

7. Fubini Theorem

Definition 59 Let \((\Omega_1, \mathcal{F}_1)\) and \((\Omega_2, \mathcal{F}_2)\) be two measurable spaces. Let \(E \subseteq \Omega_1 \times \Omega_2\). For all \(\omega_1 \in \Omega_1\), we call \(\omega_1\)-section of \(E\) in \(\Omega_2\), the set:

\[E^{\omega_1} \triangleq \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in E\}\]

Exercise 1. Let \((\Omega_1, \mathcal{F}_1)\), \((\Omega_2, \mathcal{F}_2)\) and \((S, \Sigma)\) be three measurable spaces, and \(f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \rightarrow (S, \Sigma)\) be a measurable map. Given \(\omega_1 \in \Omega_1\), define:

\[\Gamma^{\omega_1} \triangleq \{E \subseteq \Omega_1 \times \Omega_2 , E^{\omega_1} \in \mathcal{F}_2\}\]

1. Show that for all \(\omega_1 \in \Omega_1\), \(\Gamma^{\omega_1}\) is a \(\sigma\)-algebra on \(\Omega_1 \times \Omega_2\).
2. Show that for all \(\omega_1 \in \Omega_1\), \(\mathcal{F}_1 \sqcup \mathcal{F}_2 \subseteq \Gamma^{\omega_1}\).
3. Show that for all \(\omega_1 \in \Omega_1\) and \(E \in \mathcal{F}_1 \otimes \mathcal{F}_2\), we have \(E^{\omega_1} \in \mathcal{F}_2\).
4. Given \(\omega_1 \in \Omega_1\), show that \(\omega \rightarrow f(\omega_1, \omega)\) is measurable.
5. Show that $\theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \rightarrow (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ defined by $\theta(\omega_2, \omega_1) = (\omega_1, \omega_2)$ is a measurable map.

6. Given $\omega_2 \in \Omega_2$, show that $\omega \rightarrow f(\omega, \omega_2)$ is measurable.

**Theorem 29** Let $(S, \Sigma), (\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be three measurable spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \rightarrow (S, \Sigma)$ be a measurable map. For all $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, the map $\omega \rightarrow f(\omega_1, \omega)$ is measurable w.r. to $\mathcal{F}_2$ and $\Sigma$, and $\omega \rightarrow f(\omega, \omega_2)$ is measurable w.r. to $\mathcal{F}_1$ and $\Sigma$.

**Exercise 2.** Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces with $\text{card} I \geq 2$. Let $f : (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i) \rightarrow (E, \mathcal{B}(E))$ be a measurable map, where $(E, d)$ is a metric space. Let $i_1 \in I$. Put $E_1 = \Omega_{i_1}$, $\mathcal{E}_1 = \mathcal{F}_{i_1}$, $E_2 = \Pi_{i \in I \setminus \{i_1\}} \Omega_i$, $\mathcal{E}_2 = \otimes_{i \in I \setminus \{i_1\}} \mathcal{F}_i$.

1. Explain why $f$ can be viewed as a map defined on $E_1 \times E_2$.
2. Show that $f : (E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2) \rightarrow (E, \mathcal{B}(E))$ is measurable.
3. For all $\omega_i \in \Omega_i$, show that the map $\omega \rightarrow f(\omega_i, \omega)$ defined on $\Pi_{i \in I \setminus \{i_1\}} \Omega_i$ is measurable w.r. to $\otimes_{i \in I \setminus \{i_1\}} \mathcal{F}_i$ and $\mathcal{B}(E)$.

**Definition 60** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a **finite measure space**, or we say that $\mu$ is a **finite measure**, if and only if $\mu(\Omega) < +\infty$.

**Definition 61** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a **$\sigma$-finite measure space**, or $\mu$ a **$\sigma$-finite measure**, if and only if there exists a sequence $(\Omega_n)_{n \geq 1}$ in $\mathcal{F}$ such that $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < +\infty$, for all $n \geq 1$.

**Exercise 3.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

1. Show that $(\Omega, \mathcal{F}, \mu)$ is $\sigma$-finite if and only if there exists a sequence $(\Omega_n)_{n \geq 1}$ in $\mathcal{F}$ such that $\Omega = \biguplus_{n=1}^{+\infty} \Omega_n$, and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$. 

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2. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then $\mu$ has values in $\mathbb{R}^+$. 

3. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then it is $\sigma$-finite.

4. Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-decreasing map. Show that the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), dF)$ is $\sigma$-finite, where $dF$ is the Stieltjes measure associated with $F$.

**Exercise 4.** Let $(\Omega_1, \mathcal{F}_1)$ be a measurable space, and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be a $\sigma$-finite measure space. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\omega_1 \in \Omega_1$, define:

$$
\Phi_E(\omega_1) \triangleq \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2(x)
$$

Let $\mathcal{D}$ be the set of subsets of $\Omega_1 \times \Omega_2$, defined by:

$$
\mathcal{D} \triangleq \{ E \in \mathcal{F}_1 \otimes \mathcal{F}_2 : \Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})) \text{ is measurable} \}
$$

1. Explain why for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, the map $\Phi_E$ is well defined.
2. Show that $\mathcal{F}_1 \sqcup \mathcal{F}_2 \subseteq \mathcal{D}$.

3. Show that if $\mu_2$ is finite, $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.

4. Show that if $E_n \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $n \geq 1$ and $E_n \uparrow E$, then $\Phi_{E_n} \uparrow \Phi_E$.

5. Show that if $\mu_2$ is finite then $\mathcal{D}$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

6. Show that if $\mu_2$ is finite, then the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

7. Let $(\Omega_2^n)_{n \geq 1}$ in $\mathcal{F}_2$ be such that $\Omega_2^n \uparrow \Omega_2$ and $\mu_2(\Omega_2^n) < +\infty$. Define $\mu_2^n = \mu_2|_{\Omega_2^n} = \mu_2(\bullet \cap \Omega_2^n)$. For $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we put:

$$\Phi_E^n(\omega_1) \triangleq \int_{\Omega_2} 1_{E}(\omega_1, x) d\mu_2^n(x)$$

Show that $\Phi_E^n : (\Omega_1, \mathcal{F}_1) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, and:

$$\Phi_E^n(\omega_1) = \int_{\Omega_2} 1_{\Omega_2^n}(x) 1_E(\omega_1, x) d\mu_2(x)$$
Deduce that $\Phi_E \uparrow \Phi_E$.

8. Show that the map $\Phi_E : (\Omega_1, F_1) \to (\mathbb{R}, B(\mathbb{R}))$ is measurable, for all $E \in F_1 \otimes F_2$.

9. Let $s$ be a simple function on $(\Omega_1 \times \Omega_2, F_1 \otimes F_2)$. Show that the map $\omega \to \int_{\Omega_2} s(\omega, x) d\mu_2(x)$ is well defined and measurable with respect to $F_1$ and $B(\mathbb{R})$.

10. Show the following theorem:

**Theorem 30** Let $(\Omega_1, F_1)$ be a measurable space, and $(\Omega_2, F_2, \mu_2)$ be a $\sigma$-finite measure space. Then for all non-negative and measurable map $f : (\Omega_1 \times \Omega_2, F_1 \otimes F_2) \to [0, +\infty]$, the map:

$$\omega \to \int_{\Omega_2} f(\omega, x) d\mu_2(x)$$

is measurable with respect to $F_1$ and $B(\mathbb{R})$. 

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Exercise 5. Let $(Ω_i, F_i)_{i ∈ I}$ be a family of measurable spaces, with $\text{card} I \geq 2$. Let $i_0 ∈ I$, and suppose that $μ_0$ is a $σ$-finite measure on $(Ω_{i_0}, F_{i_0})$. Show that if $f : (Π_{i ∈ I} Ω_i, ⊗_{i ∈ I} F_i) → [0, +∞]$ is a non-negative and measurable map, then:

$$ω → \int_{Ω_{i_0}} f(ω, x) dμ_0(x)$$

defined on $Π_{i ∈ I \setminus \{i_0\}} Ω_i$, is measurable w.r. to $⊗_{i ∈ I \setminus \{i_0\}} F_i$ and $B(\mathbb{R})$.

Exercise 6. Let $(Ω_1, F_1, μ_1)$ and $(Ω_2, F_2, μ_2)$ be two $σ$-finite measure spaces. For all $E ∈ F_1 ⊗ F_2$, we define:

$$μ_1 ⊗ μ_2(E) \triangleq \int_{Ω_1} \left( \int_{Ω_2} 1_E(x, y) dμ_2(y) \right) dμ_1(x)$$

1. Explain why $μ_1 ⊗ μ_2 : F_1 ⊗ F_2 → [0, +∞]$ is well defined.
2. Show that $μ_1 ⊗ μ_2$ is a measure on $F_1 ⊗ F_2$. 

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3. Show that if $A \times B \in \mathcal{F}_1 \Pi \mathcal{F}_2$, then:

$$\mu_1 \otimes \mu_2(A \times B) = \mu_1(A)\mu_2(B)$$

**Exercise 7.** Further to ex. (6), suppose that $\mu : \mathcal{F}_1 \otimes \mathcal{F}_2 \to [0, +\infty]$ is another measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ with $\mu(A \times B) = \mu_1(A)\mu_2(B)$, for all measurable rectangle $A \times B$. Let $(\Omega_n^1)_{n \geq 1}$ and $(\Omega_n^2)_{n \geq 1}$ be sequences in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that $\Omega_n^1 \uparrow \Omega_1$, $\Omega_n^2 \uparrow \Omega_2$, $\mu_1(\Omega_n^1) < +\infty$ and $\mu_2(\Omega_n^2) < +\infty$. Define, for all $n \geq 1$:

$$\mathcal{D}_n \overset{\Delta}{=} \{ E \in \mathcal{F}_1 \otimes \mathcal{F}_2 : \mu(E \cap (\Omega_n^1 \times \Omega_n^2)) = \mu_1 \otimes \mu_2(E \cap (\Omega_n^1 \times \Omega_n^2)) \}$$

1. Show that for all $n \geq 1$, $\mathcal{F}_1 \Pi \mathcal{F}_2 \subseteq \mathcal{D}_n$.

2. Show that for all $n \geq 1$, $\mathcal{D}_n$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

3. Show that $\mu = \mu_1 \otimes \mu_2$.

4. Show that $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is a $\sigma$-finite measure space.
5. Show that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have:

$$
\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left( \int_{\Omega_1} 1_E(x, y) d\mu_1(x) \right) d\mu_2(y)
$$

**Exercise 8.** Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be $n$ $\sigma$-finite measure spaces, $n \geq 2$. Let $i_0 \in \{1, \ldots, n\}$ and put $E_1 = \Omega_{i_0}$, $E_2 = \Pi_{i \neq i_0} \Omega_i$, $\mathcal{E}_1 = \mathcal{F}_{i_0}$ and $\mathcal{E}_2 = \otimes_{i \neq i_0} \mathcal{F}_i$. Put $\nu_1 = \mu_{i_0}$, and suppose that $\nu_2$ is a $\sigma$-finite measure on $(E_2, \mathcal{E}_2)$ such that for all measurable rectangle $\Pi_{i \neq i_0} A_i \in \Pi_{i \neq i_0} \mathcal{F}_i$, we have $\nu_2(\Pi_{i \neq i_0} A_i) = \Pi_{i \neq i_0} \mu_i(A_i)$.

1. Show that $\nu_1 \otimes \nu_2$ is a $\sigma$-finite measure on the measure space $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:

$$
\nu_1 \otimes \nu_2(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)
$$

2. Show by induction the existence of a measure $\mu$ on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, 

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such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:

$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

3. Show the uniqueness of such measure, denoted $\mu_1 \otimes \ldots \otimes \mu_n$.

4. Show that $\mu_1 \otimes \ldots \otimes \mu_n$ is $\sigma$-finite.

5. Let $i_0 \in \{1, \ldots, n\}$. Show that $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i) = \mu_1 \otimes \ldots \otimes \mu_n$.

**Definition 62** Let $(\Omega_1, F_1, \mu_1), \ldots, (\Omega_n, F_n, \mu_n)$ be $n$ $\sigma$-finite measure spaces, with $n \geq 2$. We call **product measure** of $\mu_1, \ldots, \mu_n$, the unique measure on $F_1 \otimes \ldots \otimes F_n$, denoted $\mu_1 \otimes \ldots \otimes \mu_n$, such that for all measurable rectangles $A_1 \times \ldots \times A_n$ in $F_1 \Pi \ldots \Pi F_n$, we have:

$$\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

This measure is itself $\sigma$-finite.
Exercise 9. Prove that the following definition is legitimate:

**Definition 63** We call **Lebesgue measure** in $\mathbb{R}^n$, $n \geq 1$, the unique measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, denoted $dx$, $dx^n$ or $dx_1 \ldots dx_n$, such that for all $a_i \leq b_i$, $i = 1, \ldots, n$, we have:

$$dx([a_1, b_1] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i)$$

Exercise 10.

1. Show that $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx^n)$ is a $\sigma$-finite measure space.

2. For $n, p \geq 1$, show that $dx^{n+p} = dx^n \otimes dx^p$.

Exercise 11. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be $\sigma$-finite.
1. Let $s$ be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that:

$$
\int_{\Omega_1 \times \Omega_2} sd\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left( \int_{\Omega_2} sd\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left( \int_{\Omega_1} sd\mu_1 \right) d\mu_2
$$

2. Show the following:

**Theorem 31 (Fubini)** Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two $\sigma$-finite measure spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ be a non-negative and measurable map. Then:

$$
\int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left( \int_{\Omega_2} f d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left( \int_{\Omega_1} f d\mu_1 \right) d\mu_2
$$

**Exercise 12.** Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be $n$ $\sigma$-finite measure spaces, $n \geq 2$. Let $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ be a non-negative, measurable map. Let $\sigma$ be a permutation of $\mathbb{N}_n$, i.e. a bijection from $\mathbb{N}_n$ to itself.

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1. For all $\omega \in \Pi_{i \neq 1} \Omega_i$, define:

$$J_1(\omega) \triangleq \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

Explain why $J_1 : (\Pi_{i \neq 1} \Omega_i, \otimes_{i \neq 1} \mathcal{F}_i) \to [0, +\infty]$ is a well defined, non-negative and measurable map.

2. Suppose $J_k : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \mathcal{F}_i) \to [0, +\infty]$ is a non-negative, measurable map, for $1 \leq k < n - 2$. Define:

$$J_{k+1}(\omega) \triangleq \int_{\Omega_{\sigma(k+1)}} J_k(\omega, x) d\mu_{\sigma(k+1)}(x)$$

and show that:

$J_{k+1} : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \mathcal{F}_i) \to [0, +\infty]$ is also well-defined, non-negative and measurable.
3. Propose a rigorous definition for the following notation:
\[
\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f \, d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}
\]

**Exercise 13.** Further to ex. (12), Let \((f_p)_{p \geq 1}\) be a sequence of non-negative and measurable maps:
\[
f_p : (\Omega_1 \times \ldots \times \Omega_n, F_1 \otimes \ldots \otimes F_n) \to [0, +\infty]
\]
such that \(f_p \uparrow f\). Define similarly:
\[
J_p^1(\omega) \triangleq \int_{\Omega_{\sigma(1)}} f_p(\omega, x) \, d\mu_{\sigma(1)}(x)
\]
\[
J_{k+1}^p(\omega) \triangleq \int_{\Omega_{\sigma(k+1)}} J_k^p(\omega, x) \, d\mu_{\sigma(k+1)}(x), \quad 1 \leq k < n - 2
\]

1. Show that \(J_1^p \uparrow J_1\).

2. Show that if \(J_k^p \uparrow J_k\), then \(J_{k+1}^p \uparrow J_{k+1}, \quad 1 \leq k < n - 2\).

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3. Show that:
\[
\int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} f_{p} d\mu_{\sigma(1)} \cdots d\mu_{\sigma(n)} \uparrow \int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} f_{p} d\mu_{\sigma(1)} \cdots d\mu_{\sigma(n)}
\]

4. Show that the map \( \mu : \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n \rightarrow [0, +\infty] \), defined by:
\[
\mu(E) = \int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} 1_{E} d\mu_{\sigma(1)} \cdots d\mu_{\sigma(n)}
\]
is a measure on \( \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n \).

5. Show that for all \( E \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n \), we have:
\[
\mu_1 \otimes \cdots \otimes \mu_n (E) = \int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} 1_{E} d\mu_{\sigma(1)} \cdots d\mu_{\sigma(n)}
\]

6. Show the following:
**Theorem 32** Let $(Ω_1,F_1,μ_1),\ldots,(Ω_n,F_n,μ_n)$ be $n$ σ-finite measure spaces, with $n \geq 2$. Let $f : (Ω_1 \times \ldots \times Ω_n,F_1 \otimes \ldots \otimes F_n) → [0, +∞]$ be a non-negative and measurable map. Let $σ$ be a permutation of $N_n$. Then:

\[
\int_{Ω_1 \times \ldots \times Ω_n} f \, dμ_1 \otimes \ldots \otimes μ_n = \int_{Ω_σ(1)} f \, dμ_σ(1) \ldots \int_{Ω_σ(n)} f \, dμ_σ(n)
\]

**Exercise 14.** Let $(Ω,F,μ)$ be a measure space. Define:

\[
L^1 ≜ \{ f : Ω → \mathbb{R} \mid \exists g ∈ L^1_\mathbb{R}(Ω,F,μ) \land f = g \, \mu\text{-a.s.}\}
\]

1. Show that if $f ∈ L^1$, then $|f| < +∞, \mu\text{-a.s.}$

2. Suppose there exists $A ⊆ Ω$, such that $A ∉ F$ and $A ⊆ N$ for some $N ∈ F$ with $μ(N) = 0$. Show that $1_A ∈ L^1$ and $1_A$ is not measurable with respect to $F$ and $B(\mathbb{R})$.

3. Explain why if $f ∈ L^1$, the integrals $\int |f|dμ$ and $\int fdμ$ may not be well defined.

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4. Suppose that \( f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is a measurable map with \( \int |f| \, d\mu < +\infty \). Show that \( f \in L^1 \).

5. Show that if \( f \in L^1 \) and \( f = f_1 \) \( \mu \)-a.s. then \( f_1 \in L^1 \).

6. Suppose that \( f \in L^1 \) and \( g_1, g_2 \in L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \) are such that \( f = g_1 \) \( \mu \)-a.s. and \( f = g_2 \) \( \mu \)-a.s.. Show that \( \int g_1 \, d\mu = \int g_2 \, d\mu \).

7. Propose a definition of the integral \( \int f \, d\mu \) for \( f \in L^1 \) which extends the integral defined on \( L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \).

**Exercise 15.** Further to ex. (14), Let \( (f_n)_{n \geq 1} \) be a sequence in \( L^1 \), and \( f, h \in L^1 \), with \( f_n \to f \) \( \mu \)-a.s. and for all \( n \geq 1 \), \( |f_n| \leq h \) \( \mu \)-a.s..

1. Show the existence of \( N_1 \in \mathcal{F}, \mu(N_1) = 0 \), such that for all \( \omega \in N_1^c \), \( f_n(\omega) \to f(\omega) \), and for all \( n \geq 1 \), \( |f_n(\omega)| \leq h(\omega) \).

2. Show the existence of \( g_n, g, h_1 \in L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \) and \( N_2 \in \mathcal{F}, \mu(N_2) = 0 \), such that for all \( \omega \in N_2^c \), \( g(\omega) = f(\omega) \), \( h(\omega) = h_1(\omega) \), and for all \( n \geq 1 \), \( g_n(\omega) = f_n(\omega) \).
3. Show the existence of $N \in \mathcal{F}$, $\mu(N) = 0$, such that for all $\omega \in N^c$, $g_n(\omega) \to g(\omega)$, and for all $n \geq 1$, $|g_n(\omega)| \leq h_1(\omega)$.

4. Show that the Dominated Convergence Theorem can be applied to $g_n1_{N^c}, g1_{N^c}$ and $h_11_{N^c}$.

5. Recall the definition of $\int |f_n - f|d\mu$ when $f, f_n \in L^1$.

6. Show that $\int |f_n - f|d\mu \to 0$.

**Exercise 16.** Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two $\sigma$-finite measure spaces. Let $f$ be an element of $L^1_{\mathcal{R}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Let $\theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ be the map defined by $\theta(\omega_2, \omega_1) = (\omega_1, \omega_2)$ for all $(\omega_2, \omega_1) \in \Omega_2 \times \Omega_1$.

1. Let $A = \{\omega_1 \in \Omega_1 : \int_{\Omega_2} |f(\omega_1, x)|d\mu_2(x) < +\infty\}$. Show that $A \in \mathcal{F}_1$ and $\mu_1(A^c) = 0$.

2. Show that $f(\omega_1, \cdot) \in L^1_{\mathcal{R}}(\Omega_2, \mathcal{F}_2, \mu_2)$ for all $\omega_1 \in A$. 

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3. Show that \( \bar{I}(\omega_1) = \int_{\Omega_2} f(\omega_1, x) d\mu_2(x) \) is well defined for all \( \omega_1 \in A \). Let \( I \) be an arbitrary extension of \( \bar{I} \), on \( \Omega_1 \).

4. Define \( J = I 1_A \). Show that:
\[
J(\omega) = 1_A(\omega) \int_{\Omega_2} f^+(\omega, x) d\mu_2(x) - 1_A(\omega) \int_{\Omega_2} f^-(\omega, x) d\mu_2(x)
\]

5. Show that \( J \) is \( \mathcal{F}_1 \)-measurable and \( \mathbb{R} \)-valued.

6. Show that \( J \in L^1(\Omega_1, \mathcal{F}_1, \mu_1) \) and that \( J = I \) \( \mu_1 \)-a.s.

7. Propose a definition for the integral:
\[
\int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x)
\]

8. Show that \( \int_{\Omega_1} (1_A \int_{\Omega_2} f^+ d\mu_2) d\mu_1 = \int_{\Omega_1 \times \Omega_2} f^+ d\mu_1 \otimes \mu_2 \).
9. Show that:
\[
\int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 \quad (1)
\]

10. Show that if \( f \in L^1_{C}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2) \), then the map \( \omega_1 \rightarrow \int_{\Omega_2} f(\omega_1, y) d\mu_2(y) \) is \( \mu_1 \)-almost surely equal to an element of \( L^1_{C}(\Omega_1, \mathcal{F}_1, \mu_1) \), and furthermore that (1) is still valid.

11. Show that if \( f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \rightarrow [0, +\infty] \) is non-negative and measurable, then \( f \circ \theta \) is non-negative and measurable, and:
\[
\int_{\Omega_2 \times \Omega_1} f \circ \theta d\mu_2 \otimes \mu_1 = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2
\]

12. Show that if \( f \in L^1_{C}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2) \), then \( f \circ \theta \) is an element of \( L^1_{C}(\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1) \), and:
\[
\int_{\Omega_2 \times \Omega_1} f \circ \theta d\mu_2 \otimes \mu_1 = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2
\]

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13. Show that if \( f \in L^1_C(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2) \), then the map 
\( \omega_2 \rightarrow \int_{\Omega_1} f(x, \omega_2) \, d\mu_1(x) \) is \( \mu_2 \)-almost surely equal to an element of \( L^1_C(\Omega_2, \mathcal{F}_2, \mu_2) \), and furthermore:
\[
\int_{\Omega_2} \left( \int_{\Omega_1} f(x, y) \, d\mu_1(x) \right) \, d\mu_2(y) = \int_{\Omega_1 \times \Omega_2} f \, d\mu_1 \otimes \mu_2
\]

**Theorem 33** Let \((\Omega_1, \mathcal{F}_1, \mu_1)\) and \((\Omega_2, \mathcal{F}_2, \mu_2)\) be two \( \sigma \)-finite measure spaces. Let \( f \in L^1_C(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2) \). Then, the map:

\[
\omega_1 \rightarrow \int_{\Omega_2} f(\omega_1, x) \, d\mu_2(x)
\]

is \( \mu_1 \)-almost surely equal to an element of \( L^1_C(\Omega_1, \mathcal{F}_1, \mu_1) \) and:

\[
\int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) \, d\mu_2(y) \right) \, d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f \, d\mu_1 \otimes \mu_2
\]
Furthermore, the map:

\[ \omega_2 \mapsto \int_{\Omega_1} f(x, \omega_2) d\mu_1(x) \]

is \( \mu_2 \)-almost surely equal to an element of \( L^1_C(\Omega_2, \mathcal{F}_2, \mu_2) \) and:

\[ \int_{\Omega_2} \left( \int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 \]

**Exercise 17.** Let \((\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)\) be \( n \) \( \sigma \)-finite measure spaces, \( n \geq 2 \). Let \( f \in L^1_C(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n) \). Let \( \sigma \) be a permutation of \( \mathbb{N}_n \).

1. For all \( \omega \in \Pi_{i \neq \sigma(1)} \Omega_i \), define:

\[ J_1(\omega) \triangleq \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x) \]

Explain why \( J_1 \) is well defined and equal to an element of \( L^1_C(\Pi_{i \neq \sigma(1)} \Omega_i, \otimes_{i \neq \sigma(1)} \mathcal{F}_i, \otimes_{i \neq \sigma(1)} \mu_i) \), \( \otimes_{i \neq \sigma(1)} \mu_i \)-almost surely.

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2. Suppose $1 \leq k < n - 2$ and that $\tilde{J}_k$ is well defined and equal to an element of:

$$L^1_C(\prod_{i \not\in \{\sigma(1), \ldots, \sigma(k)\}} \Omega_i \otimes \prod_{i \not\in \{\sigma(1), \ldots, \sigma(k)\}} \mathcal{F}_i, \otimes \prod_{i \not\in \{\sigma(1), \ldots, \sigma(k)\}} \mu_i)$$

$\otimes \prod_{i \not\in \{\sigma(1), \ldots, \sigma(k)\}} \mu_i$-almost surely. Define:

$$J_{k+1}(\omega) \triangleq \int_{\Omega_{\sigma(k+1)}} \tilde{J}_k(\omega, x) d\mu_{\sigma(k+1)}(x)$$

What can you say about $J_{k+1}$.

3. Show that:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

is a well defined complex number. (Propose a definition for it).

4. Show that:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} = \int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n$$
Solutions to Exercises

Exercise 1.

1. Let $\omega_1 \in \Omega_1$. The $\omega_1$-section of $\Omega_1 \times \Omega_2$ in $\Omega_2$, is equal to $\Omega_2 \in \mathcal{F}_2$. So $\Omega_1 \times \Omega_2 \in \Gamma^{\omega_1}$. Suppose $E \in \Gamma^{\omega_1}$. Then $E^{\omega_1} \in \mathcal{F}_2$. $\mathcal{F}_2$ being closed under complementation, $(E^{\omega_1})^c \in \mathcal{F}_2$. However, given $\omega_2 \in \Omega_2$, $\omega_2 \in (E^{\omega_1})^c$ is equivalent to $(\omega_1, \omega_2) \notin E$, i.e. $(\omega_1, \omega_2) \in E^c$. So $(E^{\omega_1})^c = (E^c)^{\omega_1}$. Hence, we see that $(E^c)^{\omega_1} \in \mathcal{F}_2$. It follows that $E^c \in \Gamma^{\omega_1}$, which is therefore closed under complementation. Let $(E_n)_{n \geq 1}$ be a sequence of elements of $\Gamma^{\omega_1}$. Let $E = \bigcup_{n=1}^{+\infty} E_n$. For all $n \geq 1$, $(E_n)^{\omega_1} \in \mathcal{F}_2$. $\mathcal{F}_2$ being closed under countable union, $\bigcup_{n=1}^{+\infty} (E_n)^{\omega_1} \in \mathcal{F}_2$. However, given $\omega_2 \in \Omega_2$, $\omega_2 \in \bigcup_{n=1}^{+\infty} (E_n)^{\omega_1}$ is equivalent to the existence of $n \geq 1$, such that $(\omega_1, \omega_2) \in E_n$. Hence, it is equivalent to $(\omega_1, \omega_2) \in \bigcup_{n=1}^{+\infty} E_n = E$. So $\bigcup_{n=1}^{+\infty} (E_n)^{\omega_1} = E^{\omega_1}$, and we see that $E^{\omega_1} \in \mathcal{F}_2$. It follows that $E \in \Gamma^{\omega_1}$, which is therefore closed under countable union. We have proved that $\Gamma^{\omega_1}$ is a $\sigma$-algebra on $\Omega_1 \times \Omega_2$. 
2. Let $\omega_1 \in \Omega_1$, and $E = A \times B \in \mathcal{F}_1 \Pi \mathcal{F}_2$ be a measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$. Suppose $\omega_1 \in A$. Then $(\omega_1, \omega_2) \in E$, if and only if $\omega_2 \in B$. So $E^{\omega_1} = B \in \mathcal{F}_2$. Suppose $\omega_1 \notin A$. Then for all $\omega_2 \in \Omega_2$, $(\omega_1, \omega_2) \notin E$. So $E^{\omega_1} = \emptyset \in \mathcal{F}_2$. In any case, $E^{\omega_1} \in \mathcal{F}_2$. It follows that $E \in \Gamma^{\omega_1}$. We have proved that $\mathcal{F}_1 \Pi \mathcal{F}_2 \subseteq \Gamma^{\omega_1}$.

3. From $\mathcal{F}_1 \Pi \mathcal{F}_2 \subseteq \Gamma^{\omega_1}$ and the fact that $\Gamma^{\omega_1}$ is a $\sigma$-algebra on $\Omega_1 \times \Omega_2$, we conclude that $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{F}_1 \Pi \mathcal{F}_2) \subseteq \Gamma^{\omega_1}$. Hence, for all $\omega_1 \in \Omega_1$ and $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $E$ is an element of $\Gamma^{\omega_1}$, or equivalently, $E^{\omega_1} \in \mathcal{F}_2$.

4. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \rightarrow (S, \Sigma)$ be a measurable map, where $(S, \Sigma)$ is a measurable space. Let $\omega_1 \in \Omega_1$, and $\phi : \Omega_2 \rightarrow S$ be the partial map $\omega \rightarrow f(\omega_1, \omega)$. Let $B \in \Sigma$. Then $\{f \in B\}$ is an element of $\mathcal{F}_1 \otimes \mathcal{F}_2$. Using 3, it follows that the $\omega_1$-section $\{f \in B\}^{\omega_1}$ of $\{f \in B\}$ is an element of $\mathcal{F}_2$. However, we have:

$$\{f \in B\}^{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \{f \in B\}\}$$
\[ \{ \omega_2 \in \Omega_2 : f(\omega_1, \omega_2) \in B \} \]
\[ = \{ \omega_2 \in \Omega_2 : \phi(\omega_2) \in B \} \]
\[ = \{ \phi \in B \} \]

Hence we see that \( \{ \phi \in B \} \in F_2 \). This being true for all \( B \in \Sigma \), we conclude that \( \phi \) is measurable. This shows that the map \( \omega \rightarrow f(\omega_1, \omega) \) is measurable.

5. Let \( \theta : (\Omega_2 \times \Omega_1, F_2 \otimes F_1) \rightarrow (\Omega_1 \times \Omega_2, F_1 \otimes F_2) \) be defined by \( \theta(\omega_2, \omega_1) = (\omega_1, \omega_2) \). From theorem (28), in order to show that \( \theta \) is measurable, it is sufficient to prove that each coordinate mapping \( \theta_1 : (\omega_2, \omega_1) \rightarrow \omega_1 \) and \( \theta_2 : (\omega_2, \omega_1) \rightarrow \omega_2 \) is measurable. This is indeed the case, since for all \( A_1 \in F_1 \) we have \( \theta_1^{-1}(A_1) = \Omega_2 \times A_1 \in F_2 \otimes F_1 \), and for all \( A_2 \in F_2 \) we have \( \theta_2^{-1}(A_2) = A_2 \times \Omega_1 \in F_2 \otimes F_1 \). So \( \theta \) is measurable.

6. Let \( \omega_2 \in \Omega_2 \). Let \( g : (\Omega_2 \times \Omega_1, F_2 \otimes F_1) \rightarrow (S, \Sigma) \) be the map defined by \( g = f \circ \theta \). Having proved in 5. that \( \theta \) is measurable, since \( f \) is itself measurable, \( g \) is a measurable map. Applying 4.
to \( g \), it follows that the map \( \omega \to g(\omega_2, \omega) \) is measurable with respect to \( \mathcal{F}_1 \) and \( \Sigma \). In other words, the map \( \omega \to f(\omega, \omega_2) \) is measurable with respect to \( \mathcal{F}_1 \) and \( \Sigma \). This completes the proof of theorem (29).

Exercise 1
Exercise 2.

1. There is an obvious bijection $\Phi$ between $E_1 \times E_2$ and $\Pi_{i \in I} \Omega_i$, defined by $\Phi(\omega_1, \omega_2)(i_1) = \omega_1$, and $\Phi(\omega_1, \omega_2)(i) = \omega_2(i)$ for $i \neq i_1$. The two sets $E_1 \times E_2$ and $\Pi_{i \in I} \Omega_i$ can therefore be identified, and $f$ can be viewed as a map defined on $E_1 \times E_2$.

2. Having identified $E_1 \times E_2$ and $\Pi_{i \in I} \Omega_i$, using exercise (10) of Tutorial 6 for the partition $I = \{i_1\} \cup (I \setminus \{i_1\})$, we obtain $\otimes_{i \in I} F_i = E_1 \otimes E_2$. So $f : (E_1 \times E_2, E_1 \otimes E_2) \rightarrow (E, B(E))$ is measurable.

3. From 2. and theorem (29), given $\omega_1 \in E_1$, the map $\omega \mapsto f(\omega_1, \omega)$ defined on $E_2$, is measurable with respect to $E_2$ and $B(E)$. In other words, given $\omega_i \in \Omega_i$, the map $\omega \mapsto f(\omega_1, \omega)$ defined on $\Pi_{i \in I \setminus \{i_1\}} \Omega_i$, is measurable w.r. to $\otimes_{i \in I \setminus \{i_1\}} F_i$ and $B(E)$.
Exercise 3.

1. Suppose there exists a sequence \((\Omega_n)_{n \geq 1}\) of pairwise disjoint elements of \(\mathcal{F}\), such that \(\Omega = \bigcup_{n=1}^{+\infty} \Omega_n\) and \(\mu(\Omega_n) < +\infty\) for all \(n \geq 1\). Define \(A_n = \bigcup_{k=1}^{n} \Omega_k\), for all \(n \geq 1\). Then:

\[
\mu(A_n) = \sum_{k=1}^{n} \mu(\Omega_k) < +\infty
\]

and furthermore, \(A_n \uparrow \Omega\). So \((\Omega, \mathcal{F}, \mu)\) is \(\sigma\)-finite. Conversely, suppose \((\Omega, \mathcal{F}, \mu)\) is \(\sigma\)-finite. Let \((A_n)_{n \geq 1}\) be a sequence in \(\mathcal{F}\), such that \(A_n \uparrow \Omega\) and \(\mu(A_n) < +\infty\) for all \(n \geq 1\). Define \(\Omega_1 = A_1\), and \(\Omega_n = A_n \setminus A_{n-1}\) for all \(n \geq 2\). Then, \((\Omega_n)_{n \geq 1}\) is a sequence of pairwise disjoint elements of \(\mathcal{F}\). Since \(\Omega_n \subseteq A_n\) for all \(n \geq 1\), we have \(\mu(\Omega_n) \leq \mu(A_n) < +\infty\). Given \(\omega \in \Omega\), since \(\Omega = \bigcup_{n=1}^{+\infty} A_n\), there exists \(n \geq 1\) such that \(\omega \in A_n\). Let \(p\) be the smallest of such \(n\). Then \(\omega \in A_p \setminus A_{p-1}\) if \(p \geq 2\), or \(\omega \in A_1\). In any case, \(\omega \in \Omega_p\). Hence, we see that \(\Omega = \bigcup_{n=1}^{+\infty} \Omega_n\) and finally \(\Omega = \bigcup_{n=1}^{+\infty} \Omega_n\). We conclude that \((\Omega, \mathcal{F}, \mu)\) is \(\sigma\)-finite, if and only
if there exists a sequence \((\Omega_n)_{n \geq 1}\) of pairwise disjoint elements of \(\mathcal{F}\), such that \(\Omega = \bigcup_{n=1}^{\infty} \Omega_n\) and \(\mu(\Omega_n) < +\infty\) for all \(n \geq 1\).

2. Suppose \((\Omega, \mathcal{F}, \mu)\) is finite. Then \(\mu(\Omega) < +\infty\). For all \(A \in \mathcal{F}\), since \(A \subseteq \Omega\), \(\mu(A) \leq \mu(\Omega) < +\infty\). So \(\mu\) takes values in \(R^+\).

3. Suppose \((\Omega, \mathcal{F}, \mu)\) is finite. Then \(\mu(\Omega) < +\infty\). Define \(\Omega_n = \Omega\) for all \(n \geq 1\). Then \((\Omega_n)_{n \geq 1}\) is a sequence in \(\mathcal{F}\) such that \(\Omega_n \uparrow \Omega\) and \(\mu(\Omega_n) < +\infty\) for all \(n \geq 1\). So \((\Omega, \mathcal{F}, \mu)\) is \(\sigma\)-finite.

4. Take \(\Omega_n = [-n, n]\) for all \(n \geq 1\). Then, \(\Omega_n \subseteq \Omega_{n+1}\) and we have \(R = \bigcup_{n=1}^{\infty} \Omega_n\). So \(\Omega_n \uparrow R\). Moreover, by definition of the Stieltjes measure (20), \(dF(\Omega_n) = F(n) - F(-n) \in R^+\). In particular, \(dF(\Omega_n) < +\infty\) for all \(n \geq 1\). We conclude that \((R, \mathcal{B}(R), dF)\) is a \(\sigma\)-finite measure space.

Exercise 3
Exercise 4.

1. Let $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$. The characteristic function $1_E$ is non-negative and measurable with respect to $\mathcal{F}_1 \otimes \mathcal{F}_2$. From theorem (29), for all $\omega_1 \in \Omega_1$, the partial function $x \mapsto 1_E(\omega_1, x)$ is measurable with respect to $\mathcal{F}_2$. It is also non-negative. It follows that the integral $\int_{\Omega_2} 1_E(\omega_1, x) \, d\mu_2(x)$ is well-defined, for all $\omega_1 \in \Omega_1$. Hence, we see that $\Phi_E$ is a well-defined map on $\Omega_1$.

2. Let $E = A \times B \in \mathcal{F}_1 \mathcal{F}_2$ be a measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$. For all $\omega_1 \in \Omega_1$, we have:

$$\Phi_E(\omega_1) = \int_{\Omega_2} 1_A(\omega_1)1_B(x) \, d\mu_2(x) = \mu_2(B)1_A(\omega_1)$$

Since $A \in \mathcal{F}_1$, the map $1_A$ is $\mathcal{F}_1$-measurable, and consequently $\Phi_E = \mu_2(B)1_A$ is $\mathcal{F}_1$-measurable. Hence, we see that $E \in \mathcal{D}$. We have proved that $\mathcal{F}_1 \mathcal{F}_2 \subseteq \mathcal{D}$.

3. Suppose $\mu_2$ is a finite measure. Let $A, B \in \mathcal{D}$ with $A \subseteq B$. For
all $\omega_1 \in \Omega_1$, from $1_B = 1_A + 1_{B \setminus A}$, we obtain:

$$
\int_{\Omega_2} 1_B(\omega_1, x) d\mu_2(x) = \int_{\Omega_2} 1_A(\omega_1, x) d\mu_2(x) + \int_{\Omega_2} 1_{B \setminus A}(\omega_1, x) d\mu_2(x)
$$

i.e. $\Phi_B(\omega_1) = \Phi_A(\omega_1) + \Phi_{B \setminus A}(\omega_1)$. $\mu_2$ being a finite measure, all $\Phi_E$’s take values in $\mathbb{R}^+$. Hence, it is legitimate to write:

$$
\Phi_{B \setminus A} = \Phi_B - \Phi_A
$$

Since $A, B \in \mathcal{D}$, both $\Phi_A$ and $\Phi_B$ are $\mathcal{F}_1$-measurable. We conclude that $\Phi_{B \setminus A}$ is $\mathcal{F}_1$-measurable, and $B \setminus A \in \mathcal{D}$. We have proved that if $A, B \in \mathcal{D}$ with $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.

4. Let $(E_n)_{n \geq 1}$ be a sequence in $\mathcal{F}_1 \otimes \mathcal{F}_2$ with $E_n \uparrow E$. In particular, $E_n \subseteq E_{n+1}$ for all $n \geq 1$, and therefore $1_{E_n} \leq 1_{E_{n+1}}$. Moreover, $E = \cup_{n=1}^{+\infty} E_n$. Let $\omega \in \Omega_1 \times \Omega_2$. If $\omega \in E$, there exists $N \geq 1$ such that $\omega \in E_N$. For all $n \geq N$, we have $1_{E_n}(\omega) = 1 = 1_E(\omega)$. If $\omega \notin E$, then $1_{E_n}(\omega) = 0 = 1_E(\omega)$, for all $n \geq 1$. In any case, $1_{E_n}(\omega) \rightarrow 1_E(\omega)$, and consequently
1_{E_n} \uparrow 1_E$. Given $\omega_1 \in \Omega_1$, we also have $1_{E_n}(\omega_1,.) \uparrow 1_E(\omega_1,.)$.

From the monotone convergence theorem (19), we obtain:

$$\int_{\Omega_2} 1_{E_n}(\omega_1,x)d\mu_2(x) \uparrow \int_{\Omega_2} 1_E(\omega_1,x)d\mu_2(x)$$

i.e. $\Phi_{E_n}(\omega_1) \uparrow \Phi_E(\omega_1)$. We conclude that $\Phi_{E_n} \uparrow \Phi_E$.

5. Suppose that $\mu_2$ is a finite measure. From 2., $\mathcal{F}_1 \Pi \mathcal{F}_2 \subseteq \mathcal{D}$, and in particular $\Omega_1 \times \Omega_2 \subseteq \mathcal{D}$. From 3., whenever $A,B \in \mathcal{D}$ are such that $A \subseteq B$, we have $B \setminus A \in \mathcal{D}$. Let $(E_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{D}$, such that $E_n \uparrow E$. For all $n \geq 1$, $\Phi_{E_n}$ is an $\mathcal{F}_1$-measurable map. Moreover from 4., $\Phi_{E_n} \uparrow \Phi_E$.

In particular, $\Phi_E = \sup_{n \geq 1} \Phi_{E_n}$ and we conclude that $\Phi_E$ is measurable with respect to $\mathcal{F}_1$. So $E \in \mathcal{D}$. We have proved that $\mathcal{D}$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

6. Suppose $\mu_2$ is a finite measure. From 5., $\mathcal{D}$ is a Dynkin system on $\Omega_1 \times \Omega_2$. From 2., we have $\mathcal{F}_1 \Pi \mathcal{F}_2 \subseteq \mathcal{D}$. The set of measurable rectangles $\mathcal{F}_1 \Pi \mathcal{F}_2$ being closed under finite intersection, from
the Dynkin system theorem (1), we see that $\mathcal{D}$ also contains the
$\sigma$-algebra generated by $\mathcal{F}_1 \amalg \mathcal{F}_2$, i.e.

$$\mathcal{F}_1 \otimes \mathcal{F}_2 \triangleq \sigma(\mathcal{F}_1 \amalg \mathcal{F}_2) \subseteq \mathcal{D}$$

We conclude that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $E$ is an element of $\mathcal{D}$, or
equivalently, the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable.

7. For all $n \geq 1$, $\mu^n_2(\Omega_2) = \mu_2(\Omega^n_2) < +\infty$. So $\mu^n_2$ is a finite
measure. It follows from 6. that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, the map
$\Phi^n_E$ defined by:

$$\Phi^n_E(\omega_1) \triangleq \int_{\Omega_2} 1_E(\omega_1, x) d\mu^n_2(x)$$

is measurable with respect to $\mathcal{F}_1$. From definition (45), we have:

$$\Phi^n_E(\omega_1) = \int_{\Omega_2} 1_{\Omega^n_2}(x) 1_E(\omega_1, x) d\mu_2(x)$$

Since $\Omega^n_2 \uparrow \Omega_2$, we have $1_{\Omega^n_2} \uparrow 1_{\Omega_2} = 1$ and consequently,
$1_{\Omega^n_2}(.) 1_E(\omega_1, .) \uparrow 1_E(\omega_1, .)$. From the monotone convergence

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theorem (19), we obtain:
\[ \int_{\Omega_2} 1_{\Omega_2}^2(x)1_E(\omega_1, x)d\mu_2(x) \uparrow \int_{\Omega_2} 1_E(\omega_1, x)d\mu_2(x) \]
i.e. \( \Phi^n_E(\omega_1) \uparrow \Phi_E(\omega_1) \), for all \( \omega_1 \in \Omega_1 \). So \( \Phi^n_E \uparrow \Phi_E \).

8. From 7., each \( \Phi_E^n \) is \( \mathcal{F}_1 \)-measurable and \( \Phi_E = \sup_{n \geq 1} \Phi^n_E \). So \( \Phi_E \) is \( \mathcal{F}_1 \)-measurable, for all \( E \in \mathcal{F}_1 \otimes \mathcal{F}_2 \).

9. Let \( s = \sum_{i=1}^n \alpha_i 1_{E_i} \) be a simple function on \((\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)\).

From theorem (29), the map \( x \to s(\omega_1, x) \) is \( \mathcal{F}_2 \)-measurable, for all \( \omega_1 \in \Omega_1 \). It is also non-negative. It follows that the integral \( \int_{\Omega_2} s(\omega_1, x)d\mu_2(x) \) is well-defined, for all \( \omega_1 \in \Omega_1 \). Moreover:
\[ \int_{\Omega_2} s(\omega_1, x)d\mu_2(x) = \sum_{i=1}^n \alpha_i \int_{\Omega_2} 1_{E_i}(\omega_1, x)d\mu_2(x) \]

Since \( E_i \in \mathcal{F}_1 \otimes \mathcal{F}_2 \), from 8., each \( \omega \to \int_{\Omega_2} 1_{E_i}(\omega, x)d\mu_2(x) \) is \( \mathcal{F}_1 \)-measurable. We conclude that \( \omega \to \int_{\Omega_2} s(\omega, x)d\mu_2(x) \) is also
\( \mathcal{F}_1 \)-measurable.

10. Let \( f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty] \) be a non-negative and measurable map. From theorem (18), there exists a sequence \((s_n)_{n \geq 1}\) of simple functions on \((\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)\) such that \( s_n \uparrow f \). In particular for all \( \omega \in \Omega_1 \), \( s_n(\omega, .) \uparrow f(\omega, .) \). From the monotone convergence theorem (19), we obtain:

\[
\int_{\Omega_2} s_n(\omega, x) d\mu_2(x) \uparrow \int_{\Omega_2} f(\omega, x) d\mu_2(x)
\]

However, from 9., each \( \omega \mapsto \int_{\Omega_2} s_n(\omega, x) d\mu_2(x) \) is \( \mathcal{F}_1 \)-measurable.

We conclude that \( \omega \mapsto \int_{\Omega_2} f(\omega, x) d\mu_2(x) \) is also measurable with respect to \( \mathcal{F}_1 \) and \( \mathcal{B}(\mathbb{R}) \). This proves theorem (30).

Exercise 4
Exercise 5. Let $f : (\Pi_{i\in I}\Omega_i, \otimes_{i\in I}\mathcal{F}_i) \to [0, +\infty]$ be a non-negative and measurable map. Define $E_1 = \Pi_{i\in I \setminus \{i_0\}}\Omega_i$ and $E_2 = \Omega_{i_0}$. Let $\mathcal{E}_1 = \otimes_{i\in I \setminus \{i_0\}}\mathcal{F}_i$ and $\mathcal{E}_2 = \mathcal{F}_{i_0}$. Using exercise (10) of Tutorial 6, having identified $E_1 \times E_2$ and $\Pi_{i\in I}\Omega_i$, we have:

$$\otimes_{i\in I}\mathcal{F}_i = (\otimes_{i\in I \setminus \{i_0\}}\mathcal{F}_i) \otimes \mathcal{F}_{i_0}$$

i.e. $\otimes_{i\in I}\mathcal{F}_i = \mathcal{E}_1 \otimes \mathcal{E}_2$. It follows that the map $f$, viewed as a map defined on $E_1 \times E_2$, is measurable with respect to $\mathcal{E}_1 \otimes \mathcal{E}_2$. $\mu_0$ being a $\sigma$-finite measure on $(E_2, \mathcal{E}_2)$, from theorem (30), we see that:

$$\omega \mapsto \int_{\Omega_0} f(\omega, x) d\mu_0(x)$$

is measurable with respect to $\mathcal{E}_1$ and $\mathcal{B}(\mathbb{R})$. In other words, it is measurable with respect to $\otimes_{i\in I \setminus \{i_0\}}\mathcal{F}_i$ and $\mathcal{B}(\mathbb{R})$.  

Exercise 5
Exercise 6.

1. Let $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$. The characteristic function $1_E$ is measurable with respect to $\mathcal{F}_1 \otimes \mathcal{F}_2$ and non-negative. $\mu_2$ being a $\sigma$-finite measure on $(\Omega_2, \mathcal{F}_2)$, applying theorem (30), we see that:

$$x \rightarrow \int_{\Omega_2} 1_E(x, y) d\mu_2(y)$$

is measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\mathbb{R})$. It is also non-negative. Hence, the integral:

$$\mu_1 \otimes \mu_2(E) \triangleq \int_{\Omega_1} \left( \int_{\Omega_2} 1_E(x, y) d\mu_2(y) \right) d\mu_1(x)$$

is well-defined, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$. So $\mu_1 \otimes \mu_2$ is a well-defined map on $\mathcal{F}_1 \otimes \mathcal{F}_2$, with values in $[0, +\infty]$.

2. Suppose $E = \emptyset$. Then $1_E = 0$ and $\mu_1 \otimes \mu_2(E) = 0$. Let $(E_n)_{n \geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{F}_1 \otimes \mathcal{F}_2$. Let
\[ E = \bigcup_{n=1}^{+\infty} E_n. \]  
Then, \( 1_E = \sum_{n=1}^{+\infty} 1_{E_n}. \) From the monotone convergence theorem (19), for all \( x \in \Omega_1, \) we have:

\[
\int_{\Omega_2} 1_E(x, y)d\mu_2(y) = \sum_{n=1}^{+\infty} \int_{\Omega_2} 1_{E_n}(x, y)d\mu_2(y)
\]

Applying the monotone convergence theorem once more:

\[
\mu_1 \otimes \mu_2(E) = \sum_{n=1}^{+\infty} \left( \int_{\Omega_2} 1_{E_n}(x, y)d\mu_2(y) \right) d\mu_1(x)
\]

i.e.

\[
\mu_1 \otimes \mu_2(E) = \sum_{n=1}^{+\infty} \mu_1 \otimes \mu_2(E_n)
\]

We have proved that \( \mu_1 \otimes \mu_2 \) is a measure on \( \mathcal{F}_1 \otimes \mathcal{F}_2. \)

3. Let \( E = A \times B \in \mathcal{F}_1 \otimes \mathcal{F}_2 \) be a measurable rectangle of \( \mathcal{F}_1 \) and
\[ F_2. \text{ For all } x \in \Omega_1, \text{ we have:} \]
\[ \int_{\Omega_2} 1_{E(x, y)} d\mu_2(y) = \int_{\Omega_2} 1_A(x) 1_B(y) d\mu_2(y) = \mu_2(B) 1_A(x) \]

It follows that:
\[ \mu_1 \otimes \mu_2(E) = \int_{\Omega_1} \mu_2(B) 1_A(x) d\mu_1(x) = \mu_1(A) \mu_2(B) \]

Exercise 6
Exercise 7.

1. By assumption, if \( E = A \times B \in \mathcal{F}_1 \Pi \mathcal{F}_2 \) is a measurable rectangle of \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), then \( \mu_1 \otimes \mu_2(E) = \mu_1(A)\mu_2(B) = \mu(E) \), i.e. \( \mu_1 \otimes \mu_2 \) and \( \mu \) coincide on \( \mathcal{F}_1 \Pi \mathcal{F}_2 \). Let \( E \in \mathcal{F}_1 \Pi \mathcal{F}_2 \). Then \( E \cap (\Omega_1^n \times \Omega_2^n) \) is still a measurable rectangle, i.e. an element of \( \mathcal{F}_1 \Pi \mathcal{F}_2 \). Hence \( \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu(E \cap (\Omega_1^n \times \Omega_2^n)) \). It follows that \( E \in \mathcal{D}_n \). So \( \mathcal{F}_1 \Pi \mathcal{F}_2 \subseteq \mathcal{D}_n \).

2. \( \Omega_1 \times \Omega_2 \in \mathcal{F}_1 \Pi \mathcal{F}_2 \subseteq \mathcal{D}_n \). Let \( E, F \in \mathcal{D}_n \) be such that \( E \subseteq F \). Then \( F = E \cup (F \setminus E) \), and consequently:
\[
\mu(F \cap (\Omega_1^n \times \Omega_2^n)) = \mu(E \cap (\Omega_1^n \times \Omega_2^n)) + \mu((F \setminus E) \cap (\Omega_1^n \times \Omega_2^n)) \quad (2)
\]
with a similar expression for \( \mu_1 \otimes \mu_2 \). Since \( E \) and \( F \) are elements of \( \mathcal{D}_n \), we also have:
\[
\mu(F \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(F \cap (\Omega_1^n \times \Omega_2^n))
\]
and:
\[
\mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))
\]

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All the terms involved being finite, it is legitimate to re-arrange and simplify equation (2) and its counterpart for $\mu_1 \otimes \mu_2$, to obtain:

$$\mu((F \setminus E) \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2((F \setminus E) \cap (\Omega_1^n \times \Omega_2^n))$$

Hence, we see that $F \setminus E \in D_n$. Let $(E_p)_{p \geq 1}$ be a sequence of elements of $D_n$, such that $E_p \uparrow E$. For all $p \geq 1$, we have:

$$\mu(E_p \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E_p \cap (\Omega_1^n \times \Omega_2^n))$$

From theorem (7), taking the limit as $p \to +\infty$, we obtain:

$$\mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))$$

It follows that $E \in D_n$. We have proved that $D_n$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

3. From 1., $\mathcal{F}_1 \supseteq \mathcal{F}_2 \subseteq D_n$. From 2., $D_n$ is in fact a Dynkin system on $\Omega_1 \times \Omega_2$. The set of measurable rectangles $\mathcal{F}_1 \uplus \mathcal{F}_2$ being closed under finite intersection, from the Dynkin system theorem (1), we conclude that $D_n$ actually contains the $\sigma$-algebra.
generated by $\mathcal{F}_1 \amalg \mathcal{F}_2$, i.e. $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{F}_1 \amalg \mathcal{F}_2) \subseteq \mathcal{D}_n$. Hence, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $E$ is an element of $\mathcal{D}_n$, or equivalently:

$$\mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))$$

Since $E \cap (\Omega_1^n \times \Omega_2^n) \uparrow E$, using theorem (7) once more, taking the limit as $n \to +\infty$, we obtain $\mu(E) = \mu_1 \otimes \mu_2(E)$. This being true for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have proved that $\mu = \mu_1 \otimes \mu_2$.

4. For all $n \geq 1$, let $E_n = \Omega_1^n \times \Omega_2^n$. Then $E_n \uparrow \Omega_1 \times \Omega_2$, and furthermore, $\mu_1 \otimes \mu_2(E_n) = \mu_1(\Omega_1^n) \mu_2(\Omega_2^n) < +\infty$. We conclude that $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is a $\sigma$-finite measure space.

5. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, define:

$$\nu(E) \triangleq \int_{\Omega_2} \left( \int_{\Omega_1} 1_E(x, y)d\mu_1(x) \right) d\mu_2(y)$$

Note that this is the same definition as that of $\mu_1 \otimes \mu_2(E)$, except that the order of integration has been changed. Similarly to exercise (6), using the monotone convergence theorem (19)
twice on infinite series, we see that $\nu$ is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$. Moreover, for all $E = A \times B \in \mathcal{F}_1 \Pi \mathcal{F}_2$ measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$, we have:

$$\nu(E) = \int_{\Omega_2} \mu_1(A)1_B(y) d\mu_2(y) = \mu_1(A)\mu_2(B)$$

So $\nu$ is another measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$, coinciding with $\mu_1 \otimes \mu_2$ on the set of measurable rectangles $\mathcal{F}_1 \Pi \mathcal{F}_2$. From 3., we see that $\nu = \mu_1 \otimes \mu_2$. We have proved that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$:

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left( \int_{\Omega_1} 1_E(x,y) d\mu_1(x) \right) d\mu_2(y)$$

Hence, as far as defining $\mu_1 \otimes \mu_2$ is concerned, the order of integration is irrelevant.

Exercise 7

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Exercise 8.

1. \((E_1, \mathcal{E}_1, \nu_1)\) and \((E_2, \mathcal{E}_2, \nu_2)\) being two \(\sigma\)-finite measure spaces, \(\nu_1 \otimes \nu_2\) is well-defined as a measure on \((E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2)\) (exercise (6)). From exercise (7), such measure is itself \(\sigma\)-finite. Having identified \(E_1 \times E_2\) with \(\Omega_1 \times \ldots \times \Omega_n\), using exercise (10) of Tutorial 6, we have:

\[
\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n = \mathcal{F}_{i_0} \otimes (\otimes_{i \neq i_0} \mathcal{F}_i) = \mathcal{E}_1 \otimes \mathcal{E}_2
\]

So \(\nu_1 \otimes \nu_2\) is a \(\sigma\)-finite measure on \((\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)\). Let \(A = A_1 \times \ldots \times A_n\) be a measurable rectangle of \(\mathcal{F}_1, \ldots, \mathcal{F}_n\). Identifying \(A\) with \(A_{i_0} \times (\Pi_{i \neq i_0} A_i)\), we have:

\[
\nu_1 \otimes \nu_2(A) = \nu_1(A_{i_0})\nu_2(\Pi_{i \neq i_0} A_i)
\]

Since by assumption, \(\nu_2(\Pi_{i \neq i_0} A_i) = \Pi_{i \neq i_0} \mu_i(A_i)\), we conclude:

\[
\nu_1 \otimes \nu_2(A) = \mu_1(A_1) \ldots \mu_n(A_n)
\]
2. If \( n = 2 \), there exists a measure \( \mu \) on \( \mathcal{F}_1 \otimes \mathcal{F}_2 \), such that for all measurable rectangle \( A_1 \times A_2 \in \mathcal{F}_1 \Pi \mathcal{F}_2 \), we have:

\[
\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)
\]

In fact, from exercise (7), such measure is unique, \( \sigma \)-finite and equal to \( \mu_1 \otimes \mu_2 \). Suppose the following induction hypothesis is true for \( n \geq 2 \):

*Given \( n \) \( \sigma \)-finite measure spaces \( (\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n) \), there exists a measure \( \mu \) on \( (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \), such that for all measurable rectangles \( A_1 \times \ldots \times A_n \), we have:

\[
\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)
\]

Moreover, such measure \( \mu \) is \( \sigma \)-finite.*

Let us prove this induction hypothesis for \( n + 1 \). Hence, suppose we have \( n + 1 \) \( \sigma \)-finite measure spaces. Take \( E_1 = \Omega_1 \) and \( E_2 = \Omega_2 \times \ldots \times \Omega_{n+1} \). Let \( \mathcal{E}_1 = \mathcal{F}_1 \) and \( \mathcal{E}_2 = \mathcal{F}_2 \otimes \ldots \otimes \mathcal{F}_{n+1} \). Put \( \nu_1 = \mu_1 \). From our induction hypothesis, there exists a \( \sigma \)-finite measure \( \nu_2 \) on \( (E_2, \mathcal{E}_2) \), such that for all measurable
rectangles \( A_2 \times \ldots \times A_{n+1} \), we have:

\[
\nu_2(A_2 \times \ldots \times A_{n+1}) = \mu_2(A_2) \ldots \mu_{n+1}(A_{n+1})
\]

All the conditions of question 1 are met: we conclude that \( \nu_1 \otimes \nu_2 \) is a \( \sigma \)-finite measure on \( (\Omega_1 \times \ldots \times \Omega_{n+1}, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_{n+1}) \) such that for all measurable rectangles \( A = A_1 \times \ldots \times A_{n+1} \):

\[
\nu_1 \otimes \nu_2(A) = \mu_1(A_1) \ldots \mu_{n+1}(A_{n+1})
\]

This proves our induction hypothesis for \( n + 1 \).

We have proved that for all \( n \geq 2 \), and \( \sigma \)-finite measure spaces \((\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)\), there exists a \( \sigma \)-finite measure \( \mu \) on \((\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)\), such that for all measurable rectangles \( A = A_1 \times \ldots \times A_n \), \( \mu(A) = \mu_1(A_1) \ldots \mu_n(A_n) \).

Note that this is a little bit stronger (\( \mu \) is \( \sigma \)-finite !), than what was required by the actual wording of the question. However the \( \sigma \)-finite property was required to carry out the induction argument, based on exercises (6) and (7).
3. Let \( \mu \) and \( \nu \) be two measures on \((\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)\), such that for all measurable rectangles \( A = A_1 \times \ldots \times A_n \):

\[
\mu(A) = \nu(A) = \mu_1(A_1) \cdots \mu_n(A_n)
\]

For all \( i = 1, \ldots, n \), let \((\Omega^p_i)_{p \geq 1}\) be a sequence of elements of \( \mathcal{F}_i \), such that \( \Omega^p_i \uparrow \Omega_i \), and \( \mu_i(\Omega^p_i) < +\infty \) for all \( p \geq 1 \). Define \( E_p = \Omega^p_1 \times \ldots \times \Omega^p_n \). Then \( E_p \uparrow \Omega_1 \times \ldots \times \Omega_n \), and for all \( p \geq 1 \), \( \mu(E_p) = \nu(E_p) < +\infty \). Define:

\[
D_p \triangleq \{ A \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n : \mu(A \cap E_p) = \nu(A \cap E_p) \}
\]

Then \( D_p \) is a Dynkin system on \( \Omega_1 \times \ldots \times \Omega_n \). Moreover, by assumption, \( \mathcal{F}_1 \Pi \ldots \Pi \mathcal{F}_n \subseteq D_p \). The set of measurable rectangles \( \mathcal{F}_1 \Pi \ldots \Pi \mathcal{F}_n \) being closed under finite intersection, from the Dynkin system theorem (1), we see that \( D_p \) actually contains the \( \sigma \)-algebra generated by \( \mathcal{F}_1 \Pi \ldots \Pi \mathcal{F}_n \), i.e.

\[
\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \triangleq \sigma(\mathcal{F}_1 \Pi \ldots \Pi \mathcal{F}_n) \subseteq D_p
\]
Solutions to Exercises

It follows that for all $A \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we have:

$$\mu(A \cap E_p) = \nu(A \cap E_p)$$

Using theorem (7), taking the limit as $p \to +\infty$, we obtain $\mu(A) = \nu(A)$. This being true for all $A \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we conclude that $\mu = \nu$. This proves the uniqueness of the measure $\mu$ on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$, denoted $\mu_1 \otimes \ldots \otimes \mu_n$, such that $\mu(A) = \mu_1(A_1) \ldots \mu_n(A_n)$, for all measurable rectangles $A = A_1 \times \ldots \times A_n$.

4. The fact that $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ is $\sigma$-finite was actually proved as part of the induction argument of 2. However, it is very easy to justify that point directly: if $(\Omega_i^p)_{p \geq 1}$ is a sequence of elements of $\mathcal{F}_i$ such that $\Omega_i^p \uparrow \Omega_i$ and $\mu(\Omega_i^p) < +\infty$ for all $p \geq 1$, defining $E_p = \Omega_1^p \times \ldots \times \Omega_n^p$, we have $E_p \uparrow \Omega_1 \times \ldots \times \Omega_n$, and furthermore:

$$\mu(E_p) = \mu_1(\Omega_1^p) \ldots \mu_n(\Omega_n^p) < +\infty$$

So $\mu_1 \otimes \ldots \otimes \mu_n$ is indeed a $\sigma$-finite measure.
5. $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i)$ is a measure on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ which coincides with $\mu_1 \otimes \ldots \otimes \mu_n$ on the measurable rectangles. From the uniqueness property proved in 3., the two measures are therefore equal, i.e. $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i) = \mu_1 \otimes \ldots \otimes \mu_n$. 

Exercise 8
Exercise 9. Showing that definition (63) is legitimate amounts to proving the existence and uniqueness of a measure $\mu$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, such that for all $a_i \leq b_i$, $i \in \mathbb{N}_n$, we have:

$$\mu([a_1, b_1] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i) \quad (3)$$

For $i \in \mathbb{N}_n$, let $(\Omega_i, \mathcal{F}_i, \mu_i)$ be the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$, where $dx$ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Each $(\Omega_i, \mathcal{F}_i, \mu_i)$ being $\sigma$-finite, from definition (62), there exists a measure $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}))$, such that for all measurable rectangles $A = A_1 \times \ldots \times A_n$, we have:

$$\mu(A) = dx(A_1) \ldots dx(A_n) \quad (4)$$

From exercise (18) of Tutorial 6, we have $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$. So $\mu$ is in fact a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Moreover, taking $A_i$ of the form $A_i = [a_i, b_i]$ for $a_i \leq b_i$, we see from (4) that equation (3) is satisfied. Hence, we have proved the existence of $\mu$. Suppose that $\nu$
is another measure on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\) satisfying the property of definition (63). Let \(\mathcal{C} = \{[a_1, b_1] \times \ldots \times [a_n, b_n] : a_i \leq b_i, \forall i \in \mathbb{N}_n\}\). Then \(\mathcal{C}\) is closed under finite intersection. Given \(p \geq 1\), let \(E_p = [-p, p]^n\), and define:

\[
\mathcal{D}_p \triangleq \{ A \in \mathcal{B}(\mathbb{R}^n) : \mu(A \cap E_p) = \nu(A \cap E_p) \}
\]

Then \(\mathcal{D}_p\) is a Dynkin system on \(\mathbb{R}^n\), and we have \(\mathcal{C} \subseteq \mathcal{D}_p\). From the Dynkin system theorem (1), we see that \(\mathcal{D}_p\) actually contains the \(\sigma\)-algebra generated by \(\mathcal{C}\), i.e. \(\sigma(\mathcal{C}) \subseteq \mathcal{D}_p\). However, we claim that \(\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^n)\). Indeed, from:

\[
\mathcal{C} \subseteq \mathcal{B}(\mathbb{R}) \uplus \ldots \uplus \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^n)
\]

we obtain \(\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbb{R}^n)\). Furthermore, if we define:

\[
\mathcal{E} \triangleq \\{[a, b] : a \leq b, a, b \in \mathbb{R}\}
\]

then every open set in \(\mathbb{R}\) can be expressed as a countable union of elements of \(\mathcal{E}\) (see the proof of theorem (6)), and it is easy to check
that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$. From theorem (26), we have:

$$\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E} \sqcup \cdots \sqcup \mathcal{E})$$

Since any element of $\mathcal{E} \sqcup \cdots \sqcup \mathcal{E}$ is of the form $A_1 \times \cdots \times A_n$ where each $A_i$ is either equal to $\mathbb{R} = \bigcup_{p=1}^{+\infty} [-p, p]$, or is an element of $\mathcal{E}$, any element of $\mathcal{E} \sqcup \cdots \sqcup \mathcal{E}$ can in fact be expressed as a countable union of elements of $\mathcal{C}$. Hence, $\mathcal{E} \sqcup \cdots \sqcup \mathcal{E} \subseteq \sigma(\mathcal{C})$ and consequently, $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{E} \sqcup \cdots \sqcup \mathcal{E}) \subseteq \sigma(\mathcal{C})$. We conclude that $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{C})^1$, and finally $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{D}_p$. It follows that for all $A \in \mathcal{B}(\mathbb{R}^n)$, we have $\mu(A \cap E_p) = \nu(A \cap E_p)$. Using theorem (7), taking the limit as $p \to +\infty$, we obtain $\mu(A) = \nu(A)$. This being true for all $A \in \mathcal{B}(\mathbb{R}^n)$, we see that $\mu = \nu$. We have proved the uniqueness of $\mu$.

**Exercise 9**

\[1\] We proved something very similar in exercise (7) of Tutorial 6.

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Exercise 10.

1. For all \( p \geq 1 \), define \( E_p = [-p, p]^n \). Then, \( E_p \uparrow \mathbb{R}^n \), and furthermore \( dx^n(E_p) = (2p)^n < +\infty \), for all \( p \geq 1 \). So \( dx^n \) is a \( \sigma \)-finite measure on \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \).

2. Let \( a_i \leq b_i \) for \( i \in \mathbb{N}_{n+p} \), and \( A = [a_1, b_1] \times \ldots \times [a_{n+p}, b_{n+p}] \). Then, \( dx^n \otimes dx^p(A) = dx^{n+p}(A) = \prod_{i=1}^{n+p} (b_i - a_i) \). From the uniqueness property of definition (63), we conclude that:

\[ dx^{n+p} = dx^n \otimes dx^p \]
Exercise 11.

1. From exercise (6) and exercise (7), for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have:

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_1} \left( \int_{\Omega_2} 1_E(x, y) d\mu_2(y) \right) d\mu_1(x)$$

together with:

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left( \int_{\Omega_1} 1_E(x, y) d\mu_1(x) \right) d\mu_2(y)$$

Hence:

$$\int_{\Omega_1 \times \Omega_2} 1_E d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left( \int_{\Omega_2} 1_E d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left( \int_{\Omega_1} 1_E d\mu_1 \right) d\mu_2$$

By linearity, it follows that if $s = \sum_{i=1}^n \alpha_i 1_{E_i}$ is a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, we have:

$$\int_{\Omega_1 \times \Omega_2} sd\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left( \int_{\Omega_2} sd\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left( \int_{\Omega_1} sd\mu_1 \right) d\mu_2$$
2. Let \( f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty] \) be a non-negative and measurable map. From theorem (18), there exists a sequence \((s_n)_{n \geq 1}\) of simple functions on \((\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)\), such that \(s_n \uparrow f\). In particular, for all \(x \in \Omega_1\), \(s_n(x, .) \uparrow f(x, .)\). From the monotone convergence theorem (19), for all \(x \in \Omega_1\), we have:

\[
\int_{\Omega_2} s_n(x, y) d\mu_2(y) \uparrow \int_{\Omega_2} f(x, y) d\mu_2(y)
\]

and applying theorem (19) once more, we obtain:

\[
\int_{\Omega_1} \left( \int_{\Omega_2} s_n(x, y) d\mu_2(y) \right) d\mu_1(x) \uparrow \int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x)
\]

and similarly:

\[
\int_{\Omega_2} \left( \int_{\Omega_1} s_n(x, y) d\mu_1(x) \right) d\mu_2(y) \uparrow \int_{\Omega_2} \left( \int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y)
\]
However, from \( s_n \uparrow f \) and the monotone convergence theorem:

\[
\int_{\Omega_1 \times \Omega_2} s_n \, d\mu_1 \otimes \mu_2 \uparrow \int_{\Omega_1 \times \Omega_2} f \, d\mu_1 \otimes \mu_2
\]

Using 1., for all \( n \geq 1 \), we have:

\[
\int_{\Omega_1 \times \Omega_2} s_n \, d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left( \int_{\Omega_2} s_n \, d\mu_2 \right) \, d\mu_1
= \int_{\Omega_2} \left( \int_{\Omega_1} s_n \, d\mu_1 \right) \, d\mu_2
\]

Hence, taking the limit as \( n \to +\infty \), we obtain:

\[
\int_{\Omega_1 \times \Omega_2} f \, d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left( \int_{\Omega_2} f \, d\mu_2 \right) \, d\mu_1
= \int_{\Omega_2} \left( \int_{\Omega_1} f \, d\mu_1 \right) \, d\mu_2
\]

This proves theorem (31).

Exercise 11
Exercise 12.

1. Let \( f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \rightarrow [0, +\infty] \) be a non-negative and measurable map. Since \( \mu_{\sigma(1)} \) is a \( \sigma \)-finite measure, from exercise (5), the map:

\[
J_1 : \omega \rightarrow \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)
\]

is well-defined on \( \prod_{i \not\in \{\sigma(1)\}} \Omega_i \), and measurable w.r. to \( \otimes_{i \not\in \{\sigma(1)\}} \mathcal{F}_i \).

2. If \( J_k : \left( \prod_{i \not\in \{\sigma(1), \ldots, \sigma(k)\}} \Omega_i, \otimes_{i \not\in \{\sigma(1), \ldots, \sigma(k)\}} \mathcal{F}_i \right) \rightarrow [0, +\infty] \) is non-negative and measurable, for \( 1 \leq k \leq n - 2 \), from exercise (5):

\[
J_{k+1} : \omega \rightarrow \int_{\Omega_{\sigma(k+1)}} J_k(\omega, x) d\mu_{\sigma(k+1)}(x)
\]

is also well-defined on \( \prod_{i \not\in \{\sigma(1), \ldots, \sigma(k+1)\}} \Omega_i \), and measurable with respect to \( \otimes_{i \not\in \{\sigma(1), \ldots, \sigma(k+1)\}} \mathcal{F}_i \).
3. The integral:
\[ I = \int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} f \, d\mu_{\sigma(1)} \cdots d\mu_{\sigma(n)} \]
can be rigorously defined as:
\[ I \triangleq \int_{\Omega_{\sigma(n)}} J_{n-1} d\mu_{\sigma(n)} \]
where \( J_{n-1} \) is given by 1. and 2.

Exercise 12
Exercise 13.

1. Since $f_p \uparrow f$, for all $\omega \in \Pi_{i \neq \sigma(1)} \Omega_i$, we have $f_p(\omega,.) \uparrow f(\omega,.)$.

From the monotone convergence theorem (19), we obtain:

$$\int_{\Omega_{\sigma(1)}} f_p(\omega,x) d\mu_{\sigma(1)}(x) \uparrow \int_{\Omega_{\sigma(1)}} f(\omega,x) d\mu_{\sigma(1)}(x)$$

i.e. $J_p^1 \uparrow J_1$.

2. Suppose $J_p^k \uparrow J_k$, $1 \leq k \leq n-2$. For all $\omega \in \Pi_{i \neq \sigma(1),\ldots,\sigma(k+1)} \Omega_i$, we have $J_p^k(\omega,.) \uparrow J_k(\omega,.)$. From the monotone convergence theorem (19), we have:

$$\int_{\Omega_{\sigma(k+1)}} J_p^k(\omega,x) d\mu_{\sigma(k+1)}(x) \uparrow \int_{\Omega_{\sigma(k+1)}} J_k(\omega,x) d\mu_{\sigma(k+1)}(x)$$

i.e. $J_{k+1}^p \uparrow J_{k+1}$.
3. From 2., \( J_{n-1}^p \uparrow J_{n-1} \). Again from theorem (19):

\[
\int_{\Omega_{\sigma(n)}} J_{n-1}^p d\mu_{\sigma(n)} \uparrow \int_{\Omega_{\sigma(n)}} J_{n-1} d\mu_{\sigma(n)}
\]

In other words:

\[
\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f_p d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \uparrow \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}
\]

4. For all \( E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \), we have:

\[
\mu(E) \triangleq \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}
\]

So \( \mu(\emptyset) = 0 \). If \( (E_p)_{p \geq 1} \) is a sequence of pairwise disjoint elements of \( \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \), and \( E = \cup_{i=1}^{+\infty} E_i \), defining for \( p \geq 1 \), \( f_p = \sum_{i=1}^{p} 1_{E_i} \), we have \( f_p \uparrow 1_E \). It follows from 3.:

\[
\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f_p d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \uparrow \mu(E)
\]
By linearity, we obtain \( \sum_{i=1}^{p} \mu(E_i) \uparrow \mu(E) \), or equivalently:

\[
\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i)
\]

We have proved that \( \mu \) is indeed a measure on \( \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \).

5. Let \( E = A_1 \times \ldots \times A_n \) be a measurable rectangle of \( (\mathcal{F}_i)_{i \in \mathbb{N}_n} \).

Then:

\[
\mu(E) = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} = \mu_1(A_1) \ldots \mu_n(A_n)
\]

From the uniqueness property of definition (62), it follows that \( \mu \) coincide with the product measure \( \mu_1 \otimes \ldots \otimes \mu_n \). Hence, for all \( E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \), we have:

\[
\mu_1 \otimes \ldots \otimes \mu_n(E) = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}
\]
6. From 5., for all \( E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \), we have:
\[
\int_{\Omega_1 \times \ldots \times \Omega_n} 1_E d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}
\]
If \( s \) is a simple function on \((\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)\), by linearity, we obtain:
\[
\int_{\Omega_1 \times \ldots \times \Omega_n} s d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} s d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}
\]
Since any \( f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \rightarrow [0, +\infty] \) non-negative and measurable, can be approximated from below by simple functions (Theorem (18)), we conclude from the monotone convergence Theorem (19) and question 3., that:
\[
\int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \cdots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}
\]
This proves Theorem (32).

Exercise 13

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Exercise 14.

1. Suppose \( f \in L^1 \). There exists \( g \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu) \) such that \( f = g \), \( \mu \)-a.s. Hence, there exists \( N \in \mathcal{F} \) with \( \mu(N) = 0 \), such that \( f(\omega) = g(\omega) \) for all \( \omega \in N^c \). However, \( g \) has values in \( \mathbb{R} \). So \( |f(\omega)| < +\infty \) for all \( \omega \in N^c \). It follows that \( |f| < +\infty \) \( \mu \)-a.s.

2. We assume the existence of \( A \subseteq \Omega \), such that \( A \not\in \mathcal{F} \) and \( A \subseteq N \), for some \( N \in \mathcal{F} \) with \( \mu(N) = 0 \). Since \( A \not\in \mathcal{F} \), \( 1_A \) is not measurable. However, for all \( \omega \in N^c \), we have \( 1_A(\omega) = 0 \). So \( 1_A = 0 \), \( \mu \)-a.s. Since \( 0 \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu) \), we see that \( 1_A \in L^1 \).

3. Suppose \( f \in L^1 \). As indicated in 2., we have no guarantee that \( f \) be a measurable map. Hence, the integrals \( \int |f| d\mu \) and \( \int f d\mu \) may not be meaningful.

4. Let \( f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) be a measurable map, such that \( \int |f| d\mu < +\infty \). In particular, we have \( \mu(|f| = +\infty) = 0 \) (see exercise (7) of Tutorial 5). Define \( g = f1_{\{|f| < +\infty\}} \). Then,
Solutions to Exercises

6. Let \( f \in L^1 \). Let \( g_1, g_2 \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu) \) with \( f = g_1 \) \( \mu \)-a.s. and \( f = g_2 \) \( \mu \)-a.s. There exist \( N_1, N_2 \in \mathcal{F} \) with \( \mu(N_1) = \mu(N_2) = 0 \), such that \( f(\omega) = g_1(\omega) \) for all \( \omega \in N_1^c \), and \( f(\omega) = g_2(\omega) \) for all \( \omega \in N_2^c \). Since \( \mu(N \cup N_1) \leq \mu(N) + \mu(N_1) = 0 \), we see that \( f_1 = g \) \( \mu \)-a.s. We conclude that \( f_1 \in L^1 \).
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all \( \omega \in N_2^c \). So \( g_1(\omega) = g_2(\omega) \) for all \( \omega \in (N_1 \cup N_2)^c \), and 
\( \mu(N_1 \cup N_2) = 0 \). So \( g_1 = g_2 \) \( \mu \)-a.s. and finally \( \int g_1d\mu = \int g_2d\mu \).

7. For all \( f \in L^1 \), we define:

\[
\int fd\mu \triangleq \int gd\mu
\]  

(5)

where \( g \) is any element of \( L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \) such that \( f = g \) \( \mu \)-a.s.

From 6., if \( g_1, g_2 \in L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \) are such that \( f = g_1 \) \( \mu \)-a.s. and 
\( f = g_2 \) \( \mu \)-a.s., then \( \int g_1d\mu = \int g_2d\mu \). So \( \int fd\mu \) is well-defined.

If \( f \in L^1 \cap L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \), then \( \int fd\mu \) as defined in (5) coincide
with \( \int fd\mu \), in its usual sense.

Exercise 14

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**Exercise 15.**

1. By assumption, \( f_n \to f \) \( \mu \)-a.s. There exists \( N \in \mathcal{F}, \mu(N) = 0 \), such that \( f_n(\omega) \to f(\omega) \) for all \( \omega \in N^c \). Also, for all \( n \geq 1 \), \( |f_n| \leq h \) \( \mu \)-a.s. There exists \( M_n \in \mathcal{F} \) with \( \mu(M_n) = 0 \) such that \( |f_n(\omega)| \leq h(\omega) \) for all \( \omega \in M_n^c \). Let \( N_1 = N \cup (\bigcup_{n \geq 1} M_n) \). Then \( N_1 \in \mathcal{F} \), and:

\[
\mu(N_1) \leq \mu(N) + \sum_{n=1}^{+\infty} \mu(M_n) = 0
\]

So \( \mu(N_1) = 0 \). Moreover, for all \( \omega \in N_1^c \), we have \( f_n(\omega) \to f(\omega) \) and for all \( n \geq 1 \), \( |f_n(\omega)| \leq h(\omega) \).

2. Since \( f \in L^1 \), there exists \( g \in L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \) such that \( f = g \) \( \mu \)-a.s. There exists \( N \in \mathcal{F} \) with \( \mu(N) = 0 \), such that \( f(\omega) = g(\omega) \) for all \( \omega \in N^c \). Similarly, there exists \( h_1 \in L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \), and a set \( M'_1 \in \mathcal{F} \) with \( \mu(M'_1) = 0 \), such that \( h(\omega) = h_1(\omega) \) for all \( \omega \in (M'_1)^c \). For all \( n \geq 1 \), there exist \( g_n \in L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu) \) and \( M_n \in \mathcal{F} \).
with \( \mu(M_n) = 0 \) such that \( g_n(\omega) = f_n(\omega) \) for all \( \omega \in M^c_n \). Let 
\[ N_2 = N \cup M_1 \cup (\bigcup_{n \geq 1} M_n) \]. Then \( N_2 \in \mathcal{F} \), \( \mu(N_2) = 0 \), and for all \( \omega \in N^c_2 \), we have \( g(\omega) = f(\omega) \), \( h_1(\omega) = h(\omega) \) and \( g_n(\omega) = f_n(\omega) \) for all \( n \geq 1 \).

3. Let \( N = N_1 \cup N_2 \) where \( N_1 \) and \( N_2 \) are given by 1. and 2. respectively. Then \( N \in \mathcal{F} \), \( \mu(N) = 0 \), and for all \( \omega \in N^c \), we have \( g_n(\omega) \to g(\omega) \) and \( |g_n(\omega)| \leq h_1(\omega) \) for all \( n \geq 1 \).

4. \( (g_n 1_{N^c})_{n \geq 1} \) is a sequence of \( \mathbb{C} \)-valued (in fact \( \mathbb{R} \)-valued) measurable maps, such that \( g_n 1_{N^c}(\omega) \to g 1_{N^c}(\omega) \) for all \( \omega \in \Omega \). Moreover, \( h_1 1_{N^c} \) is an element of \( L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu) \) such that for all \( n \geq 1 \), \( |g_n 1_{N^c}| \leq h_1 1_{N^c} \). Hence, we can apply the dominated convergence theorem (23).

5. When \( f, f_n \in L^1 \), we have \( |f_n - f| \in L^1 \), and \( \int |f_n - f|d\mu \) is defined as \( \int kd\mu \) where \( k \) is any element of \( L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu) \) such that \( |f_n - f| = k \) \( \mu \)-a.s. In fact, \( g_n - g \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu) \) and \( |f_n - f| = |g_n - g| \) \( \mu \)-a.s. So \( \int |f_n - f|d\mu = \int |g_n - g|d\mu \).
6. From 4., and the dominated convergence theorem (23), we have
\[ \lim \int 1_{\mathcal{N}}|g_n - g_n|d\mu = 0 \] and consequently, \[ \int |g_n - g|d\mu \to 0. \] It follows from 5. that \[ \int |f_n - f|d\mu \to 0. \]

Exercise 15
Exercise 16.

1. We define \( A = \{ \omega_1 \in \Omega_1 : \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) < +\infty \} \). From theorem (30), the map \( \phi : \omega_1 \to \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) \) is measurable with respect to \( \mathcal{F}_1 \) and \( \mathcal{B}(\mathbb{R}) \). It follows that:

\[
A = \phi^{-1}([-\infty, +\infty]) \in \mathcal{F}_1
\]

From theorem (31), we have:

\[
\int_{\Omega_1} \left( \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) \right) d\mu_1(\omega_1) = \int_{\Omega_1 \times \Omega_2} |f| d\mu_1 \otimes \mu_2 < +\infty
\]

Using exercise (7) (11.) of Tutorial 5, we have \( \mu_1(A^c) = 0 \).

2. For all \( \omega_1 \in A \), we have \( \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) < +\infty \). From theorem (29), the map \( f(\omega_1, \cdot) \) is measurable with respect to \( \mathcal{F}_2 \), for all \( \omega_1 \in \mathcal{F}_1 \). \( f \) being \( \mathbb{R} \)-valued, we conclude that for all \( \omega_1 \in A \), \( f(\omega_1, \cdot) \in L^1_{\mathbb{R}}(\Omega_2, \mathcal{F}_2, \mu_2) \).
3. For all $\omega_1 \in A$, the map $f(\omega_1, \cdot)$ lies in $L^1_{\mathbb{R}}(\Omega_2, \mathcal{F}_2, \mu_2)$. Hence, $I(\omega_1) = \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$ is well-defined for all $\omega_1 \in A$.

4. If $\omega \in A$, then $J(\omega) = I(\omega) = \bar{I}(\omega) = \int_{\Omega_2} f(\omega, x) d\mu_2(x)$. Hence:

$$J(\omega) = 1_A(\omega) \int_{\Omega_2} f^+(\omega, x) d\mu_2(x) - 1_A(\omega) \int_{\Omega_2} f^-(\omega, x) d\mu_2(x)$$

This equation still holds if $\omega \notin A$.

5. $\int_{\Omega_2} f^+(\omega, x) d\mu_2(x) < +\infty$ and $\int_{\Omega_2} f^-(\omega, x) d\mu_2(x) < +\infty$, for all $\omega \in A$. If $\omega \notin A$, then $J(\omega) = 0$. It follows that $J(\omega) \in \mathbb{R}$, for all $\omega \in \Omega_1$. From theorem (30), $\omega \rightarrow \int_{\Omega_2} f^+(\omega, x) d\mu_2(x)$ and $\omega \rightarrow \int_{\Omega_2} f^-(\omega, x) d\mu_2(x)$ are $\mathcal{F}_1$-measurable maps. Furthermore, $A \in \mathcal{F}_1$. So $1_A$ is also an $\mathcal{F}_1$-measurable map. From 4., we conclude that $J$ is itself $\mathcal{F}_1$-measurable.

6. For all $\omega \in \Omega_1$, using 4., we have:

$$|J(\omega)| \leq \int_{\Omega_2} f^+ d\mu_2 + \int_{\Omega_2} f^- d\mu_2 = \int_{\Omega_2} |f(\omega, x)| d\mu_2(x)$$

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and therefore:
\[ \int_{\Omega_1} |J(\omega)|d\mu_1(\omega) \leq \int_{\Omega_1} \left( \int_{\Omega_2} |f(\omega, x)|d\mu_2(x) \right) d\mu_1(\omega) < +\infty \]

Since \( J \) is \( \mathbb{R} \)-valued and \( \mathcal{F}_1 \)-measurable, \( J \in L^1_\mathbb{R}(\Omega_1, \mathcal{F}_1, \mu) \).

Furthermore, for all \( \omega \in A \), we have \( J(\omega) = I(\omega) \). Since \( \mu_1(A^c) = 0 \), we conclude that \( J = I \) \( \mu_1 \)-a.s.

7. The map \( x \to \int_{\Omega_2} f(x, y)d\mu_2(y) \) is defined for all \( x \in A \), but may not be defined for all \( x \in \Omega_1 \). Hence, strictly speaking, the integral \( \int_{\Omega_1} (\int_{\Omega_2} f d\mu_2) d\mu_1 \) may not be meaningful. However, whichever way we choose to extend \( x \to \int_{\Omega_2} f(x, y)d\mu_2(y) \) (the map \( I \)), we have \( J = I, \mu_1 \)-a.s. where \( J \in L^1_\mathbb{R}(\Omega_1, \mathcal{F}_1, \mu_1) \).

Following the previous exercise, we see that \( I \in L^1 \), and the integral \( \int_{\Omega_1} I(x)d\mu_1(x) \) can in fact be defined as:
\[ \int_{\Omega_1} \left( \int_{\Omega_2} f(x, y)d\mu_2(y) \right) d\mu_1(x) \overset{\Delta}{=} \int_{\Omega_1} J(x)d\mu_1(x) \]
8. Since $\mu_1(A^c) = 0$, we have:

$$
\int_{\Omega_1} \left( 1_A \int_{\Omega_2} f^+ d\mu_2 \right) d\mu_1 = \int_{\Omega_1} \left( \int_{\Omega_2} f^+ d\mu_2 \right) d\mu_1
$$

Using theorem (31), we conclude that:

$$
\int_{\Omega_1} \left( 1_A \int_{\Omega_2} f^+ d\mu_2 \right) d\mu_1 = \int_{\Omega_1 \times \Omega_2} f^+ d\mu_1 \otimes \mu_2
$$

9. Using 4., 8. and its counterpart for $f^-$, we obtain:

$$
\int_{\Omega_1} J(x) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f^+ d\mu_1 \otimes \mu_2 - \int_{\Omega_1 \times \Omega_2} f^- d\mu_1 \otimes \mu_2
$$

In other words:

$$
\int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2
$$

10. Suppose that $f \in L^1(\Omega_1 \times \Omega_2, F_1 \otimes F_2, \mu_1 \otimes \mu_2)$, i.e. we no longer assume that $f$ is $\mathbb{R}$-valued. Then $f = u + iv$ where
both $u$ and $v$ are elements of $L^1_{\mathbb{K}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Applying 6, the map $\omega_1 \mapsto \int_{\Omega_2} u(\omega_1, x) d\mu_2(x)$ and the map $\omega_1 \mapsto \int_{\Omega_2} v(\omega_1, x) d\mu_2(x)$ are $\mu_1$-almost surely equal to elements of $L^1_{\mathbb{K}}(\Omega_1, \mathcal{F}_1, \mu_1)$ (say $J_u$ and $J_v$ respectively). Furthermore, from (1) we have:

$$\int_{\Omega_1} \left( \int_{\Omega_2} u(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} u d\mu_1 \otimes \mu_2$$

and:

$$\int_{\Omega_1} \left( \int_{\Omega_2} v(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} v d\mu_1 \otimes \mu_2$$

It follows that $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$ is $\mu_1$-almost surely equal to $J_u + iJ_v \in L^1_{\mathbb{C}}(\Omega_1, \mathcal{F}_1, \mu_1)$, and:

$$\int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) \triangleq \int_{\Omega_1} (J_u + iJ_v) d\mu_1$$
\[
\int_{\Omega_1} J_u d\mu_1 + i \int_{\Omega_1} J_v d\mu_1 \\
= \int_{\Omega_1} \left( \int_{\Omega_2} u(x, y) d\mu_2(y) \right) d\mu_1(x) \\
+ i \int_{\Omega_1} \left( \int_{\Omega_2} v(x, y) d\mu_2(y) \right) d\mu_1(x) \\
= \int_{\Omega_1 \times \Omega_2} ud\mu_1 \otimes \mu_2 \\
+ i \int_{\Omega_1 \times \Omega_2} vd\mu_1 \otimes \mu_2 \\
= \int_{\Omega_1 \times \Omega_2} fd\mu_1 \otimes \mu_2
\]

This proves equation (1).

11. From 5. of exercise (1), the map \( \theta \) is measurable. It follows that \( f \circ \theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to [0, +\infty] \) is indeed non-negative and
measurable. Furthermore, from theorem (31), we have:
\[
\int_{\Omega_2 \times \Omega_1} f \circ \theta \, d\mu_2 \otimes \mu_1 = \int_{\Omega_2} \left( \int_{\Omega_1} f \circ \theta(\omega_2, \omega_1) \, d\mu_1(\omega_1) \right) \, d\mu_2(\omega_2)
\]
\[
= \int_{\Omega_2} \left( \int_{\Omega_1} f(\omega_1, \omega_2) \, d\mu_1(\omega_1) \right) \, d\mu_2(\omega_2)
\]

Theorem (31) \rightarrow = \int_{\Omega_1 \times \Omega_2} f \, d\mu_1 \otimes \mu_2

12. From 5. of exercise (1), the map \( \theta \) is measurable. So \( f \circ \theta \) is itself measurable. Applying 11. to \( |f| \) we obtain:
\[
\int_{\Omega_2 \times \Omega_1} |f \circ \theta| \, d\mu_2 \otimes \mu_1 = \int_{\Omega_2 \times \Omega_1} |f| \circ \theta \, d\mu_2 \otimes \mu_1
\]
\[
= \int_{\Omega_1 \times \Omega_2} |f| \, d\mu_1 \otimes \mu_2 < +\infty
\]
So \( f \circ \theta \in L^1_C(\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1) \). If \( u = Re(f) \) and

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\[ v = Im(f), \text{ using 11. once more, we obtain:} \]
\[
\int_{\Omega_2 \times \Omega_1} f \circ \theta d\mu_2 \otimes \mu_1 = \int_{\Omega_2 \times \Omega_1} u^+ \circ \theta d\mu_2 \otimes \mu_1 \\
- \int_{\Omega_2 \times \Omega_1} u^- \circ \theta d\mu_2 \otimes \mu_1 \\
+ i \int_{\Omega_2 \times \Omega_1} v^+ \circ \theta d\mu_2 \otimes \mu_1 \\
- i \int_{\Omega_2 \times \Omega_1} v^- \circ \theta d\mu_2 \otimes \mu_1 \\
= \int_{\Omega_1 \times \Omega_2} u^+ d\mu_1 \otimes \mu_2 - \int_{\Omega_1 \times \Omega_2} u^- d\mu_1 \otimes \mu_2 \\
+ i \int_{\Omega_1 \times \Omega_2} v^+ d\mu_1 \otimes \mu_2 - i \int_{\Omega_1 \times \Omega_2} v^- d\mu_1 \otimes \mu_2 \\
= \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2
\]
13. Let \( f \in L^1_C(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2) \). From 12, \( g = f \circ \theta \) is an element of \( L^1_C(\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1) \). Applying 10. to \( g \), it follows that the map \( \omega_2 \rightarrow \int_{\Omega_1} g(\omega_2, x)d\mu_1(x) \) is \( \mu_2 \)-almost surely equal to an element of \( L^1_C(\Omega_2, \mathcal{F}_2, \mu_2) \). In other words, the map \( \omega_2 \rightarrow \int_{\Omega_1} f(x, \omega_2)d\mu_1(x) \) is \( \mu_2 \)-almost surely equal to an element of \( L^1_C(\Omega_2, \mathcal{F}_2, \mu_2) \). Furthermore, we have:

\[
\int_{\Omega_2} \left( \int_{\Omega_1} f(x, y)d\mu_1(x) \right) d\mu_2(y) = \int_{\Omega_2} \left( \int_{\Omega_1} g(y, x)d\mu_1(x) \right) d\mu_2(y)
\]

From 10. \( \rightarrow = \int_{\Omega_2 \times \Omega_1} g d\mu_2 \otimes \mu_1 \)

From 12. \( \rightarrow = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 \)

This completes the proof of theorem (33).
Exercise 17.

1. Let $f \in L^1_{\mathbb{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Define $E_1 = \Pi_{i \neq \sigma(1)} \Omega_i$, $E_2 = \Omega_{\sigma(1)}$, $E_1 = \otimes_{i \neq \sigma(1)} \mathcal{F}_i$ and $E_2 = \mathcal{F}_{\sigma(1)}$. Let $\nu_1 = \otimes_{i \neq \sigma(1)} \mu_i$ and $\nu_2 = \mu_{\sigma(1)}$. Then:

$$f \in L^1_{\mathbb{C}}(E_1 \times E_2, E_1 \otimes E_2, \nu_1 \otimes \nu_2)$$

From theorem (33), the map $\omega \to \int_{E_2} f(\omega, x) d\nu_2(x)$ (defined $\nu_1$-almost surely and arbitrarily extended on $E_1$), is $\nu_1$-almost surely equal to an element of $L^1_{\mathbb{C}}(E_1, \nu_1)$. In other words:

$$J_1(\omega) \triangleq \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

is almost surely\(^2\) equal to an element of $L^1_{\mathbb{C}}(\Pi_{i \neq \sigma(1)} \Omega_i)\(^3\).

2. $J_{k+1}$ is a.s. equal to an element of $L^1_{\mathbb{C}}(\Pi_{i \neq \{\sigma(1), \ldots, \sigma(k+1)\}} \Omega_i)$.

---

\(^2\)A case of sloppy terminology: we are trying to make the whole thing readable.

\(^3\)A case of sloppy notations.
3. From 1., \( J_1(\omega) = \int_{\Omega(\sigma_1)} f(\omega, x) d\mu_{\sigma_1}(x) \) is almost surely equal to an element of \( L^1_c(\Pi \setminus \sigma_1, \Omega) \), say \( \bar{J}_1 \). Similarly, from 2., 
\[ J_2(\omega) = \int_{\Omega(\sigma_2)} \bar{J}_1(\omega, x) d\mu_{\sigma_2}(x) \] is almost surely equal to an element of \( L^1_c(\Pi \setminus \sigma_2, \Omega) \), say \( \bar{J}_2 \). By induction, we obtain a map \( J_{n-1} \) defined on \( \Omega(\sigma_n) \), and \( \mu_{\sigma_n} \)-almost surely equal to an element of \( L^1_c(\Omega(\sigma_n)) \), say \( \bar{J}_{n-1} \). We define:

\[
\int_{\Omega_\sigma(n)} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \triangleq \int_{\Omega_{\sigma(n)}} \bar{J}_{n-1} d\mu_{\sigma(n)}
\]

This multiple integral is a well-defined complex number. It is easy to check by induction that which ever choice is made of \( \bar{J}_1, \ldots, \bar{J}_{n-2} \), the map \( \bar{J}_{n-1} \) is unique up to \( \mu_{\sigma(n)} \)-almost sure equality. Hence, this multiple integral is uniquely defined.

4. From theorem (33), we have:

\[
\int_{\Pi \setminus \sigma(1), \Omega_1} \bar{J}_1(\omega) d \otimes i \neq \sigma(1) \mu_i = \int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n
\]
Following an induction argument, we obtain:
\[
\int_{\Omega_{\sigma(n)}} J_{n-1} d\mu_{\sigma(n)} = \int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n
\]
i.e.
\[
\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} = \int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n
\]
This solution is not as detailed as it could have been.

Exercise 17