8. Jensen inequality

**Definition 64** Let \( a, b \in \mathbb{R} \), with \( a < b \). Let \( \phi : ]a, b[ \to \mathbb{R} \) be an \( \mathbb{R} \)-valued function. We say that \( \phi \) is a **convex function**, if and only if, for all \( x, y \in ]a, b[ \) and \( t \in [0, 1] \), we have:

\[
\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)
\]

**Exercise 1.** Let \( a, b \in \mathbb{R} \), with \( a < b \). Let \( \phi : ]a, b[ \to \mathbb{R} \) be a map.

1. Show that \( \phi : ]a, b[ \to \mathbb{R} \) is convex, if and only if for all \( x_1, \ldots, x_n \) in \( ]a, b[ \) and \( \alpha_1, \ldots, \alpha_n \) in \( \mathbb{R}^+ \) with \( \alpha_1 + \ldots + \alpha_n = 1 \), \( n \geq 1 \), we have:

\[
\phi(\alpha_1 x_1 + \ldots + \alpha_n x_n) \leq \alpha_1 \phi(x_1) + \ldots + \alpha_n \phi(x_n)
\]

2. Show that \( \phi : ]a, b[ \to \mathbb{R} \) is convex, if and only if for all \( x, y, z \) with \( a < x < y < z < b \) we have:

\[
\phi(y) \leq \frac{z-y}{z-x} \phi(x) + \frac{y-x}{z-x} \phi(z)
\]
3. Show that $\phi : ]a, b[ \to \mathbb{R}$ is convex if and only if for all $x, y, z$
with $a < x < y < z < b$, we have:
\[
\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(y)}{z - y}
\]

4. Let $\phi : ]a, b[ \to \mathbb{R}$ be convex. Let $x_0 \in ]a, b[$, and $u, u', v, v' \in ]a, b[$
be such that $u < u' < x_0 < v < v'$. Show that for all $x \in ]x_0, v[$:
\[
\frac{\phi(u') - \phi(u)}{u' - u} \leq \frac{\phi(x) - \phi(x_0)}{x - x_0} \leq \frac{\phi(v') - \phi(v)}{v' - v}
\]
and deduce that $\lim_{x \downarrow x_0} \phi(x) = \phi(x_0)$

5. Show that if $\phi : ]a, b[ \to \mathbb{R}$ is convex, then $\phi$ is continuous.

6. Define $\phi : [0, 1] \to \mathbb{R}$ by $\phi(0) = 1$ and $\phi(x) = 0$ for all $x \in ]0, 1[$.
Show that $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$, $\forall x, y, t \in [0, 1]$, 
but that $\phi$ fails to be continuous on $[0, 1]$. 

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Definition 65  Let \((\Omega, T)\) be a topological space. We say that \((\Omega, T)\) is a **compact topological space** if and only if, for all family \((V_i)_{i \in I}\) of open sets in \(\Omega\), such that \(\Omega = \bigcup_{i \in I} V_i\), there exists a finite subset \(\{i_1, \ldots, i_n\}\) of \(I\) such that \(\Omega = V_{i_1} \cup \ldots \cup V_{i_n}\).

In short, we say that \((\Omega, T)\) is compact if and only if, from any open covering of \(\Omega\), one can extract a finite sub-covering.

Definition 66  Let \((\Omega, T)\) be a topological space, and \(K \subseteq \Omega\). We say that \(K\) is a **compact subset** of \(\Omega\), if and only if the induced topological space \((K, T|_K)\) is a compact topological space.

Exercise 2. Let \((\Omega, T)\) be a topological space.

1. Show that if \((\Omega, T)\) is compact, it is a compact subset of itself.
2. Show that \(\emptyset\) is a compact subset of \(\Omega\).
3. Show that if \(\Omega' \subseteq \Omega\) and \(K\) is a compact subset of \(\Omega'\), then \(K\) is also a compact subset of \(\Omega\).
4. Show that if \((V_i)_{i \in I}\) is a family of open sets in \(\Omega\) such that \(K \subseteq \bigcup_{i \in I} V_i\), then \(K = \bigcup_{i \in I} (V_i \cap K)\) and \(V_i \cap K\) is open in \(K\) for all \(i \in I\).

5. Show that \(K \subseteq \Omega\) is a compact subset of \(\Omega\), if and only if for any family \((V_i)_{i \in I}\) of open sets in \(\Omega\) such that \(K \subseteq \bigcup_{i \in I} V_i\), there is a finite subset \(\{i_1, \ldots, i_n\}\) of \(I\) such that \(K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}\).

6. Show that if \((\Omega, T)\) is compact and \(K\) is closed in \(\Omega\), then \(K\) is a compact subset of \(\Omega\).

EXERCISE 3. Let \(a, b \in \mathbb{R}, a < b\). Let \((V_i)_{i \in I}\) be a family of open sets in \(\mathbb{R}\) such that \([a, b] \subseteq \bigcup_{i \in I} V_i\). We define \(A\) as the set of all \(x \in [a, b]\) such that \([a, x]\) can be covered by a finite number of \(V_i\)'s. Let \(c = \sup A\).

1. Show that \(a \in A\).

2. Show that there is \(\epsilon > 0\) such that \(a + \epsilon \in A\).
3. Show that \( a < c \leq b \).

4. Show the existence of \( i_0 \in I \) and \( c', c'' \) with \( a < c' < c < c'' \), such that \( [c', c''] \subseteq V_{i_0} \).

5. Show that \( [a, c'] \) can be covered by a finite number of \( V_i \)'s.

6. Show that \( [a, c''] \) can be covered by a finite number of \( V_i \)'s.

7. Show that \( b \land c'' \leq c \) and conclude that \( c = b \).

8. Show that \( [a, b] \) is a compact subset of \( \mathbb{R} \).

**Theorem 34** Let \( a, b \in \mathbb{R}, a < b \). The closed interval \( [a, b] \) is a compact subset of \( \mathbb{R} \).
Definition 67 Let $(\Omega, T)$ be a topological space. We say that $(\Omega, T)$ is a Hausdorff topological space, if and only if for all $x, y \in \Omega$ with $x \neq y$, there exists open sets $U$ and $V$ in $\Omega$, such that:

$$x \in U, \ y \in V, \ U \cap V = \emptyset$$

Exercise 4. Let $(\Omega, T)$ be a topological space.

1. Show that if $(\Omega, T)$ is Hausdorff and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', T|_{\Omega'})$ is itself Hausdorff.

2. Show that if $(\Omega, T)$ is metrizable, then it is Hausdorff.

3. Show that any subset of $\mathbb{R}$ is Hausdorff.

4. Let $(\Omega_i, T_i)_{i \in I}$ be a family of Hausdorff topological spaces. Show that the product topological space $\Pi_{i \in I} \Omega_i$ is Hausdorff.

Exercise 5. Let $(\Omega, T)$ be a Hausdorff topological space. Let $K$ be a compact subset of $\Omega$ and suppose there exists $y \in K^c$. 

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1. Show that for all \( x \in K \), there are open sets \( V_x, W_x \) in \( \Omega \), such that \( y \in V_x, x \in W_x \) and \( V_x \cap W_x = \emptyset \).

2. Show that there exists a finite subset \( \{ x_1, \ldots, x_n \} \) of \( K \) such that \( K \subseteq W^y \) where \( W^y = W_{x_1} \cup \ldots \cup W_{x_n} \).

3. Let \( V^y = V_{x_1} \cap \ldots \cap V_{x_n} \). Show that \( V^y \) is open and \( V^y \cap W^y = \emptyset \).

4. Show that \( y \in V^y \subseteq K^c \).

5. Show that \( K^c = \cup_{y \in K} V^y \)

6. Show that \( K \) is closed in \( \Omega \).

**Theorem 35** Let \( (\Omega, T) \) be a Hausdorff topological space. For all \( K \subseteq \Omega \), if \( K \) is a compact subset, then it is closed.
Definition 68 Let $(E,d)$ be a metric space. For all $A \subseteq E$, we call \textbf{diameter} of $A$ with respect to $d$, the element of $\mathbb{R}$ denoted $\delta(A)$, defined as $\delta(A) = \sup\{d(x,y) : x, y \in A\}$, with the convention that $\delta(\emptyset) = -\infty$.

Definition 69 Let $(E,d)$ be a metric space, and $A \subseteq E$. We say that $A$ is \textbf{bounded}, if and only if $\delta(A) < +\infty$.

Exercise 6. Let $(E,d)$ be a metric space. Let $A \subseteq E$.

1. Show that $\delta(A) = 0$ if and only if $A = \{x\}$ for some $x \in E$.

2. Let $\phi : \mathbb{R} \to ]-1,1[$ be an increasing homeomorphism. Define $d''(x,y) = |x-y|$ and $d'(x,y) = |\phi(x) - \phi(y)|$, for all $x, y \in \mathbb{R}$. Show that $d'$ is a metric on $\mathbb{R}$ inducing the usual topology on $\mathbb{R}$. Show that $\mathbb{R}$ is bounded with respect to $d'$ but not with respect to $d''$. 

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3. Show that if $K \subseteq E$ is a compact subset of $E$, for all $\epsilon > 0$, there is a finite subset $\{x_1, \ldots, x_n\}$ of $K$ such that:
$$K \subseteq B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$$

4. Show that any compact subset of any metrizable topological space $(\Omega, T)$, is bounded with respect to any metric inducing the topology $T$.

**Exercise 7.** Suppose $K$ is a closed subset of $\mathbb{R}$ which is bounded with respect to the usual metric on $\mathbb{R}$.

1. Show that there exists $M \in \mathbb{R}^+$ such that $K \subseteq [-M, M]$.
2. Show that $K$ is also closed in $[-M, M]$.
3. Show that $K$ is a compact subset of $[-M, M]$.
4. Show that $K$ is a compact subset of $\mathbb{R}$.

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5. Show that any compact subset of $\mathbb{R}$ is closed and bounded.

6. Show the following:

**Theorem 36** A subset of $\mathbb{R}$ is compact if and only if it is closed, and bounded with respect to the usual metric on $\mathbb{R}$.

**Exercise 8.** Let $(\Omega, T)$ and $(S, T_S)$ be two topological spaces. Let $f : (\Omega, T) \to (S, T_S)$ be a continuous map.

1. Show that if $(W_i)_{i \in I}$ is an open covering of $f(\Omega)$, then the family $(f^{-1}(W_i))_{i \in I}$ is an open covering of $\Omega$.

2. Show that if $(\Omega, T)$ is a compact topological space, then $f(\Omega)$ is a compact subset of $(S, T_S)$.
Exercise 9.

1. Show that \((\bar{\mathbb{R}}, \mathcal{T}_{\bar{\mathbb{R}}})\) is a compact topological space.

2. Show that any compact subset of \(\mathbb{R}\) is a compact subset of \(\bar{\mathbb{R}}\).

3. Show that a subset of \(\bar{\mathbb{R}}\) is compact if and only if it is closed.

4. Let \(A\) be a non-empty subset of \(\bar{\mathbb{R}}\), and let \(\alpha = \sup A\). Show that if \(\alpha \neq -\infty\), then for all \(U \in \mathcal{T}_{\bar{\mathbb{R}}}\) with \(\alpha \in U\), there exists \(\beta \in \mathbb{R}\) with \(\beta < \alpha\) and \([\beta, \alpha] \subseteq U\). Conclude that \(\alpha \in \bar{A}\).

5. Show that if \(A\) is a non-empty closed subset of \(\bar{\mathbb{R}}\), then we have \(\sup A \in A\) and \(\inf A \in A\).

6. Consider \(A = \{x \in \mathbb{R} \mid \sin(x) = 0\}\). Show that \(A\) is closed in \(\mathbb{R}\), but that \(\sup A \notin A\) and \(\inf A \notin \bar{A}\).

7. Show that if \(A\) is a non-empty, closed and bounded subset of \(\mathbb{R}\), then \(\sup A \in \bar{A}\) and \(\inf A \in \bar{A}\).
**Exercise 10.** Let $(\Omega, T)$ be a compact, non-empty topological space. Let $f : (\Omega, T) \to (\bar{\mathbb{R}}, \mathcal{T}_{\bar{\mathbb{R}}})$ be a continuous map.

1. Show that if $f(\Omega) \subseteq \mathbb{R}$, the continuity of $f$ with respect to $\mathcal{T}_{\bar{\mathbb{R}}}$ is equivalent to the continuity of $f$ with respect to $\mathcal{T}_{\mathbb{R}}$.

2. Show the following:

**Theorem 37** Let $f : (\Omega, T) \to (\bar{\mathbb{R}}, \mathcal{T}_{\bar{\mathbb{R}}})$ be a continuous map, where $(\Omega, T)$ is a non-empty topological space. Then, if $(\Omega, T)$ is compact, $f$ attains its maximum and minimum, i.e. there exist $x_m, x_M \in \Omega$, such that:

$$f(x_m) = \inf_{x \in \Omega} f(x), \quad f(x_M) = \sup_{x \in \Omega} f(x)$$

**Exercise 11.** Let $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and differentiable on $(a, b)$, with $f(a) = f(b)$. 

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1. Show that if \( c \in [a, b] \) and \( f(c) = \sup_{x \in [a, b]} f(x) \), then \( f'(c) = 0 \).

2. Show the following:

**Theorem 38 (Rolle)** Let \( a, b \in \mathbb{R}, a < b \). Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), and differentiable on \([a, b]\), with \( f(a) = f(b) \). Then, there exists \( c \in [a, b] \) such that \( f'(c) = 0 \).

**Exercise 12.** Let \( a, b \in \mathbb{R}, a < b \). Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \([a, b]\). Define:

\[
h(x) \triangleq f(x) - (x - a) \frac{f(b) - f(a)}{b - a}
\]

1. Show that \( h \) is continuous on \([a, b]\) and differentiable on \([a, b]\).

2. Show the existence of \( c \in [a, b] \) such that:

\[
f(b) - f(a) = (b - a)f'(c)
\]
Exercise 13. Let \( a, b \in \mathbb{R}, \ a < b \). Let \( f : [a, b] \to \mathbb{R} \) be a map. Let \( n \geq 0 \). We assume that \( f \) is of class \( C^n \) on \( [a, b] \), and that \( f^{(n+1)} \) exists on \( ]a, b[ \). Define:

\[
    h(x) \triangleq f(b) - f(x) - \sum_{k=1}^{n} \frac{(b-x)^k}{k!} f^{(k)}(x) - \alpha \frac{(b-x)^{n+1}}{(n+1)!} - \alpha (b-x)^{n+1}
\]

where \( \alpha \) is chosen such that \( h(a) = 0 \).

1. Show that \( h \) is continuous on \( [a, b] \) and differentiable on \( ]a, b[ \).

2. Show that for all \( x \in ]a, b[ \):

\[
    h'(x) = \frac{(b-x)^n}{n!} (\alpha - f^{(n+1)}(x))
\]

3. Prove the following:
Theorem 39 (Taylor-Lagrange) Let $a, b \in \mathbb{R}$, $a < b$, and $n \geq 0$. Let $f : [a, b] \to \mathbb{R}$ be a map of class $C^n$ on $[a, b]$ such that $f^{(n+1)}$ exists on $]a, b[$. Then, there exists $c \in ]a, b[$ such that:

$$f(b) - f(a) = \sum_{k=1}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Exercise 14. Let $a, b \in \overline{\mathbb{R}}$, $a < b$ and $\phi : ]a, b[\to \mathbb{R}$ be differentiable.

1. Show that if $\phi$ is convex, then for all $x, y \in ]a, b[,$ $x < y,$ we have:

$$\phi'(x) \leq \phi'(y)$$

2. Show that if $x, y, z \in ]a, b[,$ with $x < y < z,$ there are $c_1, c_2 \in ]a, b[,$ with $c_1 < c_2$ and:

$$\phi(y) - \phi(x) = \phi'(c_1)(y - x)$$

$$\phi(z) - \phi(y) = \phi'(c_2)(z - y)$$

3. Show conversely that if $\phi'$ is non-decreasing, then $\phi$ is convex.
4. Show that $x \to e^x$ is convex on $\mathbb{R}$.

5. Show that $x \to -\ln(x)$ is convex on $]0, +\infty[.$

**Definition 70** we say that a finite measure space $(\Omega, \mathcal{F}, P)$ is a probability space, if and only if $P(\Omega) = 1$.

**Definition 71** Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $(S, \Sigma)$ be a measurable space. We call random variable w.r. to $(S, \Sigma)$, any measurable map $X : (\Omega, \mathcal{F}) \to (S, \Sigma)$.

**Definition 72** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X$ be a non-negative random variable, or an element of $L^1_C(\Omega, \mathcal{F}, P)$. We call expectation of $X$, denoted $E[X]$, the integral:

$$E[X] \triangleq \int_{\Omega} X dP$$
**Exercise 15.** Let \( a, b \in \bar{\mathbb{R}}, a < b \) and \( \phi : ]a, b[ \to \mathbb{R} \) be a convex map. Let \((\Omega, \mathcal{F}, P)\) be a probability space and \( X \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, P) \) be such that \( X(\Omega) \subseteq ]a, b[ \).

1. Show that \( \phi \circ X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is measurable.
2. Show that \( \phi \circ X \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, P) \), if and only if \( E[|\phi \circ X|] < +\infty \).
3. Show that if \( E[X] = a \), then \( a \in \mathbb{R} \) and \( X = a \) \( P \)-a.s.
4. Show that if \( E[X] = b \), then \( b \in \mathbb{R} \) and \( X = b \) \( P \)-a.s.
5. Let \( m = E[X] \). Show that \( m \in ]a, b[ \).
6. Define:

\[
\beta \triangleq \sup_{x \in ]a,m[} \frac{\phi(m) - \phi(x)}{m - x}
\]

Show that \( \beta \in \mathbb{R} \) and that for all \( z \in ]m, b[ \), we have:

\[
\beta \leq \frac{\phi(z) - \phi(m)}{z - m}
\]
7. Show that for all $x \in [a, b]$, we have $\phi(m) + \beta(x - m) \leq \phi(x)$.

8. Show that for all $\omega \in \Omega$, $\phi(m) + \beta(X(\omega) - m) \leq \phi(X(\omega))$.

9. Show that if $\phi \circ X \in L^1(\Omega, \mathcal{F}, P)$ then $\phi(m) \leq E[\phi \circ X]$.

**Theorem 40 (Jensen inequality)** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $a, b \in \mathbb{R}$, $a < b$ and $\phi : [a, b] \to \mathbb{R}$ be a convex map. Suppose that $X \in L^1(\Omega, \mathcal{F}, P)$ is such that $X(\Omega) \subseteq [a, b]$ and such that $\phi \circ X \in L^1(\Omega, \mathcal{F}, P)$. Then:

$$
\phi(E[X]) \leq E[\phi \circ X]
$$
Solutions to Exercises

Exercise 1.

1. Let $\phi : ]a, b[ \to \mathbb{R}$ be convex. Given $n \geq 1$, let $H_n$ be the property that for all $x_1, \ldots, x_n$ in $]a, b[$, and $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}^+$ such that $\alpha_1 + \ldots + \alpha_n = 1$, we have:

$$\phi(\alpha_1 x_1 + \ldots + \alpha_n x_n) \leq \alpha_1 \phi(x_1) + \ldots + \alpha_n \phi(x_n) \quad (1)$$

$H_1$ is obviously true. Since $\phi$ is convex, $H_2$ is also true. Given $n \geq 3$, suppose that $H_{n-1}$ has been proved. Let $x_1, \ldots, x_n$ in $]a, b[$ and $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}^+$ be such that $\alpha_1 + \ldots + \alpha_n = 1$. Define $t = \alpha_1 + \ldots + \alpha_{n-1}$. If $t = 0$, then $\alpha_i = 0$ for all $i \in \{1, \ldots, n-1\}$, and $\alpha_n = 1$. So (1) is clearly satisfied. Suppose $t \neq 0$. From our induction hypothesis $H_{n-1}$, we obtain:

$$\phi((\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1})/t) \leq (\alpha_1 \phi(x_1) + \ldots + \alpha_{n-1} \phi(x_{n-1}))/t$$

i.e. $t \phi(x) \leq \alpha_1 \phi(x_1) + \ldots + \alpha_{n-1} \phi(x_{n-1})$, where $x$ has been defined as $x = (\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1})/t$. Note that $x$ is an
element of $[a, b]$. Let $y = x_n$. Since by assumption, $\phi$ is convex and $t \in [0, 1]$, we have:

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

and thus:

$$\phi(tx + (1-t)y) \leq \alpha_1\phi(x_1) + \ldots + \alpha_{n-1}\phi(x_{n-1}) + (1-t)\phi(y)$$

Since $1-t = \alpha_n$, we see that (1) is therefore satisfied, which proves that $H_n$ is true. This induction argument shows that $H_n$ is true for all $n \geq 1$, whenever $\phi$ is convex. Conversely, if $H_n$ is true for all $n \geq 1$, then in particular $H_2$ is true, and $\phi$ is immediately convex.

2. Let $\phi: [a, b] \rightarrow \mathbb{R}$ be convex, and $x, y, z$ with $a < x < y < z < b$. Let $t = (z-y)/(z-x)$. Then $t \in [0, 1]$ and $1-t = (y-x)/(z-x)$. Moreover, we have $y = tx + (1-t)z$. $\phi$ being convex, we obtain:

$$\phi(y) \leq \frac{z-y}{z-x}\phi(x) + \frac{y-x}{z-x}\phi(z)$$

(2)
Conversely, suppose $\phi : [a, b] \rightarrow \mathbb{R}$ is a map such that (2) holds for all $x, y, z$ with $a < x < y < z < b$. Let $x, z \in [a, b]$ and $t \in [0, 1]$. Without loss of generality, we can assume that $x \leq z$. If $t = 0$, $t = 1$, or $x = z$, then we immediately have:

$$\phi(tx + (1-t)z) \leq t\phi(x) + (1-t)\phi(z)$$

(3)

Assume that $x < z$ and $t \in [0, 1]$. Define $y = tx + (1-t)z$. Then, $x < y < z$. Moreover, it is easy to check that $(z-y)/(z-x) = t$ and $(y-x)/(z-x) = 1-t$. From (2), we conclude that (3) is also satisfied. Hence, we see that $\phi$ is convex. We have proved that a map $\phi : [a, b] \rightarrow \mathbb{R}$ is convex, if and only if inequality (2) holds, whenever $a < x < y < z < b$.

3. From the previous question, $\phi : [a, b] \rightarrow \mathbb{R}$ is convex, if and only if for all $x, y, z$ with $a < x < y < z < b$, we have:

$$\phi(y) \leq \frac{z-y}{z-x}\phi(x) + \frac{y-x}{z-x}\phi(z)$$
which is equivalent to:

\[
\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(y)}{z - y}
\]  

(4)

4. Let \( \phi : ]a, b[ \to \mathbb{R} \) be convex. Let \( x_0 \in ]a, b[ \) and \( u, u', v, v' \) in \( ]a, b[ \) such that \( u < u' < x_0 < v < v' \). Let \( x \in ]x_0, v[ \). Using inequality (4), we obtain:

\[
\frac{\phi(u') - \phi(u)}{u' - u} \leq \frac{\phi(x_0) - \phi(u')}{x_0 - u'} \leq \frac{\phi(x) - \phi(x_0)}{x - x_0}
\]

and furthermore:

\[
\frac{\phi(x) - \phi(x_0)}{x - x_0} \leq \frac{\phi(v) - \phi(x)}{v - x} \leq \frac{\phi(v') - \phi(v)}{v' - v}
\]

So, in particular:

\[
\frac{\phi(u') - \phi(u)}{u' - u} \leq \frac{\phi(x) - \phi(x_0)}{x - x_0} \leq \frac{\phi(v') - \phi(v)}{v' - v}
\]
It follows that there exist $\alpha, \beta \in \mathbb{R}$, such that for all $x \in [x_0, v]$

$$\alpha(x - x_0) \leq \phi(x) - \phi(x_0) \leq \beta(x - x_0)$$

We conclude that the right-hand limit, $\lim_{x \uparrow x_0} \phi(x)$ exists, and is equal to $\phi(x_0)$.

5. Similarly to 4., for all $x \in (u', x_0]$, we have:

$$\frac{\phi(u') - \phi(u)}{u' - u} \leq \frac{\phi(x_0) - \phi(x)}{x_0 - x} \leq \frac{\phi(v') - \phi(v)}{v' - v}$$

So there exist $\alpha, \beta \in \mathbb{R}$, such that for all $x \in (u', x_0]$

$$\alpha(x_0 - x) \leq \phi(x_0) - \phi(x) \leq \beta(x_0 - x)$$

We conclude that the left-hand limit, $\lim_{x \downarrow x_0} \phi(x)$ exists, and is equal to $\phi(x_0)$. Finally, from:

$$\lim_{x \downarrow x_0} \phi(x) = \phi(x_0) = \lim_{x \uparrow x_0} \phi(x)$$

$\phi$ is continuous on $x_0$. This being true for all $x_0 \in (a, b]$, we have proved that $\phi : (a, b] \to \mathbb{R}$ is a continuous map.
6. Let $\phi : [0, 1] \to \mathbb{R}$ be defined by $\phi(0) = 1$, and $\phi(x) = 0$ for all $x \in [0, 1]$. The fact that:
$$\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y)$$
for all $t, x, y \in [0, 1]$, is clear. Yet, $\phi$ obviously fails to be continuous on $[0, 1]$. The purpose of this question is to emphasize an important point: in definition (64), we have restricted a convex function to be defined on some open interval $]a, b[$ (it needs to be an interval, as $\phi(tx + (1 - t)y)$ needs to be meaningful). If instead, we had allowed a convex function to be defined on some closed interval $[a, b]$ , it would not necessarily be continuous.

Exercise 1
Exercise 2.

1. Let \((\Omega, \mathcal{T})\) be a compact topological space. The induced topological space \((\Omega, \mathcal{T}|_\Omega)\) is nothing but \((\Omega, \mathcal{T})\) itself. So \((\Omega, \mathcal{T}|_\Omega)\) is compact, and \(\Omega\) is therefore a compact subset of itself.

2. The induced topology \(\mathcal{T}|_{\emptyset}\) is defined by \(\mathcal{T}|_{\emptyset} = \{A \cap \emptyset : A \in \mathcal{T}\}\). So \(\mathcal{T}|_{\emptyset} = \{\emptyset\}\). The topological space \((\emptyset, \{\emptyset\})\) being compact, we see that \(\emptyset\) is a compact subset of \(\Omega\).

3. Let \((\Omega, \mathcal{T})\) be a topological space and \(\Omega' \subseteq \Omega\). Let \(K\) be a compact subset of \(\Omega'\). Then \(K \subseteq \Omega'\), and the topological space \((K, (\mathcal{T}|_{\Omega'})|_K)\) is compact. However, the induced topology \((\mathcal{T}|_{\Omega'})|_K\) coincide with the induced topology \(\mathcal{T}|_K\). It follows that \((K, \mathcal{T}|_K)\) is a compact topological space, and \(K\) is therefore a compact subset of \(\Omega\).

4. Let \((V_i)_{i \in I}\) be a family of open sets in \(\Omega\), such that \(K \subseteq \bigcup_{i \in I} V_i\). If \(x \in K\), then \(x \in V_i \cap K\) for some \(i \in I\). Conversely, if
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$x \in V_i \cap K$ for some $i \in I$, then $x \in K$. So $K = \cup_{i \in I} V_i \cap K$. By definition (23) of the induced topology, each $V_i \cap K$ is an element of $\mathcal{T}_{|K}$, i.e. each $V_i \cap K$ is open in $K$.

5. Let $(\Omega, T)$ be a topological space, and $K \subseteq \Omega$. Suppose $K$ is a compact subset of $\Omega$. Let $(V_i)_{i \in I}$ be a family of open sets in $\Omega$, such that $K \subseteq \cup_{i \in I} V_i$. From 4., $K = \cup_{i \in I} V_i \cap K$, and each $V_i \cap K$ is an open set in $K$. By assumption, the topological space $(K, \mathcal{T}_{|K})$ is compact. From definition (65), it follows that there exists $\{i_1, \ldots, i_n\}$ finite subset of $I$, such that:

$$K = (V_{i_1} \cap K) \cup \ldots \cup (V_{i_n} \cap K) = (V_{i_1} \cup \ldots \cup V_{i_n}) \cap K$$

In particular, $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. Conversely, suppose that $K \subseteq \Omega$ has the property that for any family $(V_i)_{i \in I}$ of open sets in $\Omega$, such that $K \subseteq \cup_{i \in I} V_i$, there exists $\{i_1, \ldots, i_n\}$ finite subset of $I$ such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. We claim that $K$ is a compact subset of $\Omega$. Indeed, let $(W_i)_{i \in I}$ be a family of open sets in $K$ such that $K = \cup_{i \in I} W_i$. Since each $W_i$ lies in $\mathcal{T}_{|K}$, for all $i \in I$.
there exists \( V_i \in T \) such that \( W_i = V_i \cap K \). So \( K = \bigcup_{i \in I} V_i \cap K \), and in particular \( K \subseteq \bigcup_{i \in I} V_i \). By assumption, there exists \( \{i_1, \ldots, i_n\} \) finite subset of \( I \), such that \( K = V_{i_1} \cup \ldots \cup V_{i_n} \), and therefore \( K = (V_{i_1} \cup \ldots \cup V_{i_n}) \cap K = W_{i_1} \cup \ldots \cup W_{i_n} \). From definition (65), we conclude that \((K,T_K)\) is compact, i.e. \( K \) is a compact subset of \( \Omega \). We have proved that \( K \subseteq \Omega \) is a compact subset of \( \Omega \), if and only if for any family \((V_i)_{i \in I}\) of open sets in \( \Omega \) such that \( K \subseteq \bigcup_{i \in I} V_i \), there exists \( \{i_1, \ldots, i_n\} \) finite subset of \( I \), such that \( K \subseteq V_{i_1} \cup \ldots \cup V_{i_n} \).

6. Let \((\Omega, T)\) be a compact topological space. Let \( K \subseteq \Omega \), and suppose that \( K \) is closed in \( \Omega \). Let \((V_i)_{i \in I}\) be a family of open sets in \( \Omega \), such that \( K \subseteq \bigcup_{i \in I} V_i \). For all \( x \in \Omega \), either \( x \in K^c \) or \( x \in V_i \) for some \( i \in I \) (or both). So \( \Omega = (\bigcup_{i \in I} V_i) \cup K^c \). Since \( K^c \) is assumed to be open in \( \Omega \), and \((\Omega, T)\) is compact, from definition (65), there exists \( \{i_1, \ldots, i_n\} \) finite subset of \( I \), such that \( \Omega = V_{i_1} \cup \ldots \cup V_{i_n} \), or \( \Omega = (V_{i_1} \cup \ldots \cup V_{i_n}) \cup K^c \). In any case, we have \( K \subseteq V_{i_1} \cup \ldots \cup V_{i_n} \). Hence, given a family \((V_i)_{i \in I}\)
of open sets in $\Omega$, such that $K \subseteq \bigcup_{i \in I} V_i$, we have found a finite subset $\{i_1, \ldots, i_n\}$ of $I$, such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. From 5., we conclude that $K$ is a compact subset of $\Omega$. We have proved that any closed subset of a compact topological space, is itself compact (is a compact subset of it).

Exercise 2
Exercise 3.

1. By assumption, \([a, b] \subseteq \bigcup_{i \in I} V_i\) and in particular, there exists \(i \in I\) such that \(a \in V_i\). So \(\{a\} = [a, a]\) can be covered by a finite number of \(V_i\)’s. We have proved that \(a \in A\).

2. Since \(a \in V_i\) for some \(i\), and \(V_i\) is open in \(\mathbb{R}\), there exists \(\epsilon > 0\) such that \([a, a + \epsilon] \subseteq V_i\). Since \(a < b\), by choosing \(\epsilon\) small enough, we can ensure that \(a + \epsilon \in [a, b]\). Hence, we have found \(\epsilon > 0\), such that \(a + \epsilon \in [a, b]\), and \([a, a + \epsilon]\) is covered by a finite number of \(V_i\)’s. So we have found \(\epsilon > 0\), such that \(a + \epsilon \in A\).

3. Since \(c = \sup A\), \(c\) is an upper-bound of \(A\). From 2., there exists \(\epsilon > 0\), such that \(a + \epsilon \in A\). So \(a + \epsilon \leq c\) and in particular, \(a < c\). By definition, \(A\) is a subset of \([a, b]\). So \(b\) is an upper-bound of \(A\). \(c\) being the smallest of such upper-bounds, we have \(c \leq b\). We have proved that \(a < c \leq b\).

4. From 3., \(c \in [a, b] \subseteq \bigcup_{i \in I} V_i\). There exists \(i_0 \in I\) with \(c \in V_{i_0}\). \(V_{i_0}\) being open in \(\mathbb{R}\), there exist \(c', c''\) such that \(c' < c < c''\) and...
therefore, since $c, x \in A$, we have $a, x \in A$. From $a, c \in [a, x]$, we conclude that $[a, c]$ can also be covered by a finite number of $V_i$'s.

7. Since $[a, b \wedge c'] \subseteq [a, c']$, it follows from 6. that $[a, b \wedge c']$ can be covered by a finite number of $V_i$'s. Moreover, since $b \wedge c' \in [a, b]$, we see that $b \wedge c' \in A$. Hence, we have $b \wedge c' \leq c$. We know from 3. that $c \leq b$. Suppose we had $c < b$. Since $c < c'$, this would imply that $c < b \wedge c'$, which is a contradiction. It follows
that $b = c$.

8. From 7., we have $[a, b] = [a, c] \subseteq [a, c']$. From 6., $[a, c']$ can be covered by a finite number of $V_i$'s. It follows that $[a, b]$ can also be covered by a finite number of $V_i$'s. In other words, there exists a finite subset $\{i_1, \ldots, i_n\}$ of $I$, such that $[a, b] \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. Having assumed that $[a, b] \subseteq \bigcup_{i \in I} V_i$, for an arbitrary family $(V_i)_{i \in I}$ of open sets in $\mathbb{R}$, we have shown the existence of a finite subset $\{i_1, \ldots, i_n\}$ of $I$, such that $[a, b] \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. From exercise (2), we see that $[a, b]$ is a compact subset of $\mathbb{R}$.

Exercise 3
Exercise 4.

1. Let $(\Omega, T)$ be a Hausdorff topological space, and $\Omega' \subseteq \Omega$. Let $x, y \in \Omega'$ with $x \neq y$. In particular, $x, y \in \Omega$ with $x \neq y$. Since $(\Omega, T)$ is Hausdorff, there exist two open sets $U, V$ in $\Omega$, such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Define $U' = U \cap \Omega'$ and $V' = V \cap \Omega'$. Then $U'$ and $V'$ are elements of the induced topology $T|_{\Omega'}$ and furthermore, we have $x \in U'$, $y \in V'$ and $U' \cap V' = \emptyset$. Given two distinct elements $x, y$ of $\Omega'$, we have found two disjoint open sets $U'$, $V'$ in $\Omega'$, containing $x$ and $y$ respectively. This shows that the induced topological space $(\Omega', T|_{\Omega'})$ is Hausdorff.

2. Let $(\Omega, T)$ be a metrizable topological space. Let $d$ be a metric on $\Omega$, inducing the topology $T$ on $\Omega$. Let $x, y \in \Omega$ with $x \neq y$. Define $\epsilon = d(x, y)/2 > 0$, $U = B(x, \epsilon)$ and $V = B(y, \epsilon)$. Then, $U, V$ are open sets in $\Omega$, with $x \in U$ and $y \in V$. Furthermore,
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if \( z \in B(x, \epsilon) \), then \( d(x, z) < d(x, y)/2 \) and consequently:

\[
d(x, y) \leq d(x, z) + d(z, y) < d(x, y)/2 + d(z, y)
\]

from which we see that \( d(z, y) > d(x, y)/2 = \epsilon \). So \( z \notin B(y, \epsilon) \), and we have proved that \( U \cap V = \emptyset \). Given two distinct elements \( x, y \) of \( \Omega \), we have found two disjoint open sets \( U, V \) in \( \Omega \), containing \( x \) and \( y \) respectively. This shows that the metrizable topological space \((\Omega, T)\) is Hausdorff.

3. From theorem (13), the topological space \((\bar{\mathbb{R}}, T_{\bar{\mathbb{R}}})\) is metrizable. It follows from 2. that \((\mathbb{R}, T_\mathbb{R})\) is Hausdorff. From 1., any subset of \( \mathbb{R} \) (together with its induced topology) is a Hausdorff topological space.

4. Let \((\Omega_i, T_i)_{i \in I}\) be a family of Hausdorff topological spaces. Let \( \Omega = \Pi_{i \in I} \Omega_i \) and \( T = \bigotimes_{i \in I} T_i \) be the product topology on \( \Omega \) [definition (56)]. Let \( x, y \in \Omega \) with \( x \neq y \). There exists \( i_0 \in I \) such that \( x(i_0) \neq y(i_0) \). Since \((\Omega_{i_0}, T_{i_0})\) is Hausdorff, there exist \( U_{i_0}, V_{i_0} \) open sets in \( \Omega_{i_0} \), such that \( x(i_0) \in U_{i_0} \),

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$y(i_0) \in V_{i_0}$ and $U_{i_0} \cap V_{i_0} = \emptyset$. Define $U = U_{i_0} \times \Pi_{i \in I \setminus \{i_0\}} \Omega_i$ and $V = V_{i_0} \times \Pi_{i \in I \setminus \{i_0\}} \Omega_i$. Then $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Furthermore, $U$ and $V$ are rectangles of the family of topologies $(T_i)_{i \in I}$ [definition (52)], and therefore belong to the product topology $\bigotimes_{i \in I} T_i = T$. Given two distinct elements $x, y$ in $\Omega$, we have found two disjoint open sets $U, V$ in $\Omega$, containing $x$ and $y$ respectively. This shows that the product topological space $(\Omega, T)$ is Hausdorff.

Exercise 4
Exercise 5.

1. Let \( x \in K \). Since by assumption, \( y \in K^c \), we have \( x \neq y \). The topological space \((\Omega, T)\) being Hausdorff, there exist open sets \( V_x \) and \( W_x \) in \( \Omega \), such that \( y \in V_x \), \( x \in W_x \) and \( V_x \cap W_x = \emptyset \).

2. For all \( x \in K \), we have \( x \in W_x \). In particular, \( K \subseteq \bigcup_{x \in K} W_x \). \( K \) being a compact subset of \( \Omega \), and \((W_x)_{x \in K}\) being a family of open sets in \( \Omega \), there exists \( \{x_1, \ldots, x_n\} \) finite subset of \( K \), such that \( K \subseteq W_{x_1} \cup \ldots \cup W_{x_n} \), i.e. \( K \subseteq W^y = W_{x_1} \cup \ldots \cup W_{x_n} \).

3. Let \( V^y = V_{x_1} \cap \ldots \cap V_{x_n} \). All \( V_x \)'s being open in \( \Omega \), \( V^y \) is a finite intersection of open sets in \( \Omega \), and is therefore open in \( \Omega \). Suppose that \( x \in V^y \cap W^y \). Then, there exists \( i \in \{1, \ldots, n\} \) such that \( x \in W_{x_i} \). Since \( V^y \subseteq V_{x_i} \), we see that \( x \in W_{x_i} \cap V_{x_i} \), which contradicts that fact that \( W_{x_i} \cap V_{x_i} = \emptyset \). It follows that \( V^y \cap W^y = \emptyset \).

4. By construction, \( y \in V_{x_i} \) for all \( i \in \{1, \ldots, n\} \). It follows that \( y \in V_{x_1} \cap \ldots \cap V_{x_n} = V^y \). Furthermore from 2., \( K \subseteq W^y \) and

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from 3., \( V^y \cap W^y = \emptyset \). It follows that for all \( x \in V^y \), \( x \not\in K \). So \( V^y \subseteq K^c \). We have proved that \( y \in V^y \subseteq K^c \).

5. So far, for all \( y \in K^c \), we have shown the existence of an open set \( V^y \) in \( \Omega \), such that \( y \in V^y \subseteq K^c \). It is clear that \( \cup_{y \in K^c} V^y \subseteq K^c \). Conversely, for all \( y \in K^c \), we have \( y \in V^y \). So \( K^c \subseteq \cup_{y \in K^c} V^y \). We have proved that \( K^c = \cup_{y \in K^c} V^y \).

6. From 5., \( K^c \) is a union of open sets in \( \Omega \), and is therefore open in \( \Omega \). We conclude that \( K \) is a closed subset of \( \Omega \). The purpose of this exercise is to prove theorem (35).

Exercise 5
Exercise 6.

1. Suppose $A = \{x\}$ for some $x \in E$. Then $\delta(A) = \sup\{0\} = 0$.
   Conversely, suppose $\delta(A) = 0$. Then $A \neq \emptyset$, since otherwise we
   would have $\delta(A) = -\infty$. Suppose $A$ had two distinct elements
   $x$ and $y$. We would have $0 < d(x, y) \leq \delta(A)$, contradicting
   the assumption that $\delta(A) = 0$. It follows that $A$ has only one
   element. We have proved that $\delta(A) = 0$, if and only if $A = \{x\}$
   for some $x \in E$.

2. Let $\phi : \mathbb{R} \to ]-1,1[$ be an increasing homeomorphism. Let
   $d'(x, y) = |\phi(x) - \phi(y)|$. Since $\phi$ is injective, $d'(x, y) = 0$ is
   equivalent to $x = y$. So $d'$ is clearly a metric on $\mathbb{R}$. Let $A$
   be open for the usual topology on $\mathbb{R}$, i.e. $A \in \mathcal{T}_\mathbb{R}$. $\phi$
   being a homeomorphism, $\phi^{-1}$ is continuous, and therefore $\phi(A)$
   is open in $]-1,1[$. It follows that $\phi(A)$ is also open in $\mathbb{R}$. Let
   $x \in A$. Then $\phi(x) \in \phi(A)$, and there exists $\epsilon > 0$ such that
   $|\phi(x) - z| < \epsilon \Rightarrow z \in \phi(A)$. Let $y \in \mathbb{R}$ be such that
   $d'(x, y) < \epsilon$. Then $|\phi(x) - \phi(y)| < \epsilon$ and therefore $\phi(y) \in \phi(A)$. $\phi$ being
injection, we see that $y \in A$. We have found $\epsilon > 0$, such that $d'(x, y) < \epsilon \Rightarrow y \in A$. This shows that $A$ is open with respect to the metric topology induced by $d'$, i.e. $A \in T_{d'}$. This being true for all $A \in T_{\mathbb{R}}$, we have $T_{\mathbb{R}} \subseteq T_{d'}$. Conversely, let $A \in T_{d'}$. Let $x \in A$. There exists $\epsilon > 0$, such that $d'(x, y) < \epsilon \Rightarrow y \in A$. However, $\phi$ being continuous, there exists $\eta > 0$, such that $|x - y| < \eta \Rightarrow d'(x, y) < \epsilon$. Hence, we see that $|x - y| < \eta \Rightarrow y \in A$. This shows that $A$ is open with respect to the usual topology on $\mathbb{R}$, i.e. $A \in T_{\mathbb{R}}$. This being true for all $A \in T_{d'}$, we have $T_{d'} \subseteq T_{\mathbb{R}}$, and finally $T_{d'} = T_{\mathbb{R}}$. We conclude that the metric $d'$ induces the usual topology on $\mathbb{R}$.

Let $\delta'(\mathbb{R})$ be the diameter of $\mathbb{R}$ with respect to the metric $d'$. For all $x, y \in \mathbb{R}$, we have $d'(x, y) \leq 2$. It follows that $\delta'(\mathbb{R}) \leq 2$ and in particular $\delta'(\mathbb{R}) < +\infty$. So $\mathbb{R}$ is bounded with respect to the metric $d'$. However, if $d''$ denotes the usual metric on $\mathbb{R}$, and $\delta''(\mathbb{R})$ the diameter of $\mathbb{R}$ with respect to $d''$, then it is clear that $\delta''(\mathbb{R}) = +\infty$. So $\mathbb{R}$ is not bounded with respect to the usual metric on $\mathbb{R}$.
3. Let $K$ be a compact subset of $E$. Let $\epsilon > 0$. We clearly have $K \subseteq \bigcup_{x \in K} B(x, \epsilon)$. The family $\{B(x, \epsilon)\}_{x \in K}$ being a family of open sets in $E$, from exercise (2), there exists $\{x_1, \ldots, x_n\}$ finite subset of $K$, such that $K \subseteq B(x_1, \epsilon) \cup \ldots \cup B(x_n, \epsilon)$.

4. Let $(\Omega, T)$ be a metrizable topological space. Let $d$ be an arbitrary metric inducing the topology $T$. Let $K$ be a compact subset of $\Omega$. Taking $\epsilon = 1$ in 3., there exists $\{x_1, \ldots, x_n\}$ finite subset of $K$, such that $K \subseteq B(x_1, 1) \cup \ldots \cup B(x_n, 1)$. Let $x, y \in K$. There exists $i, j \in \{1, \ldots, n\}$ such that $x \in B(x_i, 1)$ and $y \in B(x_j, 1)$. It follows that:

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \leq 2 + M$$

where $M = \max_{i,j} d(x_i, x_j)$. Hence, we see that $\delta(K) \leq 2 + M$, where $\delta(K)$ is the diameter of $K$ with respect to the metric $d$. In particular, $\delta(K) < +\infty$, and $K$ is bounded with respect to the metric $d$. This is true for all $d$ inducing $T$.

**Exercise 6**

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Exercise 7.

1. Since $K$ is bounded with respect to the usual metric on $\mathbb{R}$, we have $\delta(K) < +\infty$. If $K = \emptyset$, then $K \subseteq [-M, M]$ for any $M \in \mathbb{R}^+$. Suppose $K \neq \emptyset$. Then $\delta(K) \in \mathbb{R}^+$, and for all $x, y \in K$, we have $|x - y| \leq \delta(K)$. Let $y_0 \in K$. For all $x \in K$, we have $|x| \leq \delta(K) + |y_0|$. So $K \subseteq [-M, M]$, with $M = \delta(K) + |y_0|$.

2. Let $K'$ denote the complement of $K$ in $[-M, M]$. We have $K' = [-M, M] \cap K^c$, where $K^c$ is the complement of $K$ in $\mathbb{R}$. Since by assumption $K$ is closed in $\mathbb{R}$, $K^c$ is open in $\mathbb{R}$. It follows that $[-M, M] \cap K^c$ is open with respect to the induced topology on $[-M, M]$. So $K'$ is open in $[-M, M]$, and we conclude that $K$ is closed in $[-M, M]$.

3. From theorem (34), $[-M, M]$ is a compact subset of $\mathbb{R}$. From 2., $K$ is a closed subset of $[-M, M]$. From exercise (2)[6.], we conclude that $K$ is a compact subset of $[-M, M]$.

4. From 3., $K$ is a compact subset of $[-M, M]$. It follows from

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exercise (2)[3.], that \( K \) is also a compact subset of \( \mathbb{R} \). We have proved that any closed and bounded subset of \( \mathbb{R} \), is also a compact subset of \( \mathbb{R} \).

5. Let \( K \) be a compact subset of \( \mathbb{R} \). Since \( (\mathbb{R}, \mathcal{T}_\mathbb{R}) \) is Hausdorff, from theorem (35), \( K \) is a closed subset of \( \mathbb{R} \). Moreover, from exercise (6), \( K \) is bounded with respect to any metric inducing the usual topology on \( \mathbb{R} \). In particular, it is bounded with respect to the usual metric on \( \mathbb{R} \). We have proved that any compact subset of \( \mathbb{R} \) is closed and bounded.

6. From 4., any subset of \( \mathbb{R} \) which is closed and bounded, is compact. Conversely, from 5., any compact subset of \( \mathbb{R} \) is closed and bounded. This proves theorem (36).

Exercise 7

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Exercise 8.

1. Let \((W_i)_{i \in I}\) be an open covering of \(f(\Omega)\). For all \(i \in I\), \(W_i\) is open, and \(f(\Omega) \subseteq \bigcup_{i \in I} W_i\). Let \(x \in \Omega\). Then \(f(x) \in f(\Omega)\). There exists \(i \in I\), such that \(f(x) \in W_i\), i.e. \(x \in f^{-1}(W_i)\). It follows that \(\Omega \subseteq \bigcup_{i \in I} f^{-1}(W_i)\). Moreover, \(f\) being continuous and \(W_i\) open, each \(f^{-1}(W_i)\) is open in \(\Omega\). We have proved that \((f^{-1}(W_i))_{i \in I}\) is an open covering of \(\Omega\).

2. Let \(f: (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)\) be a continuous map, where \((\Omega, \mathcal{T})\) is a compact topological space. Let \((W_i)_{i \in I}\) be a family of open sets in \(S\), such that \(f(\Omega) \subseteq \bigcup_{i \in I} W_i\). From 1., \((f^{-1}(W_i))_{i \in I}\) is a family of open sets in \(\Omega\), such that \(\Omega \subseteq \bigcup_{i \in I} f^{-1}(W_i)\). \((\Omega, \mathcal{T})\) being compact, there exists \(\{i_1, \ldots, i_n\}\) finite subset of \(I\), such that \(\Omega \subseteq f^{-1}(W_{i_1}) \cup \ldots \cup f^{-1}(W_{i_n})\). Let \(y \in f(\Omega)\). There exists \(x \in \Omega\), such that \(y = f(x)\). There exists \(k \in \{1, \ldots, n\}\), such that \(x \in f^{-1}(W_{i_k})\), i.e. \(f(x) \in W_{i_k}\). So \(y \in W_{i_k}\). We have proved that \(f(\Omega) \subseteq W_{i_k} \cup \ldots \cup W_{i_n}\). Given an arbitrary family \((W_i)_{i \in I}\) of open sets, such that \(f(\Omega) \subseteq \bigcup_{i \in I} W_i\), we have found a
finite subset \(\{i_1, \ldots, i_n\}\) of \(I\), such that \(f(\Omega) \subseteq W_{i_1} \cup \ldots \cup W_{i_n}\).

This shows that \(f(\Omega)\) is a compact subset of \((S, T_S)\).

Exercise 8
Exercise 9.

1. By construction, the topological space $(\bar{\mathbb{R}}, T_{\bar{\mathbb{R}}})$ is homeomorphic to $[-1,1]$ [definition (34)]. In particular, there exists a continuous map $h : [-1,1] \to \bar{\mathbb{R}}$. From theorem (34), the topological space $[-1,1]$ is compact. From exercise (8), we conclude that $\bar{\mathbb{R}} = h([-1,1])$ is a compact subset of $(\mathbb{R}, T_{\mathbb{R}})$. In other words, $(\mathbb{R}, T_{\mathbb{R}})$ is a compact topological space.

2. Let $K$ be a compact subset of $\mathbb{R}$. The usual topology $T_{\mathbb{R}}$ on $\mathbb{R}$ is nothing but the topology induced on $\mathbb{R}$, by the usual topology on $\bar{\mathbb{R}}$, i.e. $T_{\mathbb{R}} = (T_{\bar{\mathbb{R}}})_{\mathbb{R}}$. From exercise (2)[3], we conclude that $K$ is also a compact subset of $\bar{\mathbb{R}}$.

3. Let $K$ be a compact subset of $\bar{\mathbb{R}}$. Since $(\bar{\mathbb{R}}, T_{\bar{\mathbb{R}}})$ is metrizable, it is a Hausdorff topological space. It follows from theorem (35) that $K$ is closed in $\bar{\mathbb{R}}$. Conversely, suppose $K$ is a closed subset of $\mathbb{R}$. From 1., $(\mathbb{R}, T_{\mathbb{R}})$ is compact. We conclude from exercise (2)[6], that $K$ is a compact subset of $\bar{\mathbb{R}}$. 

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4. Let $A$ be a non-empty subset of $\overline{\mathbb{R}}$, and $\alpha = \sup A$. We assume that $\alpha \neq -\infty$ (i.e. $A$ is not reduced to $\{-\infty\}$). Let $U \in \mathcal{T}_{\overline{\mathbb{R}}}$ with $\alpha \in U$. Let $h : \mathbb{R} \to [-1, 1]$ be an increasing homeomorphism. Then, $h(U)$ is open in $[-1, 1]$, and $h(\alpha) \in h(U)$. Since $\alpha \neq -\infty$, we have $h(\alpha) \neq -1$. There exists $\epsilon > 0$, such that we have $[h(\alpha) - \epsilon, h(\alpha)] \subseteq h(U)$, together with $-1 < h(\alpha) - \epsilon$. It follows that $[\beta, \alpha] \subseteq U$, where $\beta = h^{-1}(h(\alpha) - \epsilon) \in \mathbb{R}$. Let $\bar{A}$ be the closure of $A$ in $\overline{\mathbb{R}}$ [definition 37]. If $\alpha = -\infty$, since $A \neq \emptyset$, we have $A = \{-\infty\}$. So $\alpha \in A \subseteq \bar{A}$. Suppose that $\alpha \neq -\infty$. We claim that $\alpha \in \bar{A}$. Let $U \in \mathcal{T}_{\overline{\mathbb{R}}}$ be such that $\alpha \in U$. As shown above, there exists $\beta < \alpha$, $\beta \in \mathbb{R}$, such that $[\beta, \alpha] \subseteq U$. $\alpha$ being the supremum of $A$, its is the smallest of all upper-bounds of $A$. Hence, $\beta$ cannot be such upper-bound, and there exists $c \in A$ such that $c \in [\beta, \alpha] \subseteq U$. Hence, we see that $A \cap U \neq \emptyset$. This being true for all open sets $U$ in $\mathbb{R}$ containing $\alpha$, we have proved that $\alpha \in \bar{A}$. We conclude that for any non-empty subset $A$ of $\overline{\mathbb{R}}$, we have $\alpha = \sup A \in \bar{A}$.
5. Let $A$ be a non-empty closed subset of $\bar{\mathbb{R}}$. From 4., we have $\sup A \in A$, and similarly $\inf A \in A$. $A$ being closed in $\mathbb{R}$, it coincides with its closure in $\bar{\mathbb{R}}$, i.e. $A = \bar{A}$. So $\sup A \in A$ and $\inf A \in A$. Any non-empty closed subset of $\bar{\mathbb{R}}$ contains its supremum and infimum.

6. Let $A = \{x \in \mathbb{R} : \sin x = 0\}$. The map ‘$\sin$’ being continuous, $A = \sin^{-1}(\{0\})$ is a closed subset of $\mathbb{R}$. However, $\inf A = -\infty$ and $\sup A = +\infty$, and consequently, $A$ does not contain its supremum or infimum. In 5., we showed that any non-empty closed subset of $\bar{\mathbb{R}}$ contains its supremum and infimum. This property does not hold for non-empty closed subset of $\mathbb{R}$. Indeed, $\mathbb{R}$ itself is a closed subset of itself, and does not contain its supremum or infimum. [Note that $\mathbb{R}$ is not closed in $\bar{\mathbb{R}}$].

7. Let $A$ be a non-empty closed and bounded subset of $\mathbb{R}$. From theorem (36), $A$ is a non-empty compact subset of $\mathbb{R}$. It follows that it is also a non-empty compact of subset of $\bar{\mathbb{R}}$, and consequently from theorem (35), it is a non-empty closed subset.
of $\bar{\mathbb{R}}$. We conclude from 5. that $A$ contains its supremum and infimum, i.e. $\sup A \in A$ and $\inf A \in A$.

Exercise 9
Exercise 10.

1. Let \( f : (\Omega, T) \to (\mathbb{R}, \mathcal{T}_{\mathbb{R}}) \) be a map with \( f(\Omega) \subseteq \mathbb{R} \). Suppose \( f \) is continuous with respect to \( \mathcal{T}_{\mathbb{R}} \). Let \( U \) be open in \( \mathbb{R} \). Then \( U \cap \mathbb{R} \) is open in \( \mathbb{R} \), and therefore \( f^{-1}(U) = f^{-1}(U \cap \mathbb{R}) \in \mathcal{T} \). So \( f \) is continuous with respect to \( \mathcal{T}_{\mathbb{R}} \). Conversely, suppose \( f \) is continuous with respect to \( \mathcal{T}_{\mathbb{R}} \). Let \( V \in \mathcal{T}_{\mathbb{R}} \). There exists \( U \in \mathcal{T}_{\mathbb{R}} \), such that \( V = U \cap \mathbb{R} \). So \( f^{-1}(V) = f^{-1}(U) \in \mathcal{T} \). So \( f \) is continuous with respect to \( \mathcal{T}_{\mathbb{R}} \). We have proved that whenever \( f(\Omega) \subseteq \mathbb{R} \), the continuity with respect to \( \mathcal{T}_{\mathbb{R}} \) and \( \mathcal{T}_{\mathbb{R}} \) are equivalent.

2. Let \( f : (\Omega, T) \to (\mathbb{R}, \mathcal{T}_{\mathbb{R}}) \) be a continuous map, where \((\Omega, T)\) is a non-empty compact topological space. From exercise (8), \( f(\Omega) \) is a non-empty compact subset of \( \mathbb{R} \). In particular, from theorem (35), it is a non-empty closed subset of \( \mathbb{R} \). From exercise (9)[5], we conclude that \( f(\Omega) \) contains its supremum and infimum, i.e. \( \sup f(\Omega) \in f(\Omega) \) and \( \inf f(\Omega) \in f(\Omega) \). In other
words, there exist $x_m$ and $x_M$ in $\Omega$, such that;

$$f(x_m) = \inf_{x \in \Omega} f(x), f(x_M) = \sup_{x \in \Omega} f(x)$$

This proves theorem (37).
Exercise 11.

1. Suppose \(c \in [a, b]\) and \(f(c) = \sup f([a, b])\). By assumption, \(f'(x)\) exists for all \(x \in [a, b]\). So in particular, \(f'(c)\) is well defined. For all \(x \in [a, b]\), we have \(f(x) \leq f(c)\). Hence, for all \(x \in ]c, b]\), we have \((f(x) - f(c))/(x - c) \leq 0\). Taking the limit as \(x \to c\), \(c < x\), we obtain \(f'(c) \leq 0\). Moreover, for all \(x \in ]a, c]\), we have \((f(c) - f(x))/(c - x) \geq 0\). Taking the limit as \(x \to c\), \(x < c\), we obtain \(f'(c) \geq 0\). We conclude that \(f'(c) = 0\).

2. Let \(a, b \in \mathbb{R}, a < b\). Let \(f : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\), differentiable on \(]a, b]\), with \(f(a) = f(b)\). From theorem (34), \([a, b]\) is a compact subset of \(\mathbb{R}\). \(f\) being continuous, from theorem (37), it attains its maximum and minimum on \([a, b]\). Suppose \(\sup f([a, b]) = \inf f([a, b])\). Then \(f\) is constant on \([a, b]\), and \(f'(c) = 0\) for all \(c \in ]a, b]\). Suppose that we have \(\sup f([a, b]) \neq \inf f([a, b])\). Then \(\sup f([a, b])\) and \(\inf f([a, b])\) cannot both be equal to \(f(a) = f(b)\). Changing \(f\) into \(-f\) if necessary, without loss of generality we can as-
sume that sup $f([a, b]) \neq f(a)$. Let $c \in [a, b]$ be such that $f(c) = \sup f([a, b])$. Then $f(c) \neq f(a)$ and $f(c) \neq f(b)$. So in fact, we have $c \in ]a, b[$. Since $f(c) = \sup_{x \in [a, b]} f(x)$, from 1., we conclude that $f'(c) = 0$. We have proved the existence of $c \in ]a, b[,$ such that $f'(c) = 0$. This proves theorem (38).

Exercise 11
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Exercise 12.

1. $h$ is of the form $h = f + \alpha p$, where $\alpha \in \mathbb{R}$, and $p$ is a polynomial. Since $f$ is continuous on $[a, b]$ and differentiable on $]a, b[$, the same is true of $h$.

2. We have $h(a) = f(a)$ and $h(b) = f(a)$. So $h(a) = h(b)$, and we can apply Rolle’s theorem (38). There exists $c \in ]a, b[\,$ such that $h'(c) = 0$. Since for all $x \in [a, b]$, we have:

$$h(x) = f(x) - (x - a)\frac{f(b) - f(a)}{b - a}$$

we have found $c \in ]a, b[\,$ such that:

$$f(b) - f(a) = (b - a)f'(c)$$

Exercise 12
Exercise 13.

1. $f$ is continuous on $[a, b]$, and $f'$ exists on $]a, b[$. Since $f$ is of class $C^n$, each $f^{(k)}$ is well defined and continuous on $[a, b]$, for all $k \in \{1, \ldots, n\}$. Moreover, each $f^{(k)}$ is differentiable on $[a, b]$, and in particular on $]a, b[$, for all $k \in \{1, \ldots, n-1\}$. In fact, since $f^{(n+1)}$ exist on $[a, b]$, each $f^{(k)}$ is differentiable on $]a, b[$ for all $k \in \{1, \ldots, n\}$. We conclude that $h$ is continuous on $]a, b[$, and differentiable on $]a, b[$.

2. For all $k \in \{1, \ldots, n\}$, we have:

$$\left( (b-x)^k f^{(k)} \right)' = -k(b-x)^{k-1} f^{(k)} + (b-x)^k f^{(k+1)}$$

Therefore, if we define:

$$g(x) = \sum_{k=1}^{n} \frac{(b-x)^k}{k!} f^{(k)}(x)$$

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we have:

\[
g'(x) = - \sum_{k=1}^{n} \frac{(b-x)^{k-1}}{(k-1)!} f^{(k)}(x) + \sum_{k=1}^{n} \frac{(b-x)^{k}}{k!} f^{(k+1)}(x)
\]

\[
= - \sum_{k=0}^{n-1} \frac{(b-x)^{k}}{k!} f^{(k+1)}(x) + \sum_{k=1}^{n} \frac{(b-x)^{k}}{k!} f^{(k+1)}(x)
\]

\[
= -f'(x) + \frac{(b-x)^{n}}{n!} f^{(n+1)}(x)
\]

and from:

\[
h(x) = f(b) - f(x) - g(x) - \alpha \frac{(b-x)^{n+1}}{(n+1)!}
\]

we conclude that:

\[
h'(x) = -f'(x) + f'(x) - \frac{(b-x)^{n}}{n!} f^{(n+1)}(x) + \alpha \frac{(b-x)^{n}}{n!}
\]

\[
= \frac{(b-x)^{n}}{n!} (\alpha - f^{(n+1)}(x))
\]
3. $h$ is continuous on $[a, b]$, and differentiable on $]a, b]$. Moreover, $h(b) = 0 = h(a)$. From theorem (38), there exists $c \in ]a, b]$, such that $h'(c) = 0$. Hence, from 2., there exists $c \in ]a, b]$ such that $f^{(n+1)}(c) = 0$. From $h(a) = 0$, we have:

$$f(b) - f(a) = \sum_{k=1}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c) \quad (5)$$

Given $a, b \in \mathbb{R}$, $a < b$ and $n \geq 0$, given $f : [a, b] \rightarrow \mathbb{R}$ of class $C^n$ on $[a, b]$, such that $f^{(n+1)}$ exists on $]a, b[$, we have found $c \in ]a, b]$ such that equation (5) holds. This proves theorem (39).

Exercise 13
Exercise 14.

1. Let $\phi : [a, b] \rightarrow \mathbb{R}$ be convex and differentiable. Let $x, y \in ]a, b[\, \quad x < y$. For all $z, z' \in ]x, y[$ such that $z < z'$, from exercise (1), we have:

$$
\frac{\phi(z) - \phi(x)}{z - x} \leq \frac{\phi(z') - \phi(z)}{z' - z} \leq \frac{\phi(y) - \phi(z')}{y - z'}
$$

$z'$ being fixed, taking the limit as $z \downarrow x$, we obtain:

$$
\phi'(x) \leq \frac{\phi(y) - \phi(z')}{y - z'}
$$

and finally, taking the limit as $z' \uparrow y$, $\phi'(x) \leq \phi'(y)$. We have proved that if a convex function is differentiable, its derivative is non-decreasing.

2. Let $x, y, z \in ]a, b[$ with $x < y < z$. Since $f$ is differentiable on $]a, b[$, in particular, it is continuous on $[x, y]$ and differentiable
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on \(|x, y|\). From exercise (12), there exists \(c_1 \in |x, y|\) such that:

\[
\phi(y) - \phi(x) = \phi'(c_1)(y - x)
\]  \hspace{1cm} (6)

Similarly, there exists \(c_2 \in |y, z|\), such that:

\[
\phi(z) - \phi(y) = \phi'(c_2)(z - y)
\]  \hspace{1cm} (7)

From \(x < y < x\), we conclude that \(c_1 < c_2\).

3. Let \(\phi : [a, b] \to \mathbb{R}\) be differentiable, and such that \(\phi'\) is non-decreasing. Let \(x, y, z \in [a, b]\) be such that \(x < y < z\). From 2., there exist \(c_1, c_2 \in [a, b]\), \(c_1 < c_2\), such that equations (6) and (7) are satisfied. \(\phi'\) being non-decreasing, we have \(\phi'(c_1) \leq \phi'(c_2)\).

We conclude from (6) and (7) that:

\[
\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(y)}{z - y}
\]

From exercise (1), it follows that \(\phi\) is convex. We have proved that a differentiable map on \([a, b]\), with non-decreasing derivative is convex.

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4. $x \to e^x$ is differentiable on $\mathbb{R}$, with non-decreasing derivative. It is therefore convex.

5. $x \to -\ln(x)$ is differentiable on $]0, +\infty[$, with non-decreasing derivative. It is therefore convex.

Exercise 14
Exercise 15.

1. Since \( \phi : [a, b] \to \mathbb{R} \) is convex, from exercise (1), it is continuous. It follows that \( \phi : ([a, b], \mathcal{B}([a, b])) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is measurable. Since \( X \in L^1_\mathbb{R}(\Omega, \mathcal{F}, P) \), the map \( X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is measurable. In fact, since \( X(\Omega) \subseteq [a, b] \), it is also true that \( \phi \circ X : (\Omega, \mathcal{F}) \to ([a, b], \mathcal{B}([a, b])) \) is measurable. We conclude that \( \phi \circ X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is measurable.

2. Since from 1., \( \phi \circ X \) is measurable and \( \mathbb{R} \)-valued, it is an element of \( L^1_\mathbb{R}(\Omega, \mathcal{F}, P) \), if and only if:

\[
E[|\phi \circ X|] \triangleq \int |\phi \circ X| dP < +\infty
\]

3. Suppose \( E[X] = a \). Since by assumption, \( X \in L^1_\mathbb{R}(\Omega, \mathcal{F}, P) \), \( E[X] \in \mathbb{R} \). So \( a \in \mathbb{R} \). Since \( X(\Omega) \subseteq [a, b] \), in particular \( X \geq a \). So \( X - a \geq 0 \) and \( \int (X - a) dP = 0 \). From exercise (7) [6.] of Tutorial 5, we conclude that \( X = a \) \( P \)-a.s., which contradicts \( X(\Omega) \subseteq [a, b] \).

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4. Suppose $E[X] = b$. Since by assumption, $X \in L^1_\mathbb{R}(\Omega, \mathcal{F}, P)$, $E[X] \in \mathbb{R}$. So $b \in \mathbb{R}$. Since $X(\Omega) \subseteq [a, b]$, in particular $X \leq b$. So $b - X \geq 0$ and $\int (b - X)dP = 0$. From exercise (7) [6] of Tutorial 5, we conclude that $X = b$ $P$-a.s., which contradicts $X(\Omega) \subseteq [a, b]$.

5. Let $m = E[X]$. Since $X(\Omega) \subseteq [a, b]$, we have $a < X < b$. It follows that $a \leq m \leq b$. From 3. and 4., $m = a$ or $m = b$ leads to a contradiction. We conclude that $m \in [a, b]$.

6. We define:

$$\beta \triangleq \sup_{x \in [a, m]} \frac{\phi(m) - \phi(x)}{m - x}$$

Since $a < m$, $]a, m[ \neq \emptyset$ and $\beta \neq -\infty$. Let $z \in ]m, b]$. Since $\phi$ is convex, from exercise (1), for all $x \in ]a, m]$, we have:

$$\frac{\phi(m) - \phi(x)}{m - x} \leq \frac{\phi(z) - \phi(m)}{z - m}$$
It follows that:

\[ \beta \leq \frac{\phi(z) - \phi(m)}{z - m} \]

In particular, \( \beta < +\infty \) and finally \( \beta \in \mathbb{R} \).

7. Let \( x \in ]a, b[. \) If \( x \in ]a, m[ \), then by definition of \( \beta \), we have:

\[ \frac{\phi(m) - \phi(x)}{m - x} \leq \beta \]

and consequently:

\[ \phi(m) + \beta(x - m) \leq \phi(x) \quad (8) \]

If \( x \in ]m, b[ \), then from 6., we have:

\[ \beta \leq \frac{\phi(x) - \phi(m)}{x - m} \]

and consequently, inequality (8) still holds. We conclude that inequality (8) holds for all \( x \in ]a, b[ \).

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8. For all $\omega \in \Omega$, $X(\omega) \in [a, b]$. From 7., we obtain:

$$\phi(m) + \beta(X(\omega) - m) \leq \phi(X(\omega))$$  \hspace{1cm} (9)

9. If $\phi \circ X \in L^1_\mathbb{R}(\Omega, \mathcal{F}, P)$, then $E[\phi \circ X]$ is meaningful. Taking expectations on both sides of (9), we obtain:

$$\phi(m) + \beta(E[X] - m) \leq E[\phi \circ X]$$

and since $m = E[X]$, we conclude that $\phi(m) \leq E[\phi \circ X]$. This proves theorem (40).

Exercise 15