , \; \bar{\mu}([a, b]) = F(b) - F(a)
\]

The purpose of this exercise is to prove the existence of the so called Stieltjes measure on \( \mathbb{R} \), stated in theorem (9). This is a vitally important result, as most other measures ever encountered, are derived one way or another from the Stieltjes measure on \( \mathbb{R} \).
Exercise 12.

1. Since $\mu_1([-n,n]) = F(n) - F(-n) = \mu_2([-n,n])$, $\Omega \in \mathcal{D}_n$. Suppose $A, B \in \mathcal{D}_n$, with $A \subseteq B$. We have:

$$\mu_1(B \cap [-n,n]) = \mu_2(B \cap [-n,n]) \quad (12)$$

$$\mu_1(A \cap [-n,n]) = \mu_2(A \cap [-n,n]) \quad (13)$$

Moreover, since $B = A \uplus (B \setminus A)$, for $i = 1, 2$, we have:

$$\mu_i(B \cap [-n,n]) = \mu_i(A \cap [-n,n]) + \mu_i((B \setminus A) \cap [-n,n]) \quad (14)$$

All terms involved in (12), (13) and (14) being finite, subtracting (13) from (12), and using (14), we obtain:

$$\mu_1((B \setminus A) \cap [-n,n]) = \mu_2((B \setminus A) \cap [-n,n])$$

This shows that $B \setminus A \in \mathcal{D}_n$. Let $(A_p)_{p \geq 1}$ be a sequence of elements of $\mathcal{D}_n$ such that $A_p \uparrow A$. Then $A_p \cap [-n,n] \uparrow A \cap [-n,n]$, and from theorem (7), $\mu_i(A_p \cap [-n,n]) \uparrow \mu_i(A \cap [-n,n])$ for all $i = 1, 2$. 

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\[i = 1, 2.\] However, since \(A_p \in D_n\) for all \(p \geq 1\), we have:

\[\mu_1(A_p \cap [-n, n]) = \mu_2(A_p \cap [-n, n])\]

Taking the limit as \(p \to +\infty\), we obtain:

\[\mu_1(A \cap [-n, n]) = \mu_2(A \cap [-n, n])\]

So \(A \in D_n\). Having checked (i), (ii) and (iii) of definition (1), we have proved that \(D_n\) is indeed a Dynkin system on \(\mathbb{R}\).

2. A crucial step in proving that \(D_n\) is a Dynkin system on \(\mathbb{R}\), consists in subtracting (13) from (12). One has to be very careful in avoiding meaningless expressions of the form \(+\infty - (+\infty)\). Having \(\mu_1([-n, n]) < +\infty\) and \(\mu_2([-n, n]) < +\infty\) ensures that all terms involved be finite.

3. Since \(\mu_1(\emptyset \cap [-n, n]) = 0 = \mu_2(\emptyset \cap [-n, n])\), we have \(\emptyset \in D_n\). Let \(a < b\). From exercise (1), \([a, b] \cap [-n, n]\) is an interval of the form \([a', b']\). If \(a' < b'\), then:

\[\mu_1([a', b']) = F(b') - F(a') = \mu_2([a', b'])\]

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Otherwise, $\mu_1([a', b']) = 0 = \mu_2([a', b'])$. In any case, we have $\mu_1([a', b']) = \mu_2([a', b'])$, and $[a, b] \in \mathcal{D}_n$. We have proved that $\mathcal{S} \subseteq \mathcal{D}_n$.

4. $\mathcal{S}$ being a semi-ring on $\mathbb{R}$, from definition (6), it is closed under finite intersection. Since $\mathcal{S} \subseteq \mathcal{D}_n$, $\mathcal{D}_n$ is a Dynkin system containing a set of subsets of $\mathbb{R}$, which is closed under finite intersection. According to theorem (1), $\mathcal{D}_n$ also contains the $\sigma$-algebra generated by $\mathcal{S}$. In other words, $\sigma(\mathcal{S}) \subseteq \mathcal{D}_n$. However, from theorem (6), the $\sigma$-algebra generated by $\mathcal{S}$, coincide with the Borel $\sigma$-algebra on $\mathbb{R}$, i.e. $\sigma(\mathcal{S}) = \mathcal{B}(\mathbb{R})$. It follows that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{D}_n$.

5. Let $A \in \mathcal{B}(\mathbb{R})$. from 4., we have $A \in \mathcal{D}_n$. In other words:

$$\mu_1(A \cap \mathbb{R} - n, n] = \mu_2(A \cap \mathbb{R} - n, n])$$

This being true for all $n \geq 1$, using theorem (7) and taking the limit as $n \to +\infty$, we obtain: $\mu_1(A) = \mu_2(A)$. This being true for all $A \in \mathcal{B}(\mathbb{R})$, $\mu_1 = \mu_2$. 

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6. Uniqueness follows from 5. Existence is proved in exercise (11).

Exercise 12
Exercise 13.

1. \( F \) being non-decreasing, for all \( x < x_0, \ F(x) \leq F(x_0) \). Define:

\[
\alpha \triangleq \sup_{x < x_0} F(x)
\]

Then \( \alpha \leq F(x_0) \) and in particular \( \alpha < +\infty \). It follows that given \( \epsilon > 0, \ \alpha - \epsilon < \alpha \). Being a supremum, \( \alpha \) is the smallest upper-bound of all \( F(x) \)'s for \( x < x_0 \). Hence, we see that \( \alpha - \epsilon \) cannot be such upper-bound. There exists \( x_1 < x_0 \) such that \( \alpha - \epsilon < F(x_1) \). Since \( F \) is non-decreasing, for all \( x \in ]x_1, x_0[ \), we have \( \alpha - \epsilon < F(x_1) \leq F(x) \leq \alpha \leq \alpha + \epsilon \). We conclude that for all \( \epsilon > 0 \), there exists \( x_1 < x_0 \) such that:

\[
\forall x \in ]x_1, x_0[ \quad |F(x) - \alpha| \leq \epsilon
\]

We have proved the existence of the left limit:

\[
F(x_0-) \triangleq \lim_{x < x_0, x \to x_0} F(x) = \alpha \in \mathbb{R}
\]
2. It is clear that \( \{ x_0 \} \subseteq \cap_{n=1}^{+\infty} [x_0 - 1/n, x_0] \). Conversely, suppose that \( x \in \cap_{n=1}^{+\infty} [x_0 - 1/n, x_0] \). Then for all \( n \geq 1 \), we have \( x_0 - 1/n < x \leq x_0 \). Taking the limit as \( n \to +\infty \), we obtain \( x_0 \leq x \leq x_0 \), i.e. \( x = x_0 \). So \( \cap_{n=1}^{+\infty} [x_0 - 1/n, x_0] \subseteq \{ x_0 \} \). We have proved that \( \{ x_0 \} = \cap_{n=1}^{+\infty} [x_0 - 1/n, x_0] \).

3. We have \( \{ x_0 \} = (-\infty, x_0] \cup [x_0, +\infty) \setminus \). Open intervals being open sets for the usual topology on \( \mathbb{R} \), they are also Borel sets. A \( \sigma \)-algebra being closed under finite union and complementation, we conclude that \( \{ x_0 \} \in \mathcal{B}(\mathbb{R}) \).

4. Given \( n \geq 1 \), let \( A_n = [x_0 - 1/n, x_0] \). Since \( A_{n+1} \subseteq A_n \), from 2., we have \( A_n \upharpoonright \{ x_0 \} \). Also, \( dF(A_1) = F(x_0) - F(x_0 - 1) \in \mathbb{R} \). In particular, \( dF(A_1) < +\infty \). Applying theorem (8), we obtain:

\[
\begin{align*}
dF(\{ x_0 \}) &= \lim_{n \to +\infty} dF(A_n) \\
&= F(x_0) - F(x_0 -)
\end{align*}
\]

Exercise 13
Exercise 14.

1. $]a, b[ = ]a, +\infty[\cap]\{b, +\infty[\}^c$. Open intervals being Borel sets, and a $\sigma$-algebra being closed under finite intersection and complementation, we have $]a, b[ \in \mathcal{B}(\mathbb{R})$. In virtue of definition (20), $dF(]a, b[) = F(b) - F(a)$.

2. $[a, b] = (]-\infty, a]\cup]\b, +\infty[)^c$ and is therefore a Borel set. Moreover, using exercise (13):

$$dF([a, b]) = dF(]a]) + dF([a, b]) = F(b) - F(a^-)$$

3. $]a, b]$ being open is a Borel set. Moreover, using exercise (13):

$$dF([a, b]) = dF([a, b]) - dF(]b]) = F(b^-) - F(a^-)$$

4. $]a, b[ = ]-\infty, b]\cap]\]a, +\infty[)^c$ and is therefore a Borel set. Moreover, using exercise (13):

$$dF([a, b]) = dF(]a]) + dF([a, b]) - dF(]b]) = F(b^-) - F(a^-)$$

Exercise 14
Exercise 15.

1. Suppose $\mathcal{A}$ is a topology on $\Omega$. Then $\emptyset$ and $\Omega$ are elements of $\mathcal{A}$. It follows that that $\emptyset \cap \Omega' = \emptyset$ and $\Omega \cap \Omega' = \Omega'$ are both elements of $\mathcal{A}_{|\Omega'}$. So (i) of definition (13) is satisfied for $\mathcal{A}_{|\Omega'}$. Let $A', B' \in \mathcal{A}_{|\Omega'}$. There exist $A, B \in \mathcal{A}$ such that $A' = A \cap \Omega'$ and $B' = B \cap \Omega'$. Hence, $A' \cap B' = (A \cap B) \cap \Omega'$. Since $\mathcal{A}$ is a topology, $A \cap B \in \mathcal{A}$. It follows that $A' \cap B' \in \mathcal{A}_{|\Omega'}$, and (ii) of definition (13) is satisfied for $\mathcal{A}_{|\Omega'}$. Let $(A'_i)_{i \in I}$ be a family of elements of $\mathcal{A}_{|\Omega'}$. There exists a family $(A_i)_{i \in I}$ of elements of $\mathcal{A}$, such that $A'_i = A_i \cap \Omega'$, for all $i \in I$. In particular, $\bigcup_{i \in I} A'_i = (\bigcup_{i \in I} A_i) \cap \Omega'$. Since $\mathcal{A}$ is a topology, $\bigcup_{i \in I} A_i \in \mathcal{A}$. It follows that $\bigcup_{i \in I} A'_i \in \mathcal{A}_{|\Omega'}$ and (iii) of definition (13) is satisfied for $\mathcal{A}_{|\Omega'}$. We have proved that $\mathcal{A}_{|\Omega'}$ is indeed a topology on $\Omega'$.

2. Suppose $\mathcal{A}$ is a $\sigma$-algebra on $\Omega$. Then $\Omega \in \mathcal{A}$, and we have $\Omega' = \Omega \cap \Omega' \in \mathcal{A}_{|\Omega'}$. Let $A' \in \mathcal{A}_{|\Omega'}$. There exists $A \in \mathcal{A}$ such that $A' = A \cap \Omega'$. Hence $\Omega' \setminus A' = \Omega' \cap (A')^c = \Omega' \cap A^c$. Since $\Omega' \setminus A' = \Omega' \cap (A')^c = \Omega' \cap A^c$.

---

6The notation $(A')^c$ refers to the complement of $A'$ in $\Omega$, i.e. $(A')^c = \Omega \setminus A'$. 

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\( \mathcal{A} \) is a \( \sigma \)-algebra, \( A' \in \mathcal{A} \). It follows that \( \Omega' \setminus A' \in \mathcal{A}_{\Omega'} \), and \( \mathcal{A}_{\Omega'} \) is closed under complementation in \( \Omega' \). Let \((A_n')_{n \geq 1}\) be a sequence of elements of \( \mathcal{A}_{\Omega'} \). There exists a sequence \((A_n)_{n \geq 1}\) of elements of \( \mathcal{A} \), such that \( A_n' = A_n \cap \Omega' \), for all \( n \geq 1 \). In particular, \( \bigcup_{n=1}^{+\infty} A_n' = (\bigcup_{n=1}^{+\infty} A_n) \cap \Omega' \). Since \( \mathcal{A} \) is a \( \sigma \)-algebra, \( \bigcup_{n=1}^{+\infty} A_n \in \mathcal{A} \). It follows that \( \bigcup_{n=1}^{+\infty} A_n' \in \mathcal{A}_{\Omega'} \), and \( \mathcal{A}_{\Omega'} \) is closed under countable union. We have proved that \( \mathcal{A}_{\Omega'} \) is indeed a \( \sigma \)-algebra on \( \Omega' \).

Exercise 15

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The complement of \( A' \) in \( \Omega' \) is denoted \( \Omega' \setminus A' \).
Exercise 16.

1. When working in the context of two reference sets $\Omega'$ and $\Omega$ where $\Omega' \subseteq \Omega$, given $A \subseteq \Omega'$, the notation $A^c$ and the notion of complementation can be confusing: does it refer to the complement of $A$ in $\Omega$, or the complement of $A$ in $\Omega'$? Unless otherwise specified, it is customary to keep the notation $A^c$ for the complement of $A$ relative to the large set ($A^c = \Omega \setminus A$). The complement of $A$ relative to the smaller set $\Omega'$ can still be denoted $\Omega' \setminus A$. Similarly, whenever $\mathcal{A}'$ is a set of subsets of $\Omega'$ (like $\mathcal{A}'|\Omega'$), then it is also a set of subsets of $\Omega$. Hence, a notation such as $\sigma(\mathcal{A}')$ can be ambiguous and confusing. One the one hand, $\sigma(\mathcal{A}')$ could be referring to the $\sigma$-algebra generated by $\mathcal{A}'$ on $\Omega$. One the other hand, $\sigma(\mathcal{A}')$ could be referring to the $\sigma$-algebra generated by $\mathcal{A}'$ on $\Omega'$. Hence, it is very important to specify clearly what is meant, when using a notation such as $\sigma(\mathcal{A}')$. In this exercise, $\sigma(\mathcal{A})$ is a $\sigma$-algebra on $\Omega$, whereas $\sigma(\mathcal{A}'|\Omega')$ is a $\sigma$-algebra on $\Omega'$. 

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2. Let $A \in \mathcal{A}$. Then $A \in \sigma(\mathcal{A})$ and $A \cap \Omega' \in \mathcal{A}_{\Omega'} \subseteq \sigma(\mathcal{A}_{\Omega'})$. It follows that $A \in \Gamma$, and $\mathcal{A} \subseteq \Gamma$.

3. $\sigma(\mathcal{A})$ being a $\sigma$-algebra on $\Omega$, $\Omega \in \sigma(\mathcal{A})$. $\sigma(\mathcal{A}_{\Omega'})$ being a $\sigma$-algebra on $\Omega'$, $\Omega \cap \Omega' = \Omega' \in \sigma(\mathcal{A}_{\Omega'})$. It follows that $\Omega \in \Gamma$.

Let $A \in \Gamma$. Then $A \in \sigma(\mathcal{A})$ and $A \cap \Omega' \in \sigma(\mathcal{A}_{\Omega'})$. Hence, $A^c \in \sigma(\mathcal{A})$ and $A^c \cap \Omega' = \Omega' \setminus (A \cap \Omega') \in \sigma(\mathcal{A}_{\Omega'})$. So $A^c \in \Gamma$. It follows that $\Gamma$ is closed under complementation. Let $(A_n)_{n \geq 1}$ be a sequence of elements of $\Gamma$. Then for all $n \geq 1$, $A_n \in \sigma(\mathcal{A})$ and $A_n \cap \Omega' \in \sigma(\mathcal{A}_{\Omega'})$. It follows that $\bigcup_{n=1}^{+\infty} A_n \in \sigma(\mathcal{A})$, and $(\bigcup_{n=1}^{+\infty} A_n) \cap \Omega' = \bigcup_{n=1}^{+\infty} (A_n \cap \Omega') \in \sigma(\mathcal{A}_{\Omega'})$. So $\bigcup_{n=1}^{+\infty} A_n \in \Gamma$. It follows that $\Gamma$ is closed under countable union. We have proved that $\Gamma$ is indeed a $\sigma$-algebra on $\Omega$.

4. The $\sigma$-algebra $\sigma(\mathcal{A})$ on $\Omega$ generated by $\mathcal{A}$, being the smallest $\sigma$-algebra on $\Omega$ containing $\mathcal{A}$, from $\mathcal{A} \subseteq \Gamma$, and the fact that $\Gamma$ is $\sigma$-algebra on $\Omega$, we have $\sigma(\mathcal{A}) \subseteq \Gamma$. In particular, for all $A \in \sigma(\mathcal{A})$, we have $A \cap \Omega' \in \sigma(\mathcal{A}_{\Omega'})$. Hence, we see that $\sigma(\mathcal{A})_{\Omega'} \subseteq \sigma(\mathcal{A}_{\Omega'})$. However, for all $A \in \mathcal{A}$, since $A \in \sigma(\mathcal{A})$, 

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we have $A \cap \Omega' \in \sigma(A)_{|\Omega'}$. It follows that $A_{|\Omega'} \subseteq \sigma(A)_{|\Omega'}$.

From exercise (15), $\sigma(A)_{|\Omega'}$ is a $\sigma$-algebra on $\Omega'$. The $\sigma$-algebra $\sigma(A_{|\Omega'})$ being the smallest $\sigma$-algebra on $\Omega'$ containing $A_{|\Omega'}$, we conclude that $\sigma(A_{|\Omega'}) \subseteq \sigma(A)_{|\Omega'}$. We have proved that $\sigma(A_{|\Omega'}) = \sigma(A)_{|\Omega'}$. The purpose of this exercise is to prove theorem (10).

Exercise 16
Exercise 17.

1. From theorem (10), $\mathcal{B}(\Omega|\Omega') = \sigma(T)|\Omega' = \sigma(T|\Omega') = \mathcal{B}(\Omega')$.

2. Suppose $\Omega' \in \mathcal{B}(\Omega)$. Let $A' \in \mathcal{B}(\Omega')$. Since $\mathcal{B}(\Omega') = \mathcal{B}(\Omega)|\Omega'$, there exists $A \in \mathcal{B}(\Omega)$ such that $A' = A \cap \Omega'$. A $\sigma$-algebra being closed under finite intersection, it follows that $A' \in \mathcal{B}(\Omega)$. We have proved that $\mathcal{B}(\Omega') \subseteq \mathcal{B}(\Omega)$.

3. From 1., we have $\mathcal{B}(\mathbb{R}^+) = \mathcal{B}(\mathbb{R})|\mathbb{R}^+ = \{A \cap \mathbb{R}^+, A \in \mathcal{B}(\mathbb{R})\}$

4. Since $\mathbb{R}^+ = [-\infty, 0[ \in \mathcal{B}(\mathbb{R})$, from 2. we have $\mathcal{B}(\mathbb{R}^+) \subseteq \mathcal{B}(\mathbb{R})$.

Exercise 17
Exercise 18.

1. From exercise (15), \( \mathcal{F} \) being a \( \sigma \)-algebra on \( \Omega \), \( \mathcal{F}_{|\Omega'} \) is a \( \sigma \)-algebra on \( \Omega' \). From definition (18), it follows that \((\Omega', \mathcal{F}_{|\Omega'})\) is a measurable space.

2. Suppose \( \Omega' \in \mathcal{F} \). A \( \sigma \)-algebra being closed under finite intersection, \( \mathcal{F}_{|\Omega'} = \{ A \cap \Omega', A \in \mathcal{F} \} \subseteq \mathcal{F} \).

3. If \( \Omega' \in \mathcal{F} \), from 2., \( \mathcal{F}_{|\Omega'} \subseteq \mathcal{F} \). Hence, it is legitimate to consider the restriction \( \mu|_{\mathcal{F}_{|\Omega'}} \) of the map \( \mu : \mathcal{F} \to [0, +\infty] \) to the smaller domain \( \mathcal{F}_{|\Omega'} \). Denoting such restriction by \( \mu|_{\Omega'} \), it is clearly a measure on \( \mathcal{F}_{|\Omega'} \) (definition (9)). From definition (19), it follows that \((\Omega', \mathcal{F}_{|\Omega'}, \mu|_{\Omega'})\) is a measure space.
Exercise 19.

1. Let \( x_0 \in \mathbb{R} \). If \( x_0 < 0 \), then \( \tilde{F}(x) \to 0 = \tilde{F}(x_0) \) as \( x \to x_0 \). If \( x_0 \geq 0 \), since \( F \) is right-continuous, we have:

\[
\lim_{x_0 < x, x \to x_0} \tilde{F}(x) = \lim_{x, x_0 < x \to x_0} F(x) = F(x_0) = \tilde{F}(x_0)
\]

Hence we see that \( \tilde{F} \) is itself right-continuous. Let \( a \leq b \). If \( 0 \leq a \leq b \), then \( \tilde{F}(a) = F(a) \leq F(b) = \tilde{F}(b) \). If \( a < 0 \leq b \), then \( \tilde{F}(a) = 0 \leq F(0) \leq F(b) = \tilde{F}(b) \). If \( a \leq b < 0 \), then \( \tilde{F}(a) = 0 = \tilde{F}(b) \). In any case, \( \tilde{F}(a) \leq \tilde{F}(b) \) and \( \tilde{F} \) is non-decreasing.

2. \( \mathcal{B}(\mathbb{R}^+) \subseteq \mathcal{B}(\mathbb{R}) \) and \( \mu \) is well-defined. Using exercise (13):

\[
\mu(\{0\}) = d\tilde{F}(\{0\}) = \tilde{F}(0) - \tilde{F}(0-) = F(0)
\]

Moreover, for all \( 0 \leq a \leq b \):

\[
\mu([a, b]) = d\tilde{F}([a, b]) = \tilde{F}(b) - \tilde{F}(a) = F(b) - F(a)
\]
Exercise 20.

1. For all \( 0 \leq a \leq b \), \([a, b] = [a, b] \cap \mathbb{R}^+ \in \mathcal{B}(\mathbb{R})|_{\mathbb{R}^+} = \mathcal{B}(\mathbb{R}^+)\). Moreover, we have \( \{0\} = [-1, 0] \cap \mathbb{R}^+ \in \mathcal{B}(\mathbb{R}^+) \). We have proved that \( C \subseteq \mathcal{B}(\mathbb{R}^+) \).

2. Let \( U \) be open in \( \mathbb{R}^+ \). By definition (23), there exists \( V \) open in \( \mathbb{R} \) such that \( U = V \cap \mathbb{R}^+ \). For all \( x \in V \), there exists \( \epsilon_x > 0 \) such that \( |x - \epsilon_x, x + \epsilon_x| \subseteq V \). The set of rational numbers \( Q \) being dense in \( \mathbb{R} \), we can choose \( p_x, q_x \in \mathbb{Q} \cap [x, x + \epsilon_x] \). We have \( x \in [p_x, q_x] \subseteq V \). If we define \( I = \{ [p_x, q_x], x \in V \} \), then \( I \) is a countable set (see exercise (7) for more details). For all \( i \in I \), let \( a_i = p_x \) and \( b_i = q_x \), where \( x \in V \) is such that \( i = [p_x, q_x] \). From \( V = \bigcup_{x \in V} [p_x, q_x] \), we obtain \( V = \bigcup_{i \in I} [a_i, b_i] \), and finally \( U = \bigcup_{i \in I} (\mathbb{R}^+ \cap [a_i, b_i]) \).

3. If \( 0 \leq a_i \leq b_i \), then \( \mathbb{R}^+ \cap [a_i, b_i] = [a_i, b_i] \in \mathcal{C} \). If \( a_i < 0 \leq b_i \), then \( \mathbb{R}^+ \cap [a_i, b_i] = [0, b_i] = \{0\} \cup [0, b_i] \in \sigma(\mathcal{C}) \). If \( a_i \leq b_i < 0 \), then \( \mathbb{R}^+ \cap [a_i, b_i] = \emptyset = [1, 1] \in \mathcal{C} \). In any case, \( \mathbb{R}^+ \cap [a_i, b_i] \in \sigma(\mathcal{C}) \).
4. From 2. and 3., the set $I$ being countable, we have:

$$U = \cup_{i \in I}(\mathbb{R}^+ \cap [a_i, b_i]) \in \sigma(C)$$

This being true for all $U$ open in $\mathbb{R}^+$, we have $\mathcal{I}_{\mathbb{R}^+} \subseteq \sigma(C)$. $\mathcal{B}(\mathbb{R}^+)$ being the smallest $\sigma$-algebra on $\mathbb{R}^+$ containing $\mathcal{I}_{\mathbb{R}^+}$, we obtain that $\mathcal{B}(\mathbb{R}^+) \subseteq \sigma(C)$. However from 1., $C \subseteq \mathcal{B}(\mathbb{R}^+)$. $\sigma(C)$ being the smallest $\sigma$-algebra on $\mathbb{R}^+$ containing $C$, we have $\sigma(C) \subseteq \mathcal{B}(\mathbb{R}^+)$. We have proved that $\sigma(C) = \mathcal{B}(\mathbb{R}^+)$. 

**Exercise 20**
Exercise 21.

1. \( \mu_1(\{0\} \cap [0,n]) = \mu_1(\{0\}) = \mu_2(\{0\}) = \mu_2(\emptyset \cap [0,n]) \). So \( \{0\} \in \mathcal{D}_n \). For all \( 0 \leq a \leq b \), \([a,b] \cap [0,n]\) is either empty, or is an interval of the form \([a',b']\) with \( 0 \leq a' \leq b' \). In any case, \( \mu_1([a,b] \cap [0,n]) = \mu_2([a,b] \cap [0,n]) \). It follows that \( \mathcal{C} \subseteq \mathcal{D}_n \).

Since \( \mu_1([0,n]) = \mu_1(\{0\}) + \mu_1([0,n]) = \mu_1([0,n]) \), we have \( \mathcal{R}^+ \in \mathcal{D}_n \) and both \( \mu_1([0,n]) \) and \( \mu_2([0,n]) \) are finite. Let \( A, B \in \mathcal{D}_n \) with \( A \subseteq B \). We have:

\[
\begin{align*}
\mu_1(A \cap [0,n]) &= \mu_2(A \cap [0,n]) \\
\mu_1(B \cap [0,n]) &= \mu_2(B \cap [0,n])
\end{align*}
\]

and for \( i = 1,2 \):

\[
\mu_i(B \cap [0,n]) = \mu_i(A \cap [0,n]) + \mu_i((B \setminus A) \cap [0,n])
\]

All terms being finite, we obtain:

\[
\mu_1((B \setminus A) \cap [0,n]) = \mu_2((B \setminus A) \cap [0,n])
\]
and it follows that $B \setminus A \in D_n$. Let $(A_p)_{p \geq 1}$ be a sequence of elements of $D_n$, with $A_p \uparrow A$. Then $A_p \cap [0, n] \uparrow A \cap [0, n]$. For all $p \geq 1$, we have:

$$\mu_1(A_p \cap [0, n]) = \mu_2(A_p \cap [0, n])$$

Using theorem (7), taking the limit as $p \to +\infty$, we obtain:

$$\mu_1(A \cap [0, n]) = \mu_2(A \cap [0, n])$$

and it follows that $A \in D_n$. We have proved that $D_n$ is a Dynkin system on $\mathbb{R}^+$ (definition (1)) with $C \subseteq D_n$.

2. $\mu_1([0, n]) < +\infty$ and $\mu_2([0, n]) < +\infty$ is important in ensuring that the algebra required to prove that $B \setminus A \in D_n$, is indeed meaningful.

3. Let $0 \leq a \leq b$. Then $\{0\} \cap [a, b] = \emptyset = [1, 1] \in C$. If $0 \leq c \leq d$, then $[a, b] \cap [c, d]$ can still be written as $[a', b']$ with $0 \leq a' \leq b'$, and therefore lies in $C$. It follows that $C$ is closed under finite intersection. Since $D_n$ is a Dynkin system on $\mathbb{R}^+$ such that
$\mathcal{C} \subseteq \mathcal{D}_n$, using theorem (1), we see that $\sigma(\mathcal{C}) \subseteq \mathcal{D}_n$. However, from exercise (20), $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^+)$. It follows that $\mathcal{B}(\mathbb{R}^+) \subseteq \mathcal{D}_n$. Hence, for all $A \in \mathcal{B}(\mathbb{R}^+)$, we have $\mu_1(A \cap [0, n]) = \mu_2(A \cap [0, n])$. Since $A \cap [0, n] \uparrow A$ as $n \to +\infty$, using theorem (7), we obtain $\mu_1(A) = \mu_2(A)$. This being true for all Borel set $A \in \mathcal{B}(\mathbb{R}^+)$, we have proved that $\mu_1 = \mu_2$.

4. Existence follows from exercise (19). Uniqueness is a consequence of 3.

Exercise 21

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