16. Differentiation

Definition 115  Let $(\Omega, \mathcal{T})$ be a topological space. A map $f : \Omega \to \bar{\mathbb{R}}$ is said to be lower-semi-continuous (l.s.c), if and only if:

$$\forall \lambda \in \mathbb{R}, \{\lambda < f\} \text{ is open}$$

We say that $f$ is upper-semi-continuous (u.s.c), if and only if:

$$\forall \lambda \in \mathbb{R}, \{f < \lambda\} \text{ is open}$$

Exercise 1. Let $f : \Omega \to \bar{\mathbb{R}}$ be a map, where $\Omega$ is a topological space.

1. Show that $f$ is l.s.c if and only if $\{\lambda < f\}$ is open for all $\lambda \in \bar{\mathbb{R}}$.
2. Show that $f$ is u.s.c if and only if $\{f < \lambda\}$ is open for all $\lambda \in \bar{\mathbb{R}}$.
3. Show that every open set $U$ in $\bar{\mathbb{R}}$ can be written:

$$U = V^+ \cup V^- \cup \bigcup_{i \in I}[\alpha_i, \beta_i[$$

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for some index set \( I, \alpha_i, \beta_i \in \mathbb{R}, V^+ = \emptyset \) or \( V^+ = ]\alpha, +\infty[ \), 
\((\alpha \in \mathbb{R}) \) and \( V^- = \emptyset \) or \( V^- = ]-\infty, \beta[, (\beta \in \mathbb{R}) \).

4. Show that \( f \) is continuous if and only if it is both l.s.c and u.s.c.

5. Let \( u : \Omega \to \mathbb{R} \) and \( v : \Omega \to \bar{\mathbb{R}} \). Let \( \lambda \in \mathbb{R} \). Show that:
\[
\{ \lambda < u + v \} = \bigcup_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} \{ \lambda_1 < u \} \cap \{ \lambda_2 < v \} \\
\lambda_1 + \lambda_2 = \lambda
\]

6. Show that if both \( u \) and \( v \) are l.s.c, then \( u + v \) is also l.s.c.

7. Show that if both \( u \) and \( v \) are u.s.c, then \( u + v \) is also u.s.c.

8. Show that if \( f \) is l.s.c, then \( \alpha f \) is l.s.c, for all \( \alpha \in \mathbb{R}^+ \).

9. Show that if \( f \) is u.s.c, then \( \alpha f \) is u.s.c, for all \( \alpha \in \mathbb{R}^+ \).

10. Show that if \( f \) is l.s.c, then \( -f \) is u.s.c.
11. Show that if $f$ is u.s.c, then $-f$ is l.s.c.

12. Show that if $V$ is open in $\Omega$, then $f = 1_V$ is l.s.c.

13. Show that if $F$ is closed in $\Omega$, then $f = 1_F$ is u.s.c.

**Exercise 2.** Let $(f_i)_{i \in I}$ be an arbitrary family of maps $f_i : \Omega \to \mathbb{R}$, defined on a topological space $\Omega$.

1. Show that if all $f_i$’s are l.s.c, then $f = \sup_{i \in I} f_i$ is l.s.c.

2. Show that if all $f_i$’s are u.s.c, then $f = \inf_{i \in I} f_i$ is u.s.c.

**Exercise 3.** Let $(\Omega, T)$ be a metrizable and $\sigma$-compact topological space. Let $\mu$ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $f$ be an element of $L^1_\mathbb{R}(\Omega, \mathcal{B}(\Omega), \mu)$, such that $f \geq 0$.
1. Let \((s_n)_{n \geq 1}\) be a sequence of simple functions on \((\Omega, \mathcal{B}(\Omega))\) such that \(s_n \uparrow f\). Define \(t_1 = s_1\) and \(t_n = s_n - s_{n-1}\) for all \(n \geq 2\).

Show that \(t_n\) is a simple function on \((\Omega, \mathcal{B}(\Omega))\), for all \(n \geq 1\).

2. Show that \(f\) can be written as:

\[
f = \sum_{n=1}^{+\infty} \alpha_n 1_{A_n}
\]

where \(\alpha_n \in \mathbb{R}^+ \setminus \{0\}\) and \(A_n \in \mathcal{B}(\Omega)\), for all \(n \geq 1\).

3. Show that \(\mu(A_n) < +\infty\), for all \(n \geq 1\).

4. Show that there exist \(K_n\) compact and \(V_n\) open in \(\Omega\) such that:

\[
K_n \subseteq A_n \subseteq V_n, \quad \mu(V_n \setminus K_n) \leq \frac{\epsilon}{\alpha_n 2^n + 1}
\]

for all \(\epsilon > 0\) and \(n \geq 1\).
5. Show the existence of \( N \geq 1 \) such that:
\[
\sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}
\]

6. Define \( u = \sum_{n=1}^{N} \alpha_n 1_{K_n} \). Show that \( u \) is u.s.c.

7. Define \( v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n} \). Show that \( v \) is l.s.c.

8. Show that we have \( 0 \leq u \leq f \leq v \).

9. Show that we have:
\[
v = u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n}
\]

10. Show that \( \int v d\mu \leq \int u d\mu + \epsilon < +\infty \).

11. Show that \( u \in L^1_R(\Omega, \mathcal{B}(\Omega), \mu) \).

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12. Explain why \( v \) may fail to be in \( L^1_{\mathbb{R}}(\Omega, \mathcal{B}(\Omega), \mu) \).

13. Show that \( v \) is \( \mu \)-a.s. equal to an element of \( L^1_{\mathbb{R}}(\Omega, \mathcal{B}(\Omega), \mu) \).

14. Show that \( \int (v - u) d\mu \leq \epsilon \).

15. Prove the following:

**Theorem 94 (Vitali-Caratheodory)** Let \( (\Omega, \mathcal{T}) \) be a metrizable and \( \sigma \)-compact topological space. Let \( \mu \) be a locally finite measure on \( (\Omega, \mathcal{B}(\Omega)) \) and \( f \) be an element of \( L^1_{\mathbb{R}}(\Omega, \mathcal{B}(\Omega), \mu) \). Then, for all \( \epsilon > 0 \), there exist measurable maps \( u, v : \Omega \to \mathbb{R} \), which are \( \mu \)-a.s. equal to elements of \( L^1_{\mathbb{R}}(\Omega, \mathcal{B}(\Omega), \mu) \), such that \( u \leq f \leq v \), \( u \) is u.s.c, \( v \) is l.s.c, and furthermore:

\[
\int (v - u) d\mu \leq \epsilon
\]
Definition 116 Let \((\Omega, T)\) be a topological space. We say that \((\Omega, T)\) is \textbf{connected}, if and only if the only subsets of \(\Omega\) which are both open and closed are \(\Omega\) and \(\emptyset\).

Exercise 4. Let \((\Omega, T)\) be a topological space.

1. Show that \((\Omega, T)\) is connected if and only if whenever \(\Omega = A \cup B\) where \(A, B\) are disjoint open sets, we have \(A = \emptyset\) or \(B = \emptyset\).

2. Show that \((\Omega, T)\) is connected if and only if whenever \(\Omega = A \cup B\) where \(A, B\) are disjoint closed sets, we have \(A = \emptyset\) or \(B = \emptyset\).

Definition 117 Let \((\Omega, T)\) be a topological space, and \(A \subseteq \Omega\). We say that \(A\) is a \textbf{connected subset} of \(\Omega\), if and only if the induced topological space \((A, T|_A)\) is connected.

Exercise 5. Let \(A\) be open and closed in \(\mathbb{R}\), with \(A \neq \emptyset\) and \(A^c \neq \emptyset\).
1. Let $x \in A^c$. Show that $A \cap [x, +\infty]$ or $A \cap (-\infty, x]$ is non-empty.

2. Suppose $B = A \cap [x, +\infty] \neq \emptyset$. Show that $B$ is closed and that we have $B = A \cap [x, +\infty]$. Conclude that $B$ is also open.

3. Let $b = \inf B$. Show that $b \in B$ (and in particular $b \in \mathbb{R}$).

4. Show the existence of $\epsilon > 0$ such that $]b - \epsilon, b + \epsilon[ \subseteq B$.

5. Conclude with the following:

**Theorem 95**  
*The topological space $(\mathbb{R}, T_{\mathbb{R}})$ is connected.*

**Exercise 6.** Let $(\Omega, T)$ be a topological space and $A \subseteq \Omega$ be a connected subset of $\Omega$. Let $B$ be a subset of $\Omega$ such that $A \subseteq B \subseteq \overline{A}$. We assume that $B = V_1 \uplus V_2$ where $V_1, V_2$ are disjoint open sets in $B$.

1. Show there is $U_1, U_2$ open in $\Omega$, with $V_1 = B \cap U_1, V_2 = B \cap U_2$. 

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2. Show that \( A \cap U_1 = \emptyset \) or \( A \cap U_2 = \emptyset \).

3. Suppose that \( A \cap U_1 = \emptyset \). Show that \( \bar{A} \subseteq U_1^c \).

4. Show then that \( V_1 = B \cap U_1 = \emptyset \).

5. Conclude that \( B \) and \( \bar{A} \) are both connected subsets of \( \Omega \).

**Exercise 7.** Prove the following:

**Theorem 96**  Let \( (\Omega, T), (\Omega', T') \) be two topological spaces, and \( f \) be a continuous map, \( f : \Omega \to \Omega' \). If \( (\Omega, T) \) is connected, then \( f(\Omega) \) is a connected subset of \( \Omega' \).

**Definition 118**  Let \( A \subseteq \bar{\mathbb{R}} \). We say that \( A \) is an **interval**, if and only if for all \( x, y \in A \) with \( x \leq y \), we have \( [x, y] \subseteq A \), where:

\[
[x, y] \triangleq \{ z \in \bar{\mathbb{R}} : x \leq z \leq y \}
\]
Exercise 8. Let $A \subseteq \mathbb{R}$.

1. If $A$ is an interval, and $\alpha = \inf A$, $\beta = \sup A$, show that:
   \[ \alpha, \beta \subseteq A \subseteq [\alpha, \beta] \]
   2. Show that $A$ is an interval if and only if, it is of the form $[\alpha, \beta]$, $[\alpha, \beta[, ]\alpha, \beta]$ or $]\alpha, \beta[\,$, for some $\alpha, \beta \in \mathbb{R}$.
   3. Show that an interval of the form $] - \infty, \alpha[, \,$ where $\alpha \in \mathbb{R}$, is homeomorphic to $] - 1, \alpha'[\,$, for some $\alpha' \in \mathbb{R}$.
   4. Show that an interval of the form $]\alpha, +\infty[, \,$ where $\alpha \in \mathbb{R}$, is homeomorphic to $]\alpha', 1[, \,$ for some $\alpha' \in \mathbb{R}$.
   5. Show that an interval of the form $]\alpha, \beta[, \,$ where $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$, is homeomorphic to $]-1, 1[$.
   6. Show that $]-1, 1[$ is homeomorphic to $\mathbb{R}$.
   7. Show an non-empty open interval in $\mathbb{R}$, is homeomorphic to $\mathbb{R}$.

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8. Show that an open interval in $\mathbb{R}$, is a connected subset of $\mathbb{R}$.

9. Show that an interval in $\mathbb{R}$, is a connected subset of $\mathbb{R}$.

**Exercise 9.** Let $A \subseteq \mathbb{R}$ be a non-empty connected subset of $\mathbb{R}$, and $\alpha = \inf A$, $\beta = \sup A$. We assume there exists $x_0 \in A^{c} \cap ]\alpha, \beta[$.

1. Show that $A \cap ]x_0, +\infty[ \cup A \cap ]-\infty, x_0[ \text{ is empty.}$

2. Show that $A \cap ]x_0, +\infty[ = \emptyset$ leads to a contradiction.

3. Show that $]x_0, +\infty[ \subseteq A \subseteq [\alpha, \beta]$.

4. Show the following:

**Theorem 97**  For all $A \subseteq \mathbb{R}$, $A$ is a connected subset of $\mathbb{R}$, if and only if $A$ is an interval.
Exercise 10. Prove the following:

**Theorem 98** Let \( f : \Omega \rightarrow \mathbb{R} \) be a continuous map, where \((\Omega, T)\) is a connected topological space. Let \( a, b \in \Omega \) such that \( f(a) \leq f(b) \). Then, for all \( z \in [f(a), f(b)] \), there exists \( x \in \Omega \) such that \( z = f(x) \).

Exercise 11. Let \( a, b \in \mathbb{R}, a < b \), and \( f : [a, b] \rightarrow \mathbb{R} \) be a map such that \( f'(x) \) exists for all \( x \in [a, b] \).

1. Show that \( f' : ([a, b], \mathcal{B}([a, b])) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is measurable.

2. Show that \( f' \in L^1_{\mathbb{R}}([a, b], \mathcal{B}([a, b]), dx) \) is equivalent to:
\[
\int_a^b |f'(t)| dt < +\infty
\]

3. We assume from now on that \( f' \in L^1_{\mathbb{R}}([a, b], \mathcal{B}([a, b]), dx) \). Given \( \epsilon > 0 \), show the existence of \( g : [a, b] \rightarrow \mathbb{R} \), almost surely equal...
to an element of $L^1_{\mathbb{R}}([a, b], \mathcal{B}([a, b]), dx)$, such that $f' \leq g$ and $g$
is l.s.c, with:

$$\int_a^b g(t) dt \leq \int_a^b f'(t) dt + \epsilon$$

4. By considering $g + \alpha$ for some $\alpha > 0$, show that without loss of
generality, we can assume that $f' < g$ with the above inequality
still holding.

5. We define the complex measure $\nu = \int g dx \in M^1([a, b], \mathcal{B}([a, b]))$.
Show that:

$$\forall \epsilon' > 0, \ \exists \delta > 0, \ \forall E \in \mathcal{B}([a, b]), \ dx(E) \leq \delta \Rightarrow |\nu(E)| < \epsilon'$$

6. For all $\eta > 0$ and $x \in [a, b]$, we define:

$$F_\eta(x) \triangleq \int_a^x g(t) dt - f(x) + f(a) + \eta(x - a)$$

Show that $F_\eta : [a, b] \rightarrow \mathbb{R}$ is a continuous map.
7. \( \eta \) being fixed, let \( x = \sup F_{\eta}^{-1}(\{0\}) \). Show that \( x \in [a, b] \) and \( F_{\eta}(x) = 0 \).

8. We assume that \( x \in [a, b] \). Show the existence of \( \delta > 0 \) such that for all \( t \in ]x, x + \delta[ \cap [a, b] \), we have:

\[
f'(x) < g(t) \quad \text{and} \quad \frac{f(t) - f(x)}{t - x} < f'(x) + \eta
\]

9. Show that for all \( t \in ]x, x + \delta[ \cap [a, b] \), we have \( F_{\eta}(t) > F_{\eta}(x) = 0 \).

10. Show that there exists \( t_0 \) such that \( x < t_0 < b \) and \( F_{\eta}(t_0) > 0 \).

11. Show that \( F_{\eta}(b) < 0 \) leads to a contradiction.

12. Conclude that \( F_{\eta}(b) \geq 0 \), even if \( x = b \).

13. Show that \( f(b) - f(a) \leq \int_a^b f'(t)dt \), and conclude:
Theorem 99 (Fundamental Calculus)  Let \( a, b \in \mathbb{R}, \ a < b, \) and \( f : [a, b] \rightarrow \mathbb{R} \) be a map which is differentiable at every point of \([a, b] \), and such that:

\[
\int_{a}^{b} |f'(t)| \, dt < +\infty
\]

Then, we have:

\[
f(b) - f(a) = \int_{a}^{b} f'(t) \, dt
\]

Exercise 12. Let \( \alpha > 0, \) and \( k_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by \( k_\alpha(x) = \alpha x. \)

1. Show that \( k_\alpha : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \) is measurable.

2. Show that for all \( B \in \mathcal{B}(\mathbb{R}^n), \) we have:

\[
dx(k_\alpha \in B) = \frac{1}{\alpha^n} dx(B)
\]

3. Show that for all \( \epsilon > 0 \) and \( x \in \mathbb{R}^n: \)

\[
dx(B(x, \epsilon)) = \epsilon^n dx(B(0, 1))
\]
**Definition 119** Let $\mu$ be a complex measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \geq 1$, with total variation $|\mu|$. We call **maximal function** of $\mu$, the map $M\mu : \mathbb{R}^n \to [0, +\infty]$, defined by:

$$\forall x \in \mathbb{R}^n, \quad (M\mu)(x) \triangleq \sup_{\epsilon > 0} \frac{|\mu|(B(x, \epsilon))}{dx(B(x, \epsilon))}$$

where $B(x, \epsilon)$ is the open ball in $\mathbb{R}^n$, of center $x$ and radius $\epsilon$, with respect to the usual metric of $\mathbb{R}^n$.

**Exercise 13.** Let $\mu$ be a complex measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

1. Let $\lambda \in \mathbb{R}$. Show that if $\lambda < 0$, then $\{\lambda < M\mu\} = \mathbb{R}^n$.

2. Show that if $\lambda = 0$, then $\{\lambda < M\mu\} = \mathbb{R}^n$ if $\mu \neq 0$, and $\{\lambda < M\mu\}$ is the empty set if $\mu = 0$.

3. Suppose $\lambda > 0$. Let $x \in \{\lambda < M\mu\}$. Show the existence of $\epsilon > 0$ such that $|\mu|(B(x, \epsilon)) = tdx(B(x, \epsilon))$, for some $t > \lambda$.
4. Show the existence of $\delta > 0$ such that $(\epsilon + \delta)^n < \epsilon^n t / \lambda$.

5. Show that if $y \in B(x, \delta)$, then $B(x, \epsilon) \subseteq B(y, \epsilon + \delta)$.

6. Show that if $y \in B(x, \delta)$, then:

$$|\mu|(B(y, \epsilon + \delta)) \geq \frac{\epsilon^n t}{(\epsilon + \delta)^n} dx(B(y, \epsilon + \delta)) > \lambda dx(B(y, \epsilon + \delta))$$

7. Conclude that $B(x, \delta) \subseteq \{ \lambda < M\mu \}$, and that the maximal function $M\mu : \mathbb{R}^n \rightarrow [0, +\infty]$ is l.s.c, and therefore measurable.

**Exercise 14.** Let $B_i = B(x_i, \epsilon_i)$, $i = 1, \ldots, N$, $N \geq 1$, be a finite collection of open balls in $\mathbb{R}^n$. Assume without loss of generality that $\epsilon_N \leq \ldots \leq \epsilon_1$. We define a sequence $(J_k)$ of sets by $J_0 = \{1, \ldots, N\}$ and for all $k \geq 1$:

$$J_k \triangleq \begin{cases} J_{k-1} \cap \{j : j > i_k, B_j \cap B_{i_k} = \emptyset\} & \text{if } J_{k-1} \neq \emptyset \\ \emptyset & \text{if } J_{k-1} = \emptyset \end{cases}$$
where we have put $i_k = \min J_{k-1}$, whenever $J_{k-1} \neq \emptyset$.

1. Show that if $J_{k-1} \neq \emptyset$ then $J_k \subset J_{k-1}$ (strict inclusion), $k \geq 1$.

2. Let $p = \min\{k \geq 1 : J_k = \emptyset\}$. Show that $p$ is well-defined.

3. Let $S = \{i_1, \ldots, i_p\}$. Explain why $S$ is well defined.

4. Suppose that $1 \leq k < k' \leq p$. Show that $i_{k'} \in J_k$.

5. Show that $(B_i)_{i \in S}$ is a family of pairwise disjoint open balls.

6. Let $i \in \{1, \ldots, N\} \setminus S$, and define $k_0$ to be the minimum of the set $\{k \in \mathbb{N}_p : i \notin J_k\}$. Explain why $k_0$ is well-defined.

7. Show that $i \in J_{k_0-1}$ and $i_{k_0} \leq i$.

8. Show that $B_i \cap B_{i_{k_0}} \neq \emptyset$.

9. Show that $B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}})$. 

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10. Conclude that there exists a subset $S$ of $\{1, \ldots, N\}$ such that $(B_i)_{i \in S}$ is a family of pairwise disjoint balls, and:

$$\bigcup_{i=1}^{N} B(x_i, \epsilon_i) \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_i)$$

11. Show that:

$$dx\left(\bigcup_{i=1}^{N} B(x_i, \epsilon_i)\right) \leq 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i))$$

**Exercise 15.** Let $\mu$ be a complex measure on $\mathbb{R}^n$. Let $\lambda > 0$ and $K$ be a non-empty compact subset of $\{\lambda < M\mu\}$.

1. Show that $K$ can be covered by a finite collection $B_i = B(x_i, \epsilon_i)$, $i = 1, \ldots, N$ of open balls, such that:

$$\forall i = 1, \ldots, N \ , \ \lambda dx(B_i) < |\mu|(B_i)$$
2. Show the existence of $S \subseteq \{1, \ldots, N\}$ such that:

$$dx(K) \leq 3^n \lambda^{-1} |\mu| \left( \bigcup_{x \in S} B(x, \epsilon_i) \right)$$

3. Show that $dx(K) \leq 3^n \lambda^{-1} \| \mu \|$

4. Conclude with the following:

**Theorem 100** Let $\mu$ be a complex measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \geq 1$, with maximal function $M\mu$. Then, for all $\lambda \in \mathbb{R}^+ \setminus \{0\}$, we have:

$$dx(\{ \lambda < M\mu \}) \leq 3^n \lambda^{-1} \| \mu \|$$

**Definition 120** Let $f \in L^1_c(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, and $\mu$ be the complex measure $\mu = \int f \, dx$ on $\mathbb{R}^n$, $n \geq 1$. We call maximal function of $f$, denoted $Mf$, the maximal function $M\mu$ of $\mu$. 

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Exercise 16. Let $f \in L^1_{C}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, $n \geq 1$.

1. Show that for all $x \in \mathbb{R}^n$:
   
   $$(Mf)(x) = \sup_{\epsilon > 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f| dx$$

2. Show that for all $\lambda > 0$, $dx(\{\lambda < Mf\}) \leq 3^n \lambda^{-1} \|f\|_1$.

Definition 121. Let $f \in L^1_{C}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, $n \geq 1$. We say that $x \in \mathbb{R}^n$ is a Lebesgue point of $f$, if and only if we have:

$$\lim_{\epsilon \to 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| dy = 0$$

Exercise 17. Let $f \in L^1_{C}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, $n \geq 1$.

1. Show that if $f$ is continuous at $x \in \mathbb{R}^n$, then $x$ is a Lebesgue point of $f$.  

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2. Show that if \( x \in \mathbb{R}^n \) is a Lebesgue point of \( f \), then:

\[
f(x) = \lim_{\epsilon \downarrow 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} f(y)dy
\]

**Exercise 18.** Let \( n \geq 1 \) and \( f \in L^1_c(\mathbb{R}^n, B(\mathbb{R}^n), dx) \). For all \( \epsilon > 0 \) and \( x \in \mathbb{R}^n \), we define:

\[
(T_\epsilon f)(x) \triangleq \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)|dy
\]

and we put, for all \( x \in \mathbb{R}^n \):

\[
(Tf)(x) \triangleq \limsup_{\epsilon \downarrow 0} (T_\epsilon f)(x) \triangleq \inf_{\epsilon > 0} \sup_{u \in [0, \epsilon]} (T_\epsilon f)(x)
\]

1. Given \( \eta > 0 \), show the existence of \( g \in C^c_c(\mathbb{R}^n) \) such that:

\[
\|f - g\|_1 \leq \eta
\]
2. Let $h = f - g$. Show that for all $\epsilon > 0$ and $x \in \mathbb{R}^n$:

$$
(T, h)(x) \leq \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |h| dx + |h(x)|
$$

3. Show that $Th \leq Mh + |h|$.

4. Show that for all $\epsilon > 0$, we have $T_{\epsilon}f \leq T_{\epsilon}g + T_{\epsilon}h$.

5. Show that $Tf \leq Tg + Th$.

6. Using the continuity of $g$, show that $Tg = 0$.

7. Show that $Tf \leq Mh + |h|.

8. Show that for all $\alpha > 0$, $\{2\alpha < Tf\} \subseteq \{\alpha < Mh\} \cup \{\alpha < |h|\}$.

9. Show that $dx(\{\alpha < |h|\}) \leq \alpha^{-1} \|h\|_1$.

10. Conclude that for all $\alpha > 0$ and $\eta > 0$, there is $N_{\alpha, \eta} \in B(\mathbb{R}^n)$ such that $\{2\alpha < Tf\} \subseteq N_{\alpha, \eta}$ and $dx(N_{\alpha, \eta}) \leq \eta$. 

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11. Show that for all $\alpha > 0$, there exists $N_\alpha \in \mathcal{B}(\mathbb{R}^n)$ such that 
$\{2\alpha < Tf\} \subseteq N_\alpha$ and $dx(N_\alpha) = 0$.

12. Show there is $N \in \mathcal{B}(\mathbb{R}^n)$, $dx(N) = 0$, such that $\{Tf > 0\} \subseteq N$.

13. Conclude that $Tf = 0$, $dx$–a.s.

14. Conclude with the following:

**Theorem 101** Let $f \in L^1_c(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$, $n \geq 1$. Then, $dx$–almost surely, any $x \in \mathbb{R}^n$ is a Lebesgue points of $f$, i.e.

$$dx$-a.s., \quad \lim_{\epsilon \downarrow 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| dy = 0$$

**Exercise 19.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \in \mathcal{F}$. We define $\mathcal{F}' = \mathcal{F}|_{\Omega'}$ and $\mu' = \mu|_{\mathcal{F}'}$. For all maps $f : \Omega' \rightarrow [0, +\infty]$ (or

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C), we define \( \tilde{f} : \Omega \rightarrow [0, +\infty] \) (or \( C \)), by:

\[
\tilde{f}(\omega) \triangleq \begin{cases} 
  f(\omega) & \text{if } \omega \in \Omega' \\
  0 & \text{if } \omega \notin \Omega'
\end{cases}
\]

1. Show that \( \mathcal{F}' \subseteq \mathcal{F} \) and conclude that \( \mu' \) is therefore a well-defined measure on \( (\Omega', \mathcal{F}') \).

2. Let \( A \in \mathcal{F}' \) and \( 1_A' \) be the characteristic function of \( A \) defined on \( \Omega' \). Let \( 1_A \) be the characteristic function of \( A \) defined on \( \Omega \). Show that \( 1_A' = 1_A \).

3. Let \( f : (\Omega', \mathcal{F}') \rightarrow [0, +\infty] \) be a non-negative and measurable map. Show that \( \tilde{f} : (\Omega, \mathcal{F}) \rightarrow [0, +\infty] \) is also non-negative and measurable, and that we have:

\[
\int_{\Omega'} f \, d\mu' = \int_{\Omega} \tilde{f} \, d\mu
\]
4. Let $f \in L^1_C(\Omega', \mathcal{F}', \mu')$. Show that $\tilde{f} \in L^1_C(\Omega, \mathcal{F}, \mu)$, and:

$$\int_{\Omega'} f \, d\mu' = \int_{\Omega} \tilde{f} \, d\mu$$

**Definition 122** $b : \mathbb{R}^+ \to \mathbb{C}$ is absolutely continuous, if and only if $b$ is right-continuous of finite variation, and $b$ is absolutely continuous with respect to $a(t) = t$.

**Exercise 20.** Let $b : \mathbb{R}^+ \to \mathbb{C}$ be a map.

1. Show that $b$ is absolutely continuous, if and only if there is $f \in L^1_{C_{\text{loc}}}(t)$ such that $b(t) = \int_0^t f(s) \, ds$, for all $t \in \mathbb{R}^+$.

2. Show that $b$ absolutely continuous $\Rightarrow$ $b$ continuous with $b(0) = 0$.

**Exercise 21.** Let $b : \mathbb{R}^+ \to \mathbb{C}$ be an absolutely continuous map. Let $f \in L^1_{C_{\text{loc}}}(t)$ be such that $b = f.t$. For all $n \geq 1$, we define

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Let $n \geq 1$. Show $f_n \in L^1_C(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$ and for all $t \in [0, n]$:

$$b(t) = \int_0^t f_n dx$$

2. Show the existence of $N_n \in \mathcal{B}(\mathbb{R})$ such that $dx(N_n) = 0$, and for all $t \in N_n$, $t$ is a Lebesgue point of $f_n$.

3. Show that for all $t \in \mathbb{R}$, and $\epsilon > 0$:

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} |f_n(s) - f_n(t)|ds \leq \frac{2}{dx(B(t, \epsilon))} \int_{B(t, \epsilon)} |f_n(s) - f_n(t)|ds$$

4. Show that for all $t \in N_n^c$, we have:

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} f_n(s)ds = f_n(t)$$
5. Show similarly that for all \( t \in N_n^c \), we have:
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s) ds = f_n(t)
\]

6. Show that for all \( t \in N_n^c \cap [0,n[ \), \( b'(t) \) exists and \( b'(t) = f(t) \).

7. Show the existence of \( N \in \mathcal{B}(\mathbb{R}^+) \), such that \( dx(N) = 0 \), and:
\[
\forall t \in N^c \ , \ b'(t) \text{ exists with } b'(t) = f(t)
\]

8. Conclude with the following:

\[b'(0) \text{ being a r.h.s derivative only.}\]
Theorem 102  A map \( b : \mathbb{R}^+ \rightarrow \mathbb{C} \) is absolutely continuous, if and only if there exists \( f \in L^1_{\text{loc}}(t) \) such that:

\[
\forall t \in \mathbb{R}^+, \quad b(t) = \int_0^t f(s)ds
\]

in which case, \( b \) is almost surely differentiable with \( b' = f \text{ d}x \)-a.s.
Solutions to Exercises

Exercise 1.

1. Let $f : \Omega \rightarrow \mathbb{R}$ be a map, where $\Omega$ is a topological space. Suppose that $\{ \lambda < f \}$ is open for all $\lambda \in \mathbb{R}$. Then in particular, $\{ \lambda < f \}$ is open for all $\lambda \in \mathbb{R}$. So $f$ is l.s.c. Conversely, suppose $f$ is l.s.c. Then $\{ \lambda < f \}$ is open for all $\lambda \in \mathbb{R}$, and since:

$$\{ -\infty < f \} = \bigcup_{\lambda \in \mathbb{R}} \{ \lambda < f \}$$

it follows that $\{ -\infty < f \}$ is also open. Furthermore, $\{ +\infty < f \}$ is the empty set, and in particular, $\{ +\infty < f \}$ is open. We conclude that $\{ \lambda < f \}$ is open for all $\lambda \in \mathbb{R}$. We have proved that $f$ is l.s.c if and only if $\{ \lambda < f \}$ is open for all $\lambda \in \mathbb{R}$.

2. Similarly to 1. we have:

$$\{ f < +\infty \} = \bigcup_{\lambda \in \mathbb{R}} \{ f < \lambda \}$$
and \( \{ f < -\infty \} = \emptyset \) which is open. We conclude that \( f \) is u.s.c if and only if \( \{ f < \lambda \} \) is open for all \( \lambda \in \mathbb{R} \).

3. Let \( U \) be open in \( \mathbb{R} \). If \( +\infty \in U \), let \( V^+ = [\alpha, +\infty] \) where \( \alpha \in \mathbb{R} \) is such that \( [\alpha, +\infty] \subseteq U \). Otherwise, let \( V^+ = \emptyset \). If \( -\infty \in U \), let \( V^- = (-\infty, \beta] \), where \( \beta \in \mathbb{R} \) is such that \( (-\infty, \beta] \subseteq U \). Otherwise, let \( V^- = \emptyset \). Then, we have:

\[
U = V^+ \cup V^- \cup (U \cap \mathbb{R})
\]

and \( U \cap \mathbb{R} \) is an open subset of \( \mathbb{R} \) (possibly empty). For all \( x \in U \cap \mathbb{R} \), let \( \alpha_x, \beta_x \in \mathbb{R} \) be such that \( x \in [\alpha_x, \beta_x] \subseteq U \cap \mathbb{R} \). Then, we have:

\[
U \cap \mathbb{R} = \bigcup_{x \in U \cap \mathbb{R}} [\alpha_x, \beta_x]
\]

where it is understood that if \( U \cap \mathbb{R} = \emptyset \), the corresponding union is the empty set. Taking \( I = U \cap \mathbb{R} \), we conclude that:

\[
U = V^+ \cup V^- \cup \bigcup_{i \in I} [\alpha_i, \beta_i]
\]
4. Suppose that $f$ is continuous. For all $\lambda \in \mathbb{R}$, the interval $]\lambda, +\infty]$ is an open subset of $\mathbb{R}$. It follows that $\{ \lambda < f \} = f^{-1}(]\lambda, +\infty])$ is open. This being true for all $\lambda \in \mathbb{R}$, $f$ is l.s.c. Similarly, the interval $[-\infty, \lambda]$ is an open subset of $\mathbb{R}$. It follows that $\{ f < \lambda \} = f^{-1}([-\infty, \lambda])$ is open. This being true for all $\lambda \in \mathbb{R}$, $f$ is u.s.c. Hence, if $f$ is continuous, it is both l.s.c and u.s.c. Conversely, suppose $f$ is both l.s.c. and u.s.c. Let $U$ be an open subset of $\mathbb{R}$. Using the decomposition obtained in 3. we have:

$$f^{-1}(U) = f^{-1}\left(V^+ \cup V^- \cup \bigcup_{i \in I} [\alpha_i, \beta_i] \right)$$

$$= f^{-1}(V^+) \cup f^{-1}(V^-) \cup \bigcup_{i \in I} f^{-1}([\alpha_i, \beta_i])$$

$$= f^{-1}(V^+) \cup f^{-1}(V^-) \cup \bigcup_{i \in I} \{ \alpha_i < f \} \cap \{ f < \beta_i \}$$

Since $f^{-1}(V^+)$ is either $\{ \alpha < f \}$ or $\emptyset$, and $f^{-1}(V^-)$ is either $\{ f < \beta \}$ or $\emptyset$, it follows that $f^{-1}(U)$ is a union of open sets in
Ω, and is therefore open. Having proved that \( f^{-1}(U) \) is open for all \( U \) open in \( \mathbb{R} \), we conclude that \( f \) is continuous. So \( f \) is continuous, if and only if it is both l.s.c and u.s.c.

5. Let \( u : \Omega \to \mathbb{R} \) and \( v : \Omega \to \mathbb{R} \). Let \( \lambda \in \mathbb{R} \). Note that having restricted the range of \( u \) to be a subset of \( \mathbb{R} \), the map \( u + v \) is well defined, as there can be no occurrence of \( (+\infty) + (-\infty) \).

We claim that:

\[
\{ \lambda < u + v \} = \bigcup_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} \{ \lambda_1 < u \} \cap \{ \lambda_2 < v \}
\]

It is clear that if \( \omega \in \Omega \) is such that \( \lambda_1 < u(\omega) \) and \( \lambda_2 < v(\omega) \) for some \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 + \lambda_2 = \lambda \), then \( \lambda < u(\omega) + v(\omega) \). This shows the inclusion \( \supseteq \). To show the reverse inclusion, suppose that \( \omega \in \Omega \) is such that \( \lambda < u(\omega) + v(\omega) \). Then, we have \( \lambda - u(\omega) < v(\omega) \), and there exists \( \lambda_2 \in \mathbb{R} \) such that:

\[
\lambda - u(\omega) < \lambda_2 < v(\omega)
\]
Define $\lambda_1 = \lambda - \lambda_2$. Then $\lambda_2 < v(\omega)$ and $\lambda_1 < u(\omega)$ where $\lambda_1, \lambda_2$ are elements of $\mathbb{R}$ such that $\lambda_1 + \lambda_2 = \lambda$. This shows the inclusion $\subseteq$.

6. Suppose that both $u$ and $v$ are l.s.c. Then for all $\lambda_1, \lambda_2 \in \mathbb{R}$, \{\lambda_1 < u\} and \{\lambda_2 < v\} are open subsets of $\Omega$. It follows from 5. that \{\lambda < u + v\} is also an open subset of $\Omega$, for all $\lambda \in \mathbb{R}$. So $u + v$ is l.s.c.

7. Suppose that both $u$ and $v$ are u.s.c. Similarly to 5. we have:

$$\{u + v < \lambda\} = \bigcup_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} \{u < \lambda_1\} \cap \{v < \lambda_2\}$$

and consequently \{u + v < \lambda\} is an open subset of $\Omega$, for all $\lambda \in \mathbb{R}$. So $u + v$ is u.s.c. Anticipating on questions 10. and 11., an alternative proof goes as follows: if $u$ and $v$ are u.s.c, then $-u$ and $-v$ are l.s.c. so $-u - v$ is l.s.c. and finally $u + v$ is u.s.c.
8. Suppose $f$ is l.s.c and let $\alpha \in \mathbb{R}^+$. If $\alpha = 0$, then $\alpha f = 0$ and consequently $\alpha f$ is continuous and in particular l.s.c. We assume that $\alpha > 0$. Then for all $\omega \in \Omega$, $\lambda < \alpha f(\omega)$ is equivalent to $\lambda/\alpha < f(\omega)$ (this is certainly true when $f(\omega) \in \mathbb{R}$, and one can easily check that it is still true when $f(\omega) \in \{-\infty, +\infty\}$). It follows that $\{\lambda < \alpha f\} = \{\lambda/\alpha < f\}$ and consequently $\{\lambda < \alpha f\}$ is an open subset of $\Omega$. This being true for all $\lambda \in \mathbb{R}$, we conclude that $\alpha f$ is l.s.c.

9. Suppose that $f$ is u.s.c and $\alpha \in \mathbb{R}^+$. If $\alpha = 0$ then $\alpha f$ is u.s.c. We assume that $\alpha > 0$. Then $\{\alpha f < \lambda\} = \{f < \lambda/\alpha\}$ and consequently $\{\alpha f < \lambda\}$ is open for all $\lambda \in \mathbb{R}$. So $\alpha f$ is u.s.c.

10. Suppose that $f$ is l.s.c. Then $\{-f < \lambda\} = \{-\lambda < f\}$ for all $\lambda \in \mathbb{R}$, and consequently $\{-f < \lambda\}$ is an open subset of $\Omega$. So $-f$ is u.s.c.

11. Suppose that $f$ is u.s.c. Then $\{\lambda < -f\} = \{f < -\lambda\}$ for all $\lambda \in \mathbb{R}$, and consequently $\{\lambda < -f\}$ is an open subset of $\Omega$. So
−f is l.s.c.

12. Let V be an open subset of Ω and \( f = 1_V \). Let \( \lambda \in \mathbb{R} \). If \( \lambda < 0 \) we have \( \{ \lambda < f \} = \Omega \). If \( 0 \leq \lambda < 1 \) we have \( \{ \lambda < f \} = V \). If \( 1 \leq \lambda \) we have \( \{ \lambda < f \} = \emptyset \). In any case, \( \{ \lambda < f \} \) is an open subset of Ω. So \( f \) is l.s.c. The characteristic function of an open subset of Ω is lower-semi-continuous.

13. Let \( F \) be a closed subset of Ω. Let \( \lambda \in \mathbb{R} \). Then \( \{ f < \lambda \} \) is either \( \emptyset \), \( F^c \) or Ω, depending respectively on whether \( \lambda \leq 0 \), \( 0 < \lambda \leq 1 \) and \( 1 < \lambda \). In any case, \( \{ f < \lambda \} \) is an open subset of Ω. So \( f \) is u.s.c. The characteristic function of a closed subset of Ω is upper-semi-continuous.

Exercise 1
Exercise 2.

1. Let \((f_i)_{i \in I}\) be a family of maps \(f_i : \Omega \to \bar{\mathbb{R}}\), where \(\Omega\) is a topological space. Let \(f = \sup_{i \in I} f_i\). We assume that all \(f_i\)'s are l.s.c. For all \(\lambda \in \mathbb{R}\), we claim that:

\[
\{\lambda < f\} = \bigcup_{i \in I} \{\lambda < f_i\}
\]  \hspace{1cm} (1)

Indeed, suppose that \(\omega \in \Omega\) is such that \(\lambda < f(\omega)\). Since \(f(\omega)\) is the lowest upper-bound of all \(f_i(\omega)\)'s, \(\lambda\) cannot be such an upper-bound. Hence, there exists \(i \in I\) such that \(\lambda < f_i(\omega)\). This shows the inclusion \(\subseteq\). To show the reverse inclusion, suppose \(\omega \in \Omega\) is such that \(\lambda < f_i(\omega)\) for some \(i \in I\). Since \(f_i(\omega) \leq f(\omega)\), in particular we have \(\lambda < f(\omega)\). This shows the inclusion \(\supseteq\). Having proved equation (1) and since all \(f_i\)'s are l.s.c, \(\{\lambda < f\}\) is an open subset of \(\Omega\) for all \(\lambda \in \mathbb{R}\). It follows that \(f\) is l.s.c. The supremum of l.s.c functions is l.s.c.
2. Suppose that all \( f_i \)'s are u.s.c and \( f = \inf_{i \in I} f_i \). Given \( \lambda \in \mathbb{R} \):

\[
\{ f < \lambda \} = \bigcup_{i \in I} \{ f_i < \lambda \}
\]

and consequently \( \{ f < \lambda \} \) is an open subset of \( \Omega \). It follows that \( f \) is u.s.c. The infimum of u.s.c functions is u.s.c.

Exercise 2
Exercise 3.

1. Let $(\Omega, \mathcal{T})$ be a metrizable and $\sigma$-compact topological space. Let $f \in L_1^R(\Omega, \mathcal{B}(\Omega), \mu)$, $f \geq 0$, where $\mu$ is a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. From theorem (18), there exists a sequence $(s_n)_{n \geq 1}$ of simple functions on $(\Omega, \mathcal{B}(\Omega))$ such that $s_n \uparrow f$ (i.e. $s_n \leq s_{n+1}$ for all $n \geq 1$ and $s_n \to f$ pointwise). We define $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for all $n \geq 2$. In order to show that $t_n$ is a simple function for all $n \geq 1$, we need to show that if $s, t$ are simple functions on $(\Omega, \mathcal{B}(\Omega))$ with $s \leq t$, then $t - s$ is also a simple function on $(\Omega, \mathcal{B}(\Omega))$. Since $s$ and $t$ are measurable with values in $\mathbb{R}^+$, and $s \leq t$, the map $t - s$ is also measurable with values in $\mathbb{R}^+$. From:

\[ t - s = \sum_{\alpha \in (t-s)(\Omega)} \alpha 1_{\{t-s=\alpha\}} \]

we conclude that $t - s$ is a simple function on $(\Omega, \mathcal{B}(\Omega))$.

2. Since each $t_n$ is a simple function on $(\Omega, \mathcal{B}(\Omega))$, for all $n \geq 1$
there exists an integer $p_n \geq 1$ and some $\alpha_n^1, \ldots, \alpha_n^{p_n} \in \mathbb{R}^+$ and $A_n^1, \ldots, A_n^{p_n} \in \mathcal{B}(\Omega)$ such that:

$$t_n = \sum_{k=1}^{p_n} \alpha_n^k 1_{A_n^k}$$

Note that it is always possible to assume $\alpha_n^k \neq 0$, by setting $A_n^k = \emptyset$ if necessary. Since $s_N = \sum_{n=1}^{N} t_n$ for all $N \geq 1$, from $s_N \to f$ we obtain:

$$f = \sum_{n=1}^{+\infty} t_n = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \alpha_n^k 1_{A_n^k}$$

This last sum having a countable number of (non-negative) terms, it can be re-expressed as:

$$f = \sum_{n=1}^{+\infty} \alpha_n 1_{A_n}$$
where $\alpha_n \in \mathbb{R}^+ \setminus \{0\}$ and $A_n \in \mathcal{B}(\Omega)$ for all $n \geq 1$.

3. Since $f \in L^1_\mathbb{R}(\Omega, \mathcal{B}(\Omega), \mu)$ and $f \geq 0$, from 2, we have:

$$
\sum_{n=1}^{+\infty} \alpha_n \mu(A_n) = \sum_{n=1}^{+\infty} \alpha_n \int 1_{A_n} \, d\mu \\
= \int \left( \sum_{n=1}^{+\infty} \alpha_n 1_{A_n} \right) \, d\mu \\
= \int f \, d\mu < +\infty
$$

where the second equality is obtained from the linearity of the integral and an immediate application of the monotone convergence theorem (19). Since for all $n \geq 1$ we have $\alpha_n > 0$, we conclude that $\mu(A_n) < +\infty$.

4. Let $\epsilon > 0$ and $n \geq 1$. Define $\epsilon' = \epsilon/(\alpha_n 2^{n+2})$. Since $(\Omega, \mathcal{T})$ is metrizable and $\sigma$-compact, while $\mu$ is a locally finite measure on
$(\Omega, \mathcal{B}(\Omega))$, from theorem (73) $\mu$ is a regular measure. Hence:

$$\mu(A_n) = \sup\{\mu(K) : K \subseteq A_n, K \text{ compact}\} = \inf\{\mu(V) : A_n \subseteq V, V \text{ open}\}$$

Since $\mu(A_n) < +\infty$, we have $\mu(A_n) < \mu(A_n) + \epsilon'$, and $\mu(A_n)$ being the greatest lower-bound of all $\mu(V)$’s as $V$ runs through the set of all open subsets of $\Omega$ with $A_n \subseteq V$, $\mu(A_n) + \epsilon'$ cannot be such a lower-bound. There exists $V_n$ open subset of $\Omega$ such that $A_n \subseteq V_n$, and:

$$\mu(V_n) < \mu(A_n) + \epsilon'$$

Similarly, from the fact that $\mu(A_n) - \epsilon' < \mu(A_n)$, there exists $K_n$ compact subset of $\Omega$ such that $K_n \subseteq A_n$, and:

$$\mu(A_n) - \epsilon' < \mu(K_n)$$

From $K_n \subseteq A_n$ note in particular that $\mu(K_n) < +\infty$, and con-
sequently we have $K_n \subseteq A_n \subseteq V_n$ with:

$$
\mu(V_n \setminus K_n) = \mu(V_n) - \mu(K_n) < 2\epsilon' = \frac{\epsilon}{\alpha_n^{2n+1}}
$$

5. Having proved in 3. that $\sum_{n \geq 1} \alpha_n \mu(A_n) < +\infty$, given $\epsilon > 0$ there exists $N \geq 1$ such that:

$$
\left| \sum_{n=1}^{+\infty} \alpha_n \mu(A_n) - \sum_{n=1}^{N} \alpha_n \mu(A_n) \right| \leq \frac{\epsilon}{2}
$$

or equivalently:

$$
\sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}
$$

6. Let $u = \sum_{n=1}^{N} \alpha_n 1_{K_n}$. Since $(\Omega, T)$ is metrizable, in particular it is a Hausdorff topological space. Since $K_n$ is a compact subset of $\Omega$, from theorem (35) $K_n$ is a closed subset of $\Omega$. It follows from 13. of exercise (1) that $1_{K_n}$ is upper-semi-continuous. Using 7. and 9. of exercise (1), we conclude that $u$ is also u.s.c.
7. Let \( v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n} \). Since \( V_n \) is an open subset of \( \Omega \), from 12. of exercise (1) the map \( 1_{V_n} \) is lower-semi-continuous. It follows from 6. and 8. of this same exercise that every partial sum \( \sum_{n=1}^{k} \alpha_n 1_{V_n} \) is itself l.s.c. Since \( v \) is the supremum of these partial sums, we conclude from exercise (2) that \( v \) is l.s.c.

8. Since \( K_n \subseteq A_n \subseteq V_n \) and \( \alpha_n \in \mathbb{R}^+ \) for all \( n \geq 1 \):

\[
0 \leq \sum_{n=1}^{N} \alpha_n 1_{K_n} = u \\
\leq \sum_{n=1}^{N} \alpha_n 1_{A_n} \\
\leq \sum_{n=1}^{+\infty} \alpha_n 1_{A_n} = f \\
\leq \sum_{n=1}^{+\infty} \alpha_n 1_{V_n} = v
\]
We conclude that $0 \leq u \leq f \leq v$.

9. Since $K_n \subseteq V_n$ for all $n \geq 1$, we have:

$$v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n} = \sum_{n=1}^{+\infty} \alpha_n (1_{K_n} + 1_{V_n \backslash K_n})$$

$$= \sum_{n=1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \backslash K_n}$$

$$= u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \backslash K_n}$$

10. Since $K_n \subseteq A_n$ for all $n \geq 1$, using 5. we have:

$$\sum_{n=N+1}^{+\infty} \alpha_n \mu(K_n) \leq \sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}$$

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Hence, using 9. and 4. we obtain:

\[
\int v \, d\mu = \int \left( u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n} \right) \, d\mu
\]

\[
= \int u \, d\mu + \sum_{n=N+1}^{+\infty} \alpha_n \int 1_{K_n} \, d\mu + \sum_{n=1}^{+\infty} \alpha_n \int 1_{V_n \setminus K_n} \, d\mu
\]

\[
= \int u \, d\mu + \sum_{n=N+1}^{+\infty} \alpha_n \mu(K_n) + \sum_{n=1}^{+\infty} \alpha_n \mu(V_n \setminus K_n)
\]

\[
\leq \int u \, d\mu + \frac{\epsilon}{2} + \sum_{n=1}^{+\infty} \alpha_n \cdot \frac{\epsilon}{\alpha_n 2^{n+1}}
\]

\[
= \int u \, d\mu + \epsilon
\]

where the second equality stems from the linearity of the integral and an application of the monotone convergence theorem (19).
Note that since $\mu(K_n) < +\infty$ for all $n \geq 1$, in particular:

$$\int ud\mu = \sum_{n=1}^{N} \alpha_n \mu(K_n) < +\infty$$

Hence, we conclude that:

$$\int vd\mu \leq \int ud\mu + \epsilon < +\infty$$

11. The map $u$ is $\mathbb{R}$-valued, Borel measurable with:

$$\int |u|d\mu = \int ud\mu < +\infty$$

So $u \in L^1_\mathbb{R}(\Omega, \mathcal{B}(\Omega), \mu)$.

12. The map $v$ is Borel measurable with:

$$\int |v|d\mu = \int vd\mu < +\infty$$
However, it has values in \([0, +\infty]\), i.e. \(v(\omega) = +\infty\) is possible for some \(\omega \in \Omega\). The condition \(\int v \, d\mu < +\infty\) does imply that \(v(\omega) < +\infty\) for \(\mu\)-almost every \(\omega \in \Omega\). As we shall see in the next question, \(v\) is therefore \(\mu\)-almost surely equal to an element of \(L_1^R(\Omega, B(\Omega), \mu)\). But strictly speaking, it may not be itself an element of this space, because its range \(v(\Omega)\) may fail to be a subset of \(\mathbb{R}\).

13. Since \(\int v \, d\mu < +\infty\), we have \(v < +\infty\) \(\mu\)-a.s since:

\[
( +\infty ) \cdot \mu(\{v = +\infty\}) = \int_{\{v = +\infty\}} v \, d\mu \leq \int v \, d\mu < +\infty
\]

Hence, if \(N = \{v = +\infty\}\), we have \(N \in B(\Omega)\) and \(\mu(N) = 0\). Let \(v^* = v 1_N^c\). Then \(v^*\) has values in \(\mathbb{R}\), is Borel measurable and:

\[
\int |v^*| \, d\mu = \int v 1_N^c \, d\mu = \int v \, d\mu < +\infty
\]

So \(v^* \in L_1^R(\Omega, B(\Omega), \mu)\). Since \(v^* = v\) \(\mu\)-a.s. we conclude that \(v\) is \(\mu\)-almost surely equal to an element of \(L_1^R(\Omega, B(\Omega), \mu)\).
14. Note that from 8. we have $0 \leq u \leq v$ and consequently $v - u$ is non-negative and measurable, and the integral $\int (v - u) d\mu$ makes sense. In fact, even if $u \leq v$ did not hold, since $u \in L^1$ and $v$ is almost surely equal to an element of $L^1$, it would be possible to give meaning to $\int (v - u) d\mu$ in the obvious way. Now from 10. we have:

$$\int ud\mu + \int (v - u) d\mu = \int vd\mu$$

$$\leq \int ud\mu + \epsilon$$

and since $\int ud\mu < +\infty$ we conclude that $\int (v - u) d\mu \leq \epsilon$.

15. Having considered a metrizable and $\sigma$-compact topological space $(\Omega, T)$ and a locally finite measure $\mu$ on $(\Omega, B(\Omega))$, given $\epsilon > 0$ and $f \in L^1_\mathbb{R}(\Omega, B(\Omega), \mu)$ with $f \geq 0$, we have found two measurable maps $u, v : \Omega \to [0, +\infty]$ (where in fact $u$ has values in $\mathbb{R}^+$), which are $\mu$-almost surely equal to elements of $L^1_\mathbb{R}(\Omega, B(\Omega), \mu)$
(in fact \(u\) is itself an element of \(L^1\)) and such that \(u \leq f \leq v\), \(u\) is u.s.c, \(v\) is l.s.c. and:

\[
\int (v - u) d\mu \leq \epsilon
\]

Now let \(f \in L^1_\mathbb{R}(\Omega, \mathcal{B}(\Omega), \mu)\) which we no longer assume to be non-negative. Let \(f^+\) and \(f^-\) be respectively the positive and negative parts of \(f\). Then \(f = f^+ - f^-\) and given \(\epsilon > 0\), it is possible to apply the result of this exercise to \(f^+\) and \(f^-\) separately, with \(\epsilon/2\) instead of \(\epsilon\). Hence, there exist four measurable maps \(u^+, v^+, u^-, v^-\) where \(u^+, u^-\) have values in \(\mathbb{R}^+\) and \(v^+, v^-\) have values in \([0, +\infty]\), which are \(\mu\)-almost surely equal elements of \(L^1\), and satisfy the conditions \(u^+ \leq f^+ \leq v^+, u^- \leq f^- \leq v^-\), \(u^+, u^-\) are u.s.c, \(v^+, v^-\) are l.s.c, and:

\[
\int (v^+ - u^+) d\mu \leq \frac{\epsilon}{2}
\]
Solutions to Exercises together with:

\[ \int (v^- - u^-)d\mu \leq \frac{\epsilon}{2} \]

We define \( u = u^+ - v^- \) and \( v = v^+ - u^- \). Since \( u^+, u^- \) have values in \( \mathbb{R} \), given \( \omega \in \Omega \), the differences \( u^+(\omega) - v^- (\omega) \) and \( v^+(\omega) - u^- (\omega) \) are always well-defined elements of \( \overline{\mathbb{R}} \). It follows that \( u, v : \Omega \rightarrow \overline{\mathbb{R}} \) are well-defined measurable maps. Furthermore, it is clear that both \( u \) and \( v \) are \( \mu \)-almost surely equal to an element of \( L^1 \). From \( u^+ \leq f^+ \leq v^+ \), \( u^- \leq f^- \leq v^- \) and \( f = f^+ - f^- \) we obtain \( u \leq f \leq v \). Furthermore, since \( u^+ \) is \( \mathbb{R} \)-valued and u.s.c while \( v^- \) is l.s.c, from exercise (1) \( u = u^+ - v^- \) is u.s.c, and similarly \( v = v^+ - u^- \) is l.s.c. Finally, since \( u \leq f \leq v \) and \( f \) is \( \mathbb{R} \)-valued, given \( \omega \in \Omega \) the difference \( v(\omega) - u(\omega) \) is always a well-defined element of \([0, +\infty]\). So \( v - u \) is a well-defined non-negative and measurable map, and the integral \( \int (v - u)d\mu \) is meaningful. We have:

\[ \int (v - u)d\mu = \int (v^+ - u^- - u^+ + v^-)d\mu \]
\[ = \int (v^+ - u^+ + v^- - u^-) d\mu \]
\[ = \int (v^+ - u^+) d\mu + \int (v^- - u^-) d\mu \]
\[ \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

This completes the proof of theorem (94).
Exercise 4.

1. Let \((\Omega, T)\) be a topological space. Suppose it is connected and \(\Omega = A \cup B\) where \(A, B\) are disjoint open sets. Then \(A^c = B\) so \(A\) is closed and consequently \(A\) is both open and closed. Hence, \(\Omega\) being connected, we have \(A = \emptyset\) or \(A = \Omega\), i.e. \(A = \emptyset\) or \(B = \emptyset\). Conversely, suppose \(\Omega = A \cup B\) with \(A, B\) disjoint open sets implies that \(A = \emptyset\) or \(B = \emptyset\). Then if \(A\) is both open and closed in \(\Omega\), with have \(\Omega = A \cup A^c\) where \(A, A^c\) are disjoint open sets. So \(A = \emptyset\) or \(A^c = \emptyset\), i.e. \(A = \emptyset\) or \(A = \Omega\). This shows that \(\Omega\) is connected. We have proved that \(\Omega\) is connected if and only if whenever \(\Omega = A \cup B\) with \(A, B\) disjoint open sets, we have \(A = \emptyset\) or \(B = \emptyset\).

2. If \(\Omega = A \cup B\) with \(A, B\) disjoint open sets, then \(\Omega = A^c \cup B^c\) with \(A^c, B^c\) disjoint closed sets, and conversely if \(\Omega = A \cup B\) with \(A, B\) disjoint closed sets, then \(\Omega = A^c \cup B^c\) with \(A^c, B^c\)
disjoint open sets. Hence, the statements:

(i) $\Omega = A \cup B$, $A, B$ disjoint and open $\Rightarrow A = \emptyset$ or $B = \emptyset$

(ii) $\Omega = A \cup B$, $A, B$ disjoint and closed $\Rightarrow A = \emptyset$ or $B = \emptyset$

are equivalent. We conclude from 1, that $\Omega$ is connected, if and only if whenever $\Omega = A \cup B$ with $A, B$ disjoint closed sets, we have $A = \emptyset$ or $B = \emptyset$.

Exercise 4
Exercise 5.

1. Let $A$ be an open and closed subset of $\mathbb{R}$, with $A \neq \emptyset$ and $A^c \neq \emptyset$. Let $x \in A^c$. We have:

$$A = (A \cap ]-\infty, x]) \cup (A \cap [x, +\infty[)$$

and since $A \neq \emptyset$, we have $A \cap ]-\infty, x] \neq \emptyset$ or $A \cap [x, +\infty[ \neq \emptyset$.

2. Let $B = A \cap [x, +\infty[$ and suppose $B \neq \emptyset$. Both $A$ and $[x, +\infty[$ are closed subsets of $\mathbb{R}$. So $B$ is a closed subset of $\mathbb{R}$. However, since $x \in A^c$, we have:

$$B = A \cap [x, +\infty[$$

$$= (A \cap \{x\}) \cup (A \cap [x, +\infty[)$$

$$= A \cap [x, +\infty[$$

and since both $A$ and $[x, +\infty[$ are open subsets of $\mathbb{R}$, $B$ is also an open subset of $\mathbb{R}$. Note that the assumption $B \neq \emptyset$ has not been used so far.
3. Let \( b = \inf B \). We have proved in exercise (9) (part 5) of Tutorial 8 that if \( B \) is a non-empty closed subset of \( \mathbb{R} \), then \( \inf B \in B \). Unfortunately, this result does not apply to non-empty closed subsets of \( \mathbb{R} \) (indeed \( \mathbb{R} \), is a non-empty closed subset of \( \mathbb{R} \) and \( \inf \mathbb{R} = -\infty \not\in \mathbb{R} \)). So we cannot apply exercise (9) of Tutorial 8, at least not without a little bit of care. However, the following can be done: since \( B \neq \emptyset \), there exists \( y \in B = A \cap [x, +\infty[ \). Then it is clear that \( B^* = A \cap [x, y] \) is a non-empty closed subset of \( \mathbb{R} \), and consequently since \( b = \inf B^* \), applying exercise (9) of Tutorial 8, we have \( b \in B^* \). So \( b \in B \subseteq \mathbb{R} \). For those who wish to have a more detailed argument, the following can be said: the fact that \( B^* \neq \emptyset \) is a consequence of \( y \in B^* \). If we define \( b^* = \inf B^* \), the fact that \( b^* = b \) can be shown as follows: since \( B^* \subseteq B \), any lower-bound of \( B \) is also a lower-bound of \( B^* \), and consequently \( b \) is a lower-bound of \( B^* \) which shows that \( b \leq b^* \). To show the reverse inequality, consider \( u \in B \). Then if \( u \leq y \) we have \( u \in B^* \) and therefore \( b^* \leq u \). But if \( y < u \), then \( b^* \leq y < u \) and we see

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that $b^* \leq u$ is true in all cases. So $b^*$ is a lower-bound of $B$ which shows that $b^* \leq b$. We have proved that $b = b^*$. To show that $B^*$ is a closed subset of $\mathbb{R}$, we first argue that it is a closed subset of $\mathbb{R}$ since $A$ is closed and $[x, y]$ is closed. However, the topology of $\mathbb{R}$ is induced by the topology of $\mathbb{R}$ itself. It is a simple exercise to show that any closed subset of $\mathbb{R}$ can be written as $F \cap \mathbb{R}$ where $F$ is a closed subset of $\mathbb{R}$. Hence, there is a closed subset $F$ of $\mathbb{R}$ such that $B^* = F \cap \mathbb{R}$. But then:

\[
B^* = A \cap [x, y] \\
= A \cap [x, y] \cap [x, y] \\
= B^* \cap [x, y] \\
= (F \cap \mathbb{R}) \cap [x, y] \\
= F \cap [x, y]
\]

and since $[x, y]$ is also closed in $\mathbb{R}$, we conclude that $B^*$ is indeed closed in $\mathbb{R}$. This concludes our proof that $b \in B$. All this may seem like a lot of work, made necessary by our desperate attempt
to apply exercise (9) of Tutorial 8. For those who believe that a direct proof is more convenient, here is the following: Since $B = A \cap [x,+\infty]$, it is clear that $x$ is a lower bound of $B$ and consequently $x \leq b$. To show that $b \in B$, we only need to show that $b \in A$. Since $B \neq \emptyset$, there exist $y \in B \subseteq \mathbb{R}$ and from $b \leq y$ we obtain in particular $b < +\infty$. Hence, there exists a sequence $(t_n)_{n \geq 1}$ in $\mathbb{R}$ such that $t_n \downarrow b$ (i.e. $t_n \to b$ with $b < t_{n+1} \leq t_n$ for all $n \geq 1$). Since $b < t_n$, it is impossible that $t_n$ be a lower-bound of $B$. Hence, for all $n \geq 1$ there exists some $x_n \in B \subseteq A$ such that $b \leq x_n < t_n$. From $t_n \to b$ we see that $x_n \to b$ and since $x_n \in A$ while $A$ is a closed subset of $\mathbb{R}$, we conclude that $b \in A$. This completes our second proof of $b \in B$.

4. Having proved in 2. that $B$ is an open subset of $\mathbb{R}$, since $b \in B$ there exists $\epsilon > 0$ such that $]b - \epsilon, b + \epsilon[ \subseteq B$.

5. To show that $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ is connected, we need to show that if $A$ is an open and closed subset of $\mathbb{R}$, then $A = \emptyset$ or $A = \mathbb{R}$. Suppose this is not the case and $A \neq \emptyset$ together with $A^c \neq \emptyset$. We have
shown in 2. that \( A \cap [x, +\infty] \neq \emptyset \) or \( A \cap ]-\infty, x[ \neq \emptyset \). If we assume that \( B = A \cap [x, +\infty] \) and \( B \neq \emptyset \), then \( b = \inf B \in \mathbb{R} \) and we have proved in 4. that there exists \( \epsilon > 0 \) such that \( ]b-\epsilon, b+\epsilon[ \subseteq B \). This is a contradiction. Indeed, since \( b - \epsilon/2 < b \), the fact that \( b - \epsilon/2 \in B \) contradicts the fact that \( b \) is a lower-bound of \( B \).

So the only possible case is that \( C \neq \emptyset \) where \( C = A \cap ]-\infty, x[ \). However, if \( c = \sup C \), then a similar proof to that of 3. will show that \( c \in C \) (in particular \( c \in \mathbb{R} \)) and \( C \) being open in \( \mathbb{R} \), there exists \( \epsilon > 0 \) with \( ]c-\epsilon, c+\epsilon[ \subseteq C \), leading to a contradiction.

Hence, we see that all possible cases lead to a contradiction. We conclude that the initial assumption is absurd, i.e. that \( A = \emptyset \) or \( A = \mathbb{R} \). So \((\mathbb{R}, T_\mathbb{R})\) is a connected topological space, which completes the proof of theorem (95).

**Exercise 5**
Exercise 6.

1. Let \((\Omega, T)\) be a topological space and \(A \subseteq \Omega\) be a connected subset of \(\Omega\). Let \(B\) be a subset of \(\Omega\) such that \(A \subseteq B \subseteq \bar{A}\), where \(\bar{A}\) is the closure of \(A\) in \(\Omega\). Let \(V_1, V_2\) be disjoint open subsets of \(B\) such that \(B = V_1 \cup V_2\). From definition (23) of the induced topology \(T|_B\), there exist \(U_1, U_2\) open subsets of \(\Omega\) such that \(V_1 = B \cap U_1\) and \(V_2 = B \cap U_2\).

2. Since \(A \subseteq B\), using 1. we have:
   \[
   A = A \cap B = A \cap (V_1 \cup V_2) = A \cap [(B \cap U_1) \cup (B \cap U_2)] = (A \cap B \cap U_1) \cup (A \cap B \cap U_2) = (A \cap U_1) \cup (A \cap U_2)
   \]

Now since \(U_1, U_2\) are open subsets of \(\Omega\), \(A \cap U_1\) and \(A \cap U_2\) are open subsets of \(A\). Furthermore, since \(V_1\) and \(V_2\) are disjoint,
we have $V_1 \cap V_2 = B \cap U_1 \cap U_2 = \emptyset$. and in particular since $A \subseteq B$, $A \cap U_1 \cap U_2 = \emptyset$. So $A \cap U_1$ and $A \cap U_2$ are disjoint open subsets of $A$ with $A = (A \cap U_1) \cup (A \cap U_2)$. Having assumed that $A$ is a connected subset of $\Omega$, the topological space $(A, T_A)$ is connected and consequently using exercise (4), it follows that $A \cap U_1 = \emptyset$ or $A \cap U_2 = \emptyset$.

3. Suppose that $A \cap U_1 = \emptyset$. Let $x \in \bar{A}$. Then for all $U$ open subsets of $\Omega$ with $x \in U$, we have $A \cap U \neq \emptyset$. Hence, since $U_1$ is an open subset of $\Omega$ and $A \cap U_1 = \emptyset$, it is necessary that $x \notin U_1$. So $x \in U_1^c$ and we have proved that $\bar{A} \subseteq U_1^c$.

4. Having assumed that $B \subseteq \bar{A}$, it follows from 3. that $B \subseteq U_1^c$, i.e. $V_1 = B \cap U_1 = \emptyset$.

5. From 3. and 4. we have seen that if $A \cap U_1 = \emptyset$, then $V_1 = \emptyset$. Similarly, if $A \cap U_2 = \emptyset$, then $V_2 = \emptyset$. However, we have shown in 2. that $A \cap U_1 = \emptyset$ or $A \cap U_2 = \emptyset$. So $V_1 = \emptyset$ or $V_2 = \emptyset$. Having considered $B \subseteq \Omega$ such that $A \subseteq B \subseteq \bar{A}$, and $V_1, V_2$
disjoint open subsets of $B$ such that $B = V_1 \uplus V_2$, we have proved that $V_1 = \emptyset$ or $V_2 = \emptyset$. From exercise (4), this shows that the topological space $(B, T_B)$ is connected, or equivalently that $B$ is a connected subset of $\Omega$. Hence, if $A$ is a connected subset of $\Omega$ and $A \subseteq B \subseteq \bar{A}$, then $B$ is also a connected subset of $\Omega$. In particular, $\bar{A}$ is a connected subset of $\Omega$.

Exercise 6
Exercise 7. Let $(\Omega, \mathcal{T})$ and $(\Omega', \mathcal{T}')$ be two topological spaces, and $f$ be a continuous map $f: \Omega \to \Omega'$. We assume that $(\Omega, \mathcal{T})$ is connected. We claim that $f(\Omega)$ is a connected subset of $\Omega'$, or equivalently that the topological space $(f(\Omega), \mathcal{T}'_{f(\Omega)})$ is connected. In order to prove this, we shall use exercise (4) and consider $A, B$ two disjoint open subsets of $f(\Omega)$ such that $f(\Omega) = A \cup B$. There exist $U', V'$ open subsets of $\Omega'$ such that $A = f(\Omega) \cap U'$ and $B = f(\Omega) \cap V'$. Since $f$ is continuous, $f^{-1}(U')$ and $f^{-1}(V')$ are open subsets of $\Omega$. Furthermore, it is clear that:

$$f^{-1}(U') = f^{-1}(f(\Omega) \cap U') = f^{-1}(A)$$

and similarly $f^{-1}(V') = f^{-1}(B)$. So $f^{-1}(A)$ and $f^{-1}(B)$ are open subsets of $\Omega$. Since $A$ and $B$ are disjoint, $f^{-1}(A)$ and $f^{-1}(B)$ are also disjoint. Since $f(\Omega) = A \cup B$, for all $x \in \Omega$ we have $f(x) \in A$ or $f(x) \in B$. So $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. It follows that $f^{-1}(A)$ and $f^{-1}(B)$ are two disjoint open subsets of $\Omega$, such that $\Omega = f^{-1}(A) \cup f^{-1}(B)$. Since $\Omega$ is connected, from exercise (4) it follows that $f^{-1}(A) = \emptyset$ or $f^{-1}(B) = \emptyset$. Suppose that $f^{-1}(A) = \emptyset$. 

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We claim that $A = \emptyset$. Otherwise there exists $y \in A \subseteq f(\Omega)$. Let $x \in \Omega$ be such that $y = f(x)$. Then $f(x) \in A$ and consequently $x \in f^{-1}(A)$ which contradicts $f^{-1}(A) = \emptyset$. So $f^{-1}(A) = \emptyset$ implies that $A = \emptyset$, and similarly $f^{-1}(B) = \emptyset$ implies that $B = \emptyset$. It follows that $A = \emptyset$ or $B = \emptyset$. Having assumed that $f(\Omega) = A \cup B$ where $A, B$ are disjoint open subsets of $f(\Omega)$, we have proved that $A = \emptyset$ or $B = \emptyset$. From exercise (4), this shows that the topological space $(f(\Omega), T_{f(\Omega)})$ is connected, or equivalently that $f(\Omega)$ is a connected subset of $\Omega'$. This completes the proof of theorem (96).

Exercise 7
Exercise 8.

1. Let $A \subseteq \mathbb{R}$ and suppose that $A$ is an interval. Let $\alpha = \inf A$ and $\beta = \sup A$. We claim that:

$$]\alpha, \beta[ \subseteq A \subseteq [\alpha, \beta]$$

If $A = \emptyset$, then $\alpha = +\infty$ and $\beta = -\infty$, so there is nothing to prove. So we assume that $A \neq \emptyset$. Then there is $x \in A$, and we have $\alpha \leq x$ as well as $x \leq \beta$. In particular, $\alpha \leq \beta$. Let $z \in A$. Since $\alpha$ is a lower-bound of $A$, $\alpha \leq z$. Since $\beta$ is an upper-bound of $A$, $z \leq \beta$. So $z \in [\alpha, \beta]$ and we have proved that $A \subseteq [\alpha, \beta]$. Suppose $z \in ]\alpha, \beta[$. From $\alpha < z$ we see that $z$ cannot be a lower-bound of $A$ ($\alpha$ is the greatest of such lower-bounds). There exists $x \in A$ such that $\alpha \leq x < z$. From $z < \beta$ we see that $z$ cannot be an upper-bound of $A$. There exists $y \in A$ such that $z < y \leq \beta$. From $x < z < y$ we obtain in particular $z \in [x, y]$. Since $x, y \in A$ and $A$ is assumed to be an interval, it follows from definition (118) that $z \in A$. We have proved that $]\alpha, \beta[ \subseteq A$. 

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2. Let \( A \subseteq \bar{\mathbb{R}} \). Suppose that \( A \) is of the form \([\alpha, \beta], [\alpha, \beta[, [\alpha, \beta] \) or \([\alpha, \beta] \) for some \( \alpha, \beta \in \mathbb{R} \). Suppose there exist \( x, y \in A \) with \( x \leq y \). Then for all \( z \in [x, y] \) we have \( x \leq z \leq y \). If \( \alpha \leq x \) then \( \alpha \leq z \). If \( \alpha < x \) then \( \alpha < z \). If \( y \leq \beta \) then \( z \leq \beta \). If \( y < \beta \) then \( z < \beta \). In any case, we see that \( z \in A \). This shows that \([x, y] \subseteq A\) for all \( x, y \in A, x \leq y \), and consequently from definition (118), \( A \) is an interval. Note that \( A \) can be the empty set without anything being flawed in the argument just given. Conversely, suppose that \( A \) is an interval. From 1. we have:

\[ \alpha, \beta \subseteq A \subseteq [\alpha, \beta] \]

where \( \alpha = \inf A \) and \( \beta = \sup A \). We shall distinguish four cases: suppose \( \alpha \in A \) and \( \beta \in A \). Then:

\[ [\alpha, \beta] = [\alpha, \beta] \cup \{\alpha\} \cup \{\beta\} \subseteq A \subseteq [\alpha, \beta] \]

and consequently \( A = [\alpha, \beta] \). Suppose \( \alpha \in A \) and \( \beta \notin A \). Then:

\[ [\alpha, \beta] = [\alpha, \beta] \cup \{\alpha\} \subseteq A \subseteq [\alpha, \beta] \setminus \{\beta\} = [\alpha, \beta] \]
and consequently $A = [\alpha, \beta]$. Suppose $\alpha \notin A$ and $\beta \in A$. Then:

$$[\alpha, \beta] = \alpha, \beta \cup \{\beta\} \subseteq A \subseteq [\alpha, \beta] \setminus \{\alpha\} = [\alpha, \beta]$$

and consequently $A = [\alpha, \beta]$. Finally suppose $\alpha \notin A$ and $\beta \notin A$:

$$[\alpha, \beta] \subseteq A \subseteq [\alpha, \beta] \setminus \{\alpha, \beta\} = [\alpha, \beta]$$

and consequently $A = [\alpha, \beta]$. Hence, we have proved that $A$ is of the form $[\alpha, \beta], [\alpha, \beta], [\alpha, \beta]$ or $[\alpha, \beta]$. Note that if $A = \emptyset$, there is nothing flawed in the argument just given.

3. Let $A = ]-\infty, \alpha[$ where $\alpha \in \mathbb{R}$. Consider $\phi: \mathbb{R} \to ]-1, 1[$ defined by $\phi(x) = x/(1 + |x|)$. Then $\phi$ is a bijection with $\phi^{-1}(y) = y/(1 - |y|)$. Let $\psi = \phi|_{A}$ be the restriction of $\phi$ to $A$. Then $\psi$ is injective, and it is therefore a bijection from $A$ to $\psi(A)$. We claim that $\psi(A) = ]-1, \phi(\alpha)[$. Since $|\phi(x)| < 1$ for all $x \in \mathbb{R}$, it is clear that $\psi(A) \subseteq ]-1, 1[$. Since $\phi(x) = 1 - 1/(1 + x)$ for $x > 0$ and $\phi(x) = 1 + 1/(1 - x)$ for $x < 0$, it is clear that $\phi$ is increasing. So $\psi(A) \subseteq ]-1, \phi(\alpha)[$. To show the reverse
inclusion, consider \( y \in ]-1, \phi(\alpha) [ \). Since \( \phi^{-1} \) is also increasing, from \( y < \phi(\alpha) \) we obtain \( \phi^{-1}(y) < \alpha \). Hence, \( \phi^{-1}(y) \in A \) and \( y = \psi(\phi^{-1}(y)) \in \psi(A) \). We have proved that \( \psi(A) = ]-1, \phi(\alpha) [ \) and \( \psi \) is consequently a bijection from \( A \) to \( ]-1, \phi(\alpha) [ \). Since \( \phi \) is continuous, \( \psi = \phi|_A \) is also continuous. Since \( \phi^{-1} \) is continuous, \( \psi^{-1} = (\phi^{-1})|_{\psi(A)} \) is also continuous. We conclude that \( \psi : A \to ]-1, \phi(\alpha) [ \) is a homeomorphism. We have proved that for all \( \alpha \in \mathbb{R}, ]-\infty, \alpha [ \) is homeomorphic to \( ]-1, \alpha' [ \) for some \( \alpha' \in \mathbb{R} \).

4. Let \( A = ]\alpha, +\infty[ \) where \( \alpha \in \mathbb{R} \). Then if \( \phi : \mathbb{R} \to ]-1, 1[ \) is defined as in 3. and \( \psi = \phi|_A \), then \( \psi(A) = ]\phi(\alpha), 1[ \) and \( \psi \) is a homeomorphism from \( A \) to \( ]\phi(\alpha), 1[ \). Hence, for all \( \alpha \in \mathbb{R}, ]\alpha, +\infty[ \) is homeomorphic to \( ]\alpha', 1[ \) for some \( \alpha' \in \mathbb{R} \).

5. Let \( A = ]\alpha, \beta[ , \alpha, \beta \in \mathbb{R}, \alpha < \beta \). Define \( \phi : ]-1, 1[ \to ]\alpha, \beta[ \) by:

\[
\phi(x) = \alpha + \frac{\beta - \alpha}{2}(x + 1)
\]

Then it is easy to show that \( \phi \) is a continuous bijection, and that
\(\phi^{-1}\) is continuous. So \(\phi : ]-1,1[ \to ]\alpha, \beta[\) is a homeomorphism.

6. \(\phi(x) = x/(1+|x|)\) is a homeomorphism between \(\mathbb{R}\) and \(-1,1[.\)

7. Let \(A\) be a non-empty open interval in \(\mathbb{R}\), i.e. a non-empty interval of \(\mathbb{R}\). Being an interval, from 2. it is of the form \([\alpha, \beta]\), \([\alpha, \beta[, \] \alpha, \beta[\) or \] \alpha, \beta[\) for some \(\alpha, \beta \in \mathbb{R}\). Suppose \(A\) is of the form \([\alpha, \beta]\). Being non-empty with have \(\alpha \leq \beta\). So \(\alpha \in [\alpha, \beta] \subseteq \mathbb{R}\). Being an open subset of \(\mathbb{R}\), there exists \(\epsilon > 0\) such that \(]\alpha - \epsilon, \alpha + \epsilon[ \subset [\alpha, \beta]\). This is a contradiction since \(\alpha \in \mathbb{R}\). So \(A\) cannot be of the form \([\alpha, \beta]\) and we prove similarly that it cannot be of the form \([\alpha, \beta[\) and \] \alpha, \beta[\) either. So \(A\) is of the form \(]\alpha, \beta[\) for some \(\alpha, \beta \in \mathbb{R}\), \(\alpha < \beta\).

Suppose \(\alpha = -\infty\) and \(\beta = +\infty\). Then \(A = \mathbb{R}\) which is clearly homeomorphic to \(\mathbb{R}\). Suppose \(\alpha = -\infty\) and \(\beta \in \mathbb{R}\). Then from 3. \(A\) is homeomorphic to \(]-1, \alpha[\) for some \(\alpha \in \mathbb{R}\), which is itself homeomorphic to \(]-1,1[\), as we have proved in 5. Having proved in 6. that \(]-1,1[\) is homeomorphic to \(\mathbb{R}\), we conclude that \(A\) is homeomorphic to \(\mathbb{R}\). Suppose \(\alpha \in \mathbb{R}\) and \(\beta = +\infty\).
Then from 4. 5. and 6. we see that $A$ is homeomorphic to $\mathbb{R}$. Suppose $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Then from 5. and 6. we see that $A$ is homeomorphic to $\mathbb{R}$. Hence, in all possible cases, we see that $A$ is homeomorphic to $\mathbb{R}$. We have proved that any non-empty open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

8. Let $A$ be an open interval of $\mathbb{R}$. If $A = \emptyset$, then the induced topology on $A$ is reduced to $\{\emptyset\}$, and $(\emptyset, \{\emptyset\})$ is a connected topological space. So $A$ is a connected subset of $\mathbb{R}$. If $A \neq \emptyset$, then from 7. $A$ is homeomorphic to $\mathbb{R}$. In particular, there exists $f : \mathbb{R} \rightarrow A$ which is continuous and surjective. From theorem (95), $\mathbb{R}$ is connected. Since $f$ is continuous, from theorem (96) $f(\mathbb{R})$ is a connected subset of $A$. Since $f$ is surjective, $f(\mathbb{R}) = A$ and consequently $A$ is connected. We have proved that any open interval of $\mathbb{R}$ is a connected subset of $\mathbb{R}$.

9. Let $A$ be an interval of $\mathbb{R}$, i.e. an interval of $\bar{\mathbb{R}}$ with $A \subseteq \mathbb{R}$. If $A = \emptyset$ then $A$ is connected. So we assume that $A \neq \emptyset$. From 1.
there exist $\alpha, \beta \in \bar{\mathbb{R}}$ such that:

$]\alpha, \beta[ \subseteq A \subseteq [\alpha, \beta]$

and since $A \neq \emptyset$ we have $\alpha \leq \beta$. Since $]\alpha, \beta[$ is an open interval in $\mathbb{R}$, from 8, it is a connected subset of $\mathbb{R}$. Suppose $\alpha = -\infty$ and $\beta = +\infty$. Then $A = \mathbb{R}$ and:

$]\alpha, \beta[ \subseteq A \subseteq [\alpha, \beta] = ]\alpha, \beta[$

Suppose $\alpha = -\infty$ and $\beta \in \mathbb{R}$. Since $A \subseteq \mathbb{R}$ we have:

$]\alpha, \beta[ \subseteq A \subseteq [\alpha, \beta] = ]\alpha, \beta[$

Suppose $\alpha \in \mathbb{R}$ and $\beta = +\infty$. Then:

$]\alpha, \beta[ \subseteq A \subseteq [\alpha, \beta] = ]\alpha, \beta[$

And finally suppose that $\alpha, \beta \in \mathbb{R}$. Then:

$]\alpha, \beta[ \subseteq A \subseteq [\alpha, \beta] = ]\alpha, \beta[$
It follows that $]a, b[ \subseteq A \subseteq [a, b]$ in all possible cases, where $]a, b[$ denotes the closure of $]a, b[$ in $\mathbb{R}$. Having proved that $]a, b[$ is a connected subset of $\mathbb{R}$, from exercise (6) we conclude that $A$ is a connected subset of $\mathbb{R}$. We have proved that any interval in $\mathbb{R}$ is a connected subset of $\mathbb{R}$.

**Exercise 8**
Exercise 9.

1. Let $A \subseteq \mathbb{R}$ be a non-empty connected subset of $\mathbb{R}$. Let $\alpha = \inf A$ and $\beta = \sup A$. We assume that there exists $x_0 \in A \cap \alpha, \beta$. In particular, we have $x_0 \in A^c$ and consequently, since $A \subseteq \mathbb{R}$:

   $$A = (A \cap ]-\infty, x_0[) \cup (A \cap [x_0, +\infty[)$$  \hfill (2)

   However, $]-\infty, x_0[$ and $[x_0, +\infty[$ being open subsets of $\mathbb{R}$, the sets $A \cap ]-\infty, x_0[$ and $A \cap [x_0, +\infty[$ are open in $A$, and they are clearly disjoint. Since $A$ is connected, it follows from exercise (4) that $A \cap ]-\infty, x_0[ = \emptyset$ or $A \cap [x_0, +\infty[ = \emptyset$.

2. Suppose $A \cap [x_0, +\infty[ = \emptyset$. From (2) we have $A = A \cap ]-\infty, x_0[$, and consequently $x_0$ is an upper-bound of $A$. Since $\beta$ is the smallest of such upper-bounds, we obtain $\beta \leq x_0$ contradicting $x_0 \in \alpha, \beta$.

3. Similarly, if $A \cap ]-\infty, x_0[ = \emptyset$, then $x_0$ is a lower-bound of $A$ and consequently $x_0 \leq \alpha$ contradicting $x_0 \in \alpha, \beta$. We have seen
in 1. that $A \cap -\infty, x_0 = \emptyset$ or $A \cap \mathbb{R} = \emptyset$. However, both of these cases lead to a contradiction. We conclude that our initial assumption was absurd, i.e. that there exists no $x_0$ in $A \cap \mathbb{R}$. In other words, $A \cap \mathbb{R} = \emptyset$ or equivalently $\alpha, \beta \subseteq A$. The fact that $A \subseteq [\alpha, \beta]$ follows immediately from the fact that $\alpha$ and $\beta$ are respectively a lower-bound and an upper-bound of $A$. We have proved that $\alpha, \beta \subseteq A \subseteq [\alpha, \beta]$.

4. Let $A \subseteq \mathbb{R}$. Suppose that $A$ is a connected subset of $\mathbb{R}$. If $A = \emptyset$ then in particular $A$ is an interval, as can be seen from definition (118). If $A \neq \emptyset$, then $A$ is a non-empty connected subset of $\mathbb{R}$, and we have just proved that $[\alpha, \beta] \subseteq A \subseteq [\alpha, \beta]$ where $\alpha = \inf A$ and $\beta = \sup A$. In a similar fashion to 2. of exercise (8) (depending on whether $\alpha, \beta$ lie in $A$ or not), we conclude that $A$ is of the form $[\alpha, \beta], [\alpha, \beta], [\alpha, \beta]$, or $[\alpha, \beta]$. From this same exercise, this is equivalent to $A$ being an interval. So any connected subset of $\mathbb{R}$ is an interval. Conversely, suppose that $A$ is an interval of $\mathbb{R}$. Then from exercise (8), $A$ is a
connected subset of $\mathbb{R}$. We have proved that for all $A \subseteq \mathbb{R}$, $A$ is connected, if and only if $A$ is an interval. This completes the proof of theorem (97).

Exercise 9
Exercise 10. Let \( f : \Omega \to \mathbf{R} \) be a continuous map, where \((\Omega, T)\) is a connected topological space. Let \( a, b \in \Omega \) with \( f(a) \leq f(b) \). From theorem (96), \( f(\Omega) \) is a connected subset of \( \mathbf{R} \). From theorem (97), \( f(\Omega) \) is therefore an interval of \( \mathbf{R} \). Since \( f(a), f(b) \) are elements of \( f(\Omega) \) and \( f(a) \leq f(b) \), it follows from definition (118) that for all \( z \in [f(a), f(b)] \) we have \( z \in f(\Omega) \). So there exists \( x \in \Omega \) such that \( z = f(x) \). This completes the proof of theorem (98).
Exercise 11.

1. Let \( a, b \in \mathbb{R}, \ a < b \). Let \( f : [a, b] \to \mathbb{R} \) be a map such that \( f'(x) \) exists for all \( x \in [a, b] \). Note in particular that \( f \) is continuous and therefore measurable. For all \( n \geq 1 \), let \( \phi_n : [a, b] \to [a, b] : \)

\[
\forall x \in [a, b], \ \phi_n(x) = \begin{cases} 
  x + \frac{(b-x)}{n}, & \text{if } x \in [a, b] \\
  \frac{b - (b-a)}{n}, & \text{if } x = b
\end{cases}
\]

Then \( \phi_n \) is well-defined on \( [a, b] \) and has indeed values in \( [a, b] \). The particular definition of \( \phi_n \) is however not very important. What we need to note is that \( \phi_n \) is Borel measurable, satisfies \( \phi_n(x) \to x \) while \( \phi_n(x) \neq x \) for all \( x \in [a, b] \). Given \( n \geq 1 \), we now define \( g_n : [a, b] \to \mathbb{R} \) as:

\[
\forall x \in [a, b], \ g_n(x) = \frac{f \circ \phi_n(x) - f(x)}{\phi_n(x) - x}
\]

Then \( g_n : ([a, b], \mathcal{B}([a, b])) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is well-defined and measurable, and furthermore \( g_n(x) \to f'(x) \) for all \( x \in [a, b] \). It fol-
follows that \( f' \) is the pointwise limit of the sequence \( (g_n)_{n \geq 1} \), and we conclude from theorem (17) that \( f' \) is itself Borel measurable.

2. Since \( f' \) is measurable and \( \mathbb{R} \)-valued, the condition:
\[
\int_a^b |f'(t)| dt < +\infty
\]
is equivalent to \( f' \in L^1_\mathbb{R}([a, b], B([a, b]), dx) \).

3. We assume that \( f' \in L^1_\mathbb{R}([a, b], B([a, b]), dx) \). Let \( \epsilon > 0 \). The topological space \([a, b]\) is metrizable and compact, and in particular \( \sigma \)-compact. The Lebesgue measure \( dx \) on \([a, b]\) is finite, and in particular locally finite. Since \( f' \in L^1_\mathbb{R}([a, b], B([a, b]), dx) \), we can apply Vitali-Caratheodory theorem (94): there exists measurable maps \( u, v : [a, b] \to \bar{\mathbb{R}} \) which are almost surely equal to elements of \( L^1 \), such that \( u \leq f' \leq v \), \( u \) is u.s.c, \( v \) is l.s.c and furthermore:
\[
\int_a^b (v(t) - u(t)) dt \leq \epsilon
\]
In particular, denoting \( g = v \), we have found \( g : [a, b] \to \mathbb{R} \) almost surely equal to an element of \( L^1 \), such that \( f' \leq g \) and \( g \) is l.s.c. Note that the integral \( \int_a^b g(t)dt \) is meaningful, and:

\[
\int_a^b g(t)dt = \int_a^b (f'(t) + g(t) - f'(t))dt \\
= \int_a^b f'(t)dt + \int_a^b (g(t) - f'(t))dt \\
\leq \int_a^b f'(t)dt + \int_a^b (v(t) - u(t))dt \\
\leq \int_a^b f'(t)dt + \epsilon
\]

4. Let \( \alpha > 0 \). Since \( f' \leq g \) we have \( f' < g + \alpha \). Indeed, suppose \( f'(x) = g(x) + \alpha \), \( x \in [a, b] \). Then \( f'(x) = g(x) = g(x) + \alpha \) and consequently \( g(x) \in \{-\infty, +\infty\} \) contradicting the fact that \( f' \) is \( \mathbb{R} \)-valued. Having proved that \( f' < g + \alpha \), note that \( g + \alpha \) is
also a lower-semi-continuous map, which furthermore is almost surely equal to an element of $L^1$, since the Lebesgue measure on $[a, b]$ is finite. Furthermore, we have:

$$
\int_a^b (g + \alpha)(t)dt = \int_a^b g(t)dt + \alpha(b - a) \\
\leq \int_a^b f'(t)dt + \epsilon + \alpha(b - a)
$$

Hence, taking $\alpha > 0$ small enough, it is possible to achieve:

$$
\int_a^b (g + \alpha)(t)dt \leq \int_a^b f'(t)dt + 2\epsilon
$$

Replacing $g$ by $g + \alpha$, we have found $g : [a, b] \to \overline{\mathbb{R}}$ almost surely equal to an element of $L^1$, which is l.s.c. and satisfies $f' < g$ together with:

$$
\int_a^b g(t)dt \leq \int_a^b f'(t)dt + 2\epsilon
$$
Since $\epsilon > 0$ was arbitrary, it is possible to find $g$ such that:

$$\int_a^b g(t)\,dt \leq \int_a^b f'(t)\,dt + \epsilon$$

In other words, without loss of generality, we have been able to find a map $g$ as in 3., with the additional condition $f' < g$.

5. Let $\nu$ be the complex measure defined by $\nu = \int g\,dx$. Note that strictly speaking, $g$ is not an element of $L^1$ (it may have values in $\{-\infty, +\infty\}$). If $h$ is an element of $L^1_{\mathbb{R}}([a, b], \mathcal{B}([a, b]), dx)$ such that $g = h\,dx$-almost surely, then for all $E \in \mathcal{B}([a, b])$, $\nu(E)$ is defined as:

$$\nu(E) = \int_E h(x)\,dx$$

Note that $\nu$ is in fact a signed measure (i.e. a complex measure with values in $\mathbb{R}$). Since $dx(E) = 0$ implies $\nu(E) = 0$, the measure $\nu$ is absolutely continuous with respect to the Lebesgue measure.
measure on \([a, b]\). From theorem (58), we have:

\[
\forall \epsilon' > 0, \exists \delta > 0, \forall E \in \mathcal{B}([a, b]), \, dx(E) \leq \delta \Rightarrow |\nu(E)| \leq \epsilon'
\]

6. Let \(\eta > 0\) and \(x \in [a, b]\). We define:

\[
F_{\eta}(x) = \int_{a}^{x} g(t)dt - f(x) + f(a) + \eta(x - a)
\]

Then \(F_{\eta} : [a, b] \to \mathbb{R}\) is well-defined, and we claim that it is continuous. It is sufficient to show that \(x \to \int_{a}^{x} g(t)dt\) is continuous. Let \(\epsilon' > 0\) be given, and consider \(\delta > 0\) such that the statement of 5. is satisfied. Let \(u, u' \in [a, b]\) such that \(|u' - u| \leq \delta\). Without loss of generality, we may assume that \(u \leq u'\). Then \(dx([u, u']) \leq \delta\) and consequently from 5., \(|\nu([u, u'])| \leq \epsilon'\). So:

\[
\left| \int_{a}^{u'} g(t)dt - \int_{a}^{u} g(t)dt \right| = \left| \int_{[a, u']} g(t)dt - \int_{[a, u]} g(t)dt \right|
\]
\[ = \left| \int_{[u, u']} g(t) dt \right| = |\nu([u, u'])| \leq \epsilon' \]

This shows that \( x \to \int_a^x g(t) dt \) is indeed continuous on \([a, b]\) (in fact uniformly continuous), and \( F_\eta : [a, b] \to \mathbb{R} \) is indeed a continuous map.

7. Given \( \eta > 0 \), let \( x = \sup F_\eta^{-1}([0]) \). It is clear that \( F_\eta(a) = 0 \) and consequently \( a \in F_\eta^{-1}([0]) \). So \( a \leq x \). Since \( F_\eta^{-1}([0]) \subseteq [a, b] \), in particular \( b \) is an upper-bound of \( F_\eta^{-1}([0]) \). So \( x \leq b \).

We have proved that \( x \in [a, b] \). In particular, \( x \in \mathbb{R} \) and for all \( n \geq 1 \) we have \( x - 1/n < x \). Since \( x \) is the lowest upper-bound of \( F_\eta^{-1}([0]) \), \( x - 1/n \) cannot be such an upper-bound. There exists \( x_n \in F_\eta^{-1}([0]) \) such that \( x - 1/n < x_n \leq x \). We have thus constructed a sequence \( (x_n)_{n \geq 1} \) in \( F_\eta^{-1}([0]) \) such that \( x_n \to x \) as \( n \to +\infty \). Since \( F_\eta(x_n) = 0 \) for all \( n \geq 1 \), from the continuity of \( F_\eta \) we obtain \( F_\eta(x) = 0 \).

8. Suppose \( x \in [a, b] \). Having proved in 4. that \( f' < g \), in particular

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\[ f'(x) < g(x). \] Since \( g \) is l.s.c, the set \( \{ f'(x) < g \} \) is an open subset of \([a, b]\), which contains \( x \). Hence, there exists \( \delta_1 > 0 \) such that:

\[ |x - \delta_1, x + \delta_1[ \cap [a, b] \subseteq \{ f'(x) < g \} \]

In particular we have:

\[ t \in ]x, x + \delta_1[ \cap [a, b] \Rightarrow f'(x) < g(t) \]

Furthermore, by definition of the derivative \( f'(x) \), since \( \eta > 0 \), there exists \( \delta_2 > 0 \) such that:

\[ t \in ]x - \delta_2, x + \delta_2[ \cap [a, b], t \neq x \Rightarrow \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \eta \]

In particular, we have:

\[ t \in ]x, x + \delta_2[ \cap [a, b] \Rightarrow \frac{f(t) - f(x)}{t - x} < f'(x) + \eta \]
Taking $\delta = \min(\delta_1, \delta_2)$, for all $t \in [x, x + \delta \cap [a, b]$ we have:

\[
f'(x) < g(t) \quad \text{and} \quad \frac{f(t) - f(x)}{t - x} < f'(x) + \eta
\]

Note that this conclusion is not very interesting if $x = b$, which is why we have assumed $x \in [a, b]$.

9. Let $t \in [x, x + \delta \cap [a, b]$. Using 8. we have:

\[
F_\eta(t) = \int_a^t g(u)du - f(t) + f(a) + \eta(t - a)
\]
\[
= F_\eta(x) + \int_x^t g(u)du + f(x) - f(t) + \eta(t - x)
\]
\[
> F_\eta(x) + \int_x^t g(u)du - f'(x)(t - x)
\]
\[
\geq F_\eta(x) + \int_x^t f'(x)du - f'(x)(t - x)
\]
\[
= F_\eta(x) = 0
\]
10. From 9. we have found $\delta > 0$ such that $F_\eta(t) > 0$ for all $t$ in the set $[x, x + \delta] \cap [a, b]$. Having assumed in 8. that $x \in [a, b]$, in particular $x < b$. So it is possible to find $t_0 \in [x, b]$ such that $t_0 \in [x, x + \delta] \cap [a, b]$. In particular $F_\eta(t_0) > 0$. We have proved the existence of $t_0 \in [x, b]$ such that $F_\eta(t_0) > 0$.

11. Suppose $F_\eta(b) < 0$. From 10. we have $t_0 \in [x, b]$ such that $F_\eta(t_0) > 0$. From 6. the map $F_\eta : [a, b] \to \mathbb{R}$ is continuous. Let $h = (F_\eta)|_{[t_0, b]}$ be the restriction of $F_\eta$ to the interval $[t_0, b]$.

Then $h$ is also continuous. From theorem (97), $[t_0, b]$ is a connected topological space. Since $0 \in [F_\eta(b), F_\eta(t_0)]$, from theorem (98) there exists $u \in [t_0, b]$ such that $F_\eta(u) = 0$. Since $x = \sup F_\eta^{-1}(\{0\})$, in particular $u \leq x$. Hence, we obtain the contradiction $x < t_0 \leq u \leq x$.

12. From 11. we see that $F_\eta(b) \geq 0$ must be true when $x \in [a, b]$. Having proved in 7. that $F_\eta(x) = 0$, if $x = b$, $F_\eta(b) = 0$ and in particular $F_\eta(b) \geq 0$ is still true. So $F_\eta(b) \geq 0$ in all cases.

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13. From $F_b(b) \geq 0$ we obtain:

$$\int_a^b g(t)dt - f(b) + f(a) + \eta(b-a) \geq 0$$

This being true for all $\eta > 0$, we have:

$$f(b) - f(a) \leq \int_a^b g(t)dt$$

Hence, using 3. we obtain:

$$f(b) - f(a) \leq \int_a^b f'(t)dt + \epsilon$$

and this being true for all $\epsilon > 0$, we have proved that:

$$f(b) - f(a) \leq \int_a^b f'(t)dt \quad (3)$$

Having considered $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ a map
such that \( f'(x) \) exists for all \( x \in [a, b] \) and:

\[
\int_a^b |f'(t)|\,dt < +\infty
\]

we have been able to prove inequality (3). Applying this result to \(-f\) instead of \(f\), we obtain:

\[
\int_a^b f'(t)\,dt \leq f(b) - f(a)
\]

and finally we conclude that:

\[
f(b) - f(a) = \int_a^b f'(t)\,dt
\]

This completes the proof of theorem (99).

Exercise 11
Exercise 12.

1. Let $\alpha > 0$ and $k_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined by $k_\alpha(x) = \alpha x$. Then $k_\alpha$ is continuous, and in particular Borel measurable.

2. Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, +\infty]$ be defined by:

$$\forall B \in \mathcal{B}(\mathbb{R}^n), \mu(B) = \alpha^n dx(\{k_\alpha \in B\})$$

where $dx$ is the Lebesgue measure on $\mathbb{R}^n$. Note that $\mu$ is well-defined since $\{k_\alpha \in B\}$ is a Borel set for all $B \in \mathcal{B}(\mathbb{R}^n)$, $k_\alpha$ being measurable. It is clear that $\mu(\emptyset) = 0$ and furthermore, if $(B_p)_{p \geq 1}$ is sequence of pairwise disjoint elements of $\mathcal{B}(\mathbb{R}^n)$ and $B = \bigcup_{p \geq 1} B_p$, we have:

$$\mu(B) = \alpha^n dx \left( k^{-1}_\alpha \left( \bigcup_{p \geq 1} B_p \right) \right)$$
= \alpha^n dx \left( \biguplus_{p \geq 1} k^{-1}_\alpha(B_p) \right)
= \alpha^n \sum_{p=1}^{+\infty} dx(k^{-1}_\alpha(B_p))
= \sum_{p=1}^{+\infty} \alpha^n dx(\{k \alpha \in B_p\})
= \sum_{p=1}^{+\infty} \mu(B_p)

So \mu is a measure on \( \mathbb{R}^n \). Let \( a_i, b_i \in \mathbb{R}, a_i \leq b_i \) for \( i \in \mathbb{N}_n \). For all \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) the inequality \( a_i \leq \alpha x_i \leq b_i \) is equivalent to \( a_i / \alpha \leq x_i \leq b_i / \alpha \). Hence:

\[ \mu([a_1, b_1] \times \ldots \times [a_n, b_n]) = \alpha^n dx \left( \alpha x \in \prod_{i=1}^{n} [a_i, b_i] \right) \]
From the uniqueness property of definition (63) we conclude that $\mu = dx$. Hence, we have proved that for all $B \in \mathcal{B}(\mathbb{R}^n)$:

$$dx(\{k_\alpha \in B\}) = \frac{1}{\alpha^n} \mu(B) = \frac{1}{\alpha^n} dx(B)$$

3. Let $\epsilon > 0$ and $x \in \mathbb{R}^n$. Let $B(x, \epsilon)$ be the open ball:

$$B(x, \epsilon) = \{y \in \mathbb{R}^n : \|x - y\| < \epsilon\}$$

where $\| \cdot \|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. Given $u \in \mathbb{R}^n$ we consider $\tau_u : \mathbb{R}^n \to \mathbb{R}^n$ the translation mapping of

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vector $u$ defined by $\tau_u(x) = u + x$. Then $\tau_u$ is clearly continuous, hence Borel measurable. Furthermore, for all $a, b \in \mathbb{R}^n$ such that $a_i \leq b_i$ for all $i \in \mathbb{N}_n$, we have:

$$dx \left( \left\{ \tau_u \in \prod_{i=1}^n [a_i, b_i] \right\} \right) = dx \left( \prod_{i=1}^n [a_i - u_i, b_i - u_i] \right)$$

$$= \prod_{i=1}^n (b_i - a_i)$$

and in a similar fashion to 2, we conclude from the uniqueness property of definition (63) that for all $B \in \mathcal{B}(\mathbb{R}^n)$:

$$dx(\{\tau_u \in B\}) = dx(B)$$

This equality expresses the idea that the Lebesgue measure is \textit{invariant by translation}. We shall see more on the subject in Tutorial 17. In the meantime, using 2, we obtain:

$$dx(B(x, \epsilon)) = dx(\{\tau_{-x} \in B(0, \epsilon)\})$$
Solutions to Exercises

\[
\begin{align*}
\ &= \, dx(B(0, \epsilon)) \\
\ &= \, dx(\{k_{1/\epsilon} \in B(0, 1)\}) \\
\ &= \, \epsilon^n dx(B(0, 1)) \\
\end{align*}
\]

So we have proved that \(dx(B(x, \epsilon)) = \epsilon^n dx(B(0, 1))\).

Exercise 12
Exercise 13.

1. Let $\mu$ be a complex measure on $\mathbb{R}^n$. Let $\lambda \in \mathbb{R}$ and suppose that $\lambda < 0$. Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Since $B(x, \epsilon)$ is an open subset of $\mathbb{R}^n$, in particular it is a Borel subset of $\mathbb{R}^n$. So $|\mu|(B(x, \epsilon))$ and $dx(B(x, \epsilon))$ are well-defined quantities of $[0, +\infty]$. In fact, from theorem (57), the total variation $|\mu|$ is a finite measure on $\mathbb{R}^n$, so $|\mu|(B(x, \epsilon))$ is an element of $\mathbb{R}^+$ (this is not relevant to the present question, but the fact that $|\mu|$ is a finite measure should not be forgotten). From the inclusions:

$$[-1/2\sqrt{n}, 1/2\sqrt{n}]^n \subseteq B(0, 1) \subseteq [-1, 1]^n$$

we obtain the crude estimates:

$$\left(\frac{1}{\sqrt{n}}\right)^n \leq dx(B(0, 1)) \leq 2^n$$

and it follows from 3. of exercise (12) that $dx(B(x, \epsilon))$ is an element of $]0, +\infty[$. Hence, we see that $|\mu|(B(x, \epsilon))/dx(B(x, \epsilon))$
is a well-defined element of $\mathbb{R}^+$. Since $(M_\mu)(x)$ is an upper-bound of all such ratios for $\epsilon > 0$, we have:
\[
\lambda < 0 \leq \frac{\mu(B(x, \epsilon))}{dx(B(x, \epsilon))} \leq (M_\mu)(x)
\]
So $x \in \{ \lambda < M_\mu \}$. This being true for all $x \in \mathbb{R}^n$, we conclude that $\{ \lambda < M_\mu \} = \mathbb{R}^n$.

2. Suppose $\lambda = 0$ and $\mu \neq 0$. There exists $E \in \mathcal{B}(\mathbb{R}^n)$ such that $\mu(E) \neq 0$. Since $|\mu(E)| \leq |\mu|(E)$, in particular $|\mu|(E) > 0$. Let $x \in \mathbb{R}^n$. Since $B(x, p) \uparrow \mathbb{R}^n$ as $p \to +\infty$, from theorem (7):
\[
0 < |\mu|(E) = \lim_{p \to +\infty} |\mu|(E \cap B(x, p))
\]
In particular, there exists $p \geq 1$ such that $|\mu|(E \cap B(x, p)) > 0$ and consequently $|\mu|(B(x, p)) > 0$. Hence, we have:
\[
0 < \frac{\mu(B(x, p))}{dx(B(x, p))} \leq (M_\mu)(x)
\]
and we have proved that \( x \in \{ \lambda < M \mu \} = \{ 0 < M \mu \} \). This being true for all \( x \in \mathbb{R}^n \), we have \( \{ \lambda < M \mu \} = \mathbb{R}^n \). Suppose now that \( \lambda = 0 \) with \( \mu = 0 \). Then \( |\mu| = 0 \) and it is clear that \( (M\mu)(x) = 0 \) for all \( x \in \mathbb{R}^n \). So \( \{ \lambda < M \mu \} = \emptyset \).

3. Suppose \( \lambda > 0 \). Let \( x \in \{ \lambda < M \mu \} \). Then \( \lambda < (M\mu)(x) \). Since \( (M\mu)(x) \) is the smallest upper-bound of all ratios:

\[
|\mu|(B(x, \epsilon))/dx(B(x, \epsilon))
\]

as \( \epsilon > 0 \), \( \lambda \) cannot be such an upper-bound. There exists \( \epsilon > 0 \) such that \( \lambda < |\mu|(B(x, \epsilon))/dx(B(x, \epsilon)) \). Defining:

\[
t = |\mu|(B(x, \epsilon))/dx(B(x, \epsilon))
\]

we have \( t > \lambda \) and \( |\mu|(B(x, \epsilon)) = tdx(B(x, \epsilon)) \).

4. Since \( 1 < t/\lambda \) we have \( \epsilon^n < \epsilon^n t/\lambda \). Furthermore, it is clear that \( \lim_{\delta \to 0} (\epsilon + \delta)^n = \epsilon^n \). Hence, we have \( (\epsilon + \delta)^n < \epsilon^n t/\lambda \), for \( \delta > 0 \) small enough.
5. Suppose \( y \in B(x, \delta) \) and let \( z \in B(x, \epsilon) \). Then:

\[
\|z - y\| \leq \|z - x\| + \|x - y\| < \epsilon + \delta
\]

So \( z \in B(y, \epsilon + \delta) \) and we have proved that \( B(x, \epsilon) \subseteq B(y, \epsilon + \delta) \).

6. Let \( y \in B(x, \delta) \). Since \( B(x, \epsilon) \subseteq B(y, \epsilon + \delta) \), we have:

\[
|\mu|(B(y, \epsilon + \delta)) \geq |\mu|(B(x, \epsilon))
\]

\[
= tdx(B(x, \epsilon))
\]

\[
= \epsilon^n tdx(B(0, 1))
\]

\[
= \frac{t^n}{(\epsilon + \delta)^n} dx(B(y, \epsilon + \delta))
\]

\[
> \lambda dx(B(y, \epsilon + \delta))
\]

where the second and third equalities stem from exercise (12).

7. For all \( y \in B(x, \delta) \), from 6. we have:

\[
\lambda < \frac{|\mu|(B(y, \epsilon + \delta))}{dx(B(y, \epsilon + \delta))} \leq (M\mu)(y)
\]

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So in particular $y \in \{\lambda < M\mu\}$ and we have proved that $B(x, \delta) \subseteq \{\lambda < M\mu\}$. Having considered $x \in \{\lambda < M\mu\}$ we have found $\delta > 0$ such that $B(x, \delta) \subseteq \{\lambda < M\mu\}$. This shows that $\{\lambda < M\mu\}$ is an open subset of $\mathbb{R}^n$, for all $\lambda \in \mathbb{R}$ with $\lambda > 0$. In fact, it follows from 1. and 2. that $\{\lambda < M\mu\}$ is also open if $\lambda \leq 0$. We conclude that $\{\lambda < M\mu\}$ is open for all $\lambda \in \mathbb{R}$, i.e. that the maximal function $M\mu$ is lower-semicontinuous. In particular, $\{\lambda < M\mu\}$ is a Borel subset of $\mathbb{R}^n$ for all $\lambda \in \mathbb{R}$ and from theorem (15), $M\mu$ is measurable.

Exercise 13
Exercise 14.

1. Let \( B_i = B(x_i, \epsilon_i) \), \( i = 1, \ldots, N \), be a finite collection of open balls in \( \mathbb{R}^n \) where we have assumed that \( \epsilon_N \leq \ldots \leq \epsilon_1 \). We define \( J_0 = \{1, \ldots, N\} \) and for all \( k \geq 1 \):

\[
J_k \triangleq \left\{ \begin{array}{ll}
J_{k-1} \cap \{ j : j > i_k \} \cap B_{i_k} = \emptyset \\
\emptyset
\end{array} \right. 
\]

where \( i_k = \min J_{k-1} \) if \( J_{k-1} \neq \emptyset \). Suppose \( k \geq 1 \) and \( J_{k-1} \neq \emptyset \). The fact that \( J_k \subseteq J_{k-1} \) is clear. However, the inclusion is strict. Indeed, since \( i_k = \min J_{k-1} \), in particular \( i_k \in J_{k-1} \). However, it is clear that \( i_k \notin J_k \). We have proved that \( J_k \subset J_{k-1} \).

2. Since \( (J_k)_{k \geq 0} \) is a strictly decreasing sequence (in the inclusion sense) and \( J_0 \) is a finite set, there exists \( k \geq 1 \) such that \( J_k = \emptyset \). It follows that \( p = \min \{k \geq 1 : J_k = \emptyset\} \), as the smallest element of a non-empty subset of \( \mathbb{N} \), is well-defined.

3. Let \( S = \{i_1, \ldots, i_p\} \) where \( i_k = \min J_{k-1} \) for all \( k \geq 1 \) with \( J_{k-1} \neq \emptyset \). In order to show that \( S \) is well-defined, we need to
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ensure that \( i_k \) is meaningful for \( k \in \mathbb{N}_p \), i.e. that \( J_{k-1} \neq \emptyset \). But if \( k \in \mathbb{N}_p \) and \( J_{k-1} = \emptyset \), since \( p \) is the smallest element of \( \{ k \geq 1 : J_k = \emptyset \} \) we obtain \( p \leq k - 1 \) and \( k \leq p \) which is a contradiction. So \( S \) is well-defined.

4. Suppose \( 1 \leq k < k' \leq p \). We have \( i_{k'} \in J_{k'-1} \subseteq J_k \). So \( i_{k'} \in J_k \).

5. The family \((B_i)_{i \in S}\) is a family of open balls. Suppose \( i, j \in S \) with \( i < j \). There exist \( 1 \leq k < k' \leq p \) such that \( i = i_k \) and \( j = i_{k'} \). From 4. we have \( j \in J_k \). This implies in particular that \( B_j \cap B_k = \emptyset \). So \( B_j \cap B_i = \emptyset \), and \((B_i)_{i \in S}\) is a family of pairwise disjoint open balls.

6. Let \( i \in \{1, \ldots, N\} \setminus S \) and \( k_0 = \min\{ k \in \mathbb{N}_p : i \not\in J_k \} \). In order to show that \( k_0 \) is well-defined, we need to check that \( \{ k \in \mathbb{N}_p : i \not\in J_k \} \) is not empty. This is clear from the fact that \( J_p = \emptyset \). So \( k_0 \) is well-defined. Note that this conclusion holds for any \( i \in \{1, \ldots, N\} \).

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7. \( k_0 \) being the smallest element of \( \{ k \in \mathbb{N} : i \notin J_k \} \), \( k_0 - 1 \) does not lie in this set. So either \( k_0 - 1 = 0 \) or \( i \in J_{k_0 - 1} \). Since \( J_0 = \{1, \ldots, N\} \), in any case we have \( i \in J_{k_0 - 1} \). In particular \( J_{k_0 - 1} \neq \emptyset \). So \( i_{k_0} \) is defined as the smallest element of \( J_{k_0 - 1} \). From \( i \in J_{k_0 - 1} \) we obtain \( i_{k_0} \leq i \).

8. Since \( J_{k_0 - 1} \neq \emptyset \), we have:

\[
J_{k_0} = J_{k_0 - 1} \cap \{ j : j > i_{k_0}, B_j \cap B_{i_{k_0}} = \emptyset \}
\]

\( k_0 \) being the smallest element of \( \{ k \in \mathbb{N} : i \notin J_k \} \), in particular it is an element of this set and consequently we know that \( i \notin J_{k_0} \). However, we have proved in 7. that \( i \in J_{k_0 - 1} \). Furthermore, we know that \( i_{k_0} \leq i \) and since by assumption \( i \in \{1, \ldots, N\} \setminus S \), in particular \( i \) is not an element of \( S \). So \( i \neq i_{k_0} \) and therefore \( i_{k_0} < i \). Since \( i \notin J_{k_0} \) we conclude that \( B_i \cap B_{i_{k_0}} \neq \emptyset \).

9. From 8. we have \( B_i \cap B_{i_{k_0}} = B(x_i, \epsilon_i) \cap B(x_{i_{k_0}}, \epsilon_{i_{k_0}}) \neq \emptyset \). Let \( x \) be an arbitrary element of \( B_i \cap B_{i_{k_0}} \). Then for all \( y \in B_i \), since

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\( i_{k_0} < i \) and \( \epsilon_N \leq \ldots \leq \epsilon_1 \), we have:
\[
\| y - x_{i_{k_0}} \| \leq \| y - x_i \| + \| x_i - x \| + \| x - x_{i_{k_0}} \| < \epsilon_i + \epsilon_i + \epsilon_{i_{k_0}} \leq 3\epsilon_{i_{k_0}}
\]
So \( y \in B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}}) \) and we have proved \( B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}}) \).

10. For all \( i \in \{1, \ldots, N\} \setminus S \), we found \( k_0 \in \mathbb{N}_p \) such that \( B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}}) \). In other words, if we denote \( j(i) = i_{k_0} \), there exists some \( j(i) \in S \) such that we have \( B_i \subseteq B(x_{j(i)}, 3\epsilon_{j(i)}) \).
Hence:
\[
\begin{align*}
\bigcup_{i=1}^{N} B(x_i, \epsilon_i) & = \bigcup_{i \in S} B(x_i, \epsilon_i) \cup \left( \bigcup_{i \notin S} B(x_i, \epsilon_i) \right) \\
& \subseteq \bigcup_{i \in S} B(x_i, \epsilon_i) \cup \left( \bigcup_{i \notin S} B(x_{j(i)}, 3\epsilon_{j(i)}) \right)
\end{align*}
\]
So \( S = \{i_1, \ldots, i_p\} \) is a subset of \( \{1, \ldots, N\} \) such that \( (B_i)_{i \in S} \) is a family of pairwise disjoint open balls, and:

\[
\bigcup_{i=1}^{N} B(x_i, \epsilon_i) \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_i)
\]

11. Using 10. and exercise (12), we have:

\[
dx\left(\bigcup_{i=1}^{N} B(x_i, \epsilon_i)\right) \leq \sum_{i \in S} dx(B(x_i, 3\epsilon_i))
\]
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\[
= \sum_{i \in S} 3^n \epsilon_i^n dx(B(0,1)) \\
= 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i))
\]

where the second inequality stems from the fact that a measure is always sub-additive, as can be seen from exercise (13) of Tutorial 5.

Exercise 14
Exercise 15.

1. Let $\mu$ be a complex measure on $\mathbb{R}^n$. Let $\lambda > 0$ and $K$ be a non-empty compact subset of $\{\lambda < M\mu\}$. Let $x \in K$. Then $x \in \{\lambda < M\mu\}$, i.e. $\lambda < (M\mu)(x)$. Since $(M\mu)(x)$ is the smallest upper-bound of all ratios:

$$\frac{|\mu|(B(x, \epsilon))}{dx(B(x, \epsilon))}$$

as $\epsilon > 0$, it is impossible for $\lambda$ to be such an upper-bound. There exists $\epsilon_x > 0$ such that:

$$\lambda < \frac{|\mu|(B(x, \epsilon_x))}{dx(B(x, \epsilon_x))} \quad (4)$$

Now it is clear that $K \subseteq \cup_{x \in K} B(x, \epsilon_x)$. Since $K$ is compact, there exist $N \geq 1$ and $x_1, \ldots, x_N \in K$ such that:

$$K \subseteq B(x_1, \epsilon_{x_1}) \cup \ldots \cup B(x_N, \epsilon_{x_N})$$

Defining $\epsilon_i = \epsilon_{x_i}$ and $B_i = B(x_i, \epsilon_i)$, the collection $(B_i)_{i \in \mathbb{N}_N}$ is therefore a covering of $K$. From (4), for all $i = 1, \ldots, N$ we
have $\lambda dx(B_i) < |\mu|(B_i)$.

2. By re-indexing the $B_i$’s if necessary, without loss of generality we can assume that $\epsilon_N \leq \ldots \leq \epsilon_1$. From exercise (14), there exists a subset $S$ of $\{1, \ldots, N\}$ such that the $B_i$’s for $i \in S$ are pairwise disjoint, and furthermore:

$$dx \left( \bigcup_{i=1}^{N} B(x_i, \epsilon_i) \right) \leq 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i))$$

Hence, since $K \subseteq \bigcup_{i=1}^{N} B_i$, using 1. we obtain:

$$dx(K) \leq dx \left( \bigcup_{i=1}^{N} B(x_i, \epsilon_i) \right) \leq 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i)) < 3^n \sum_{i \in S} \frac{1}{\lambda} |\mu|(B(x_i, \epsilon_i))$$
\[ = \frac{3^n}{\lambda} |\mu| \left( \bigcup_{x \in S} B(x_i, \epsilon_i) \right) \]

where the last equality stems from the fact that all the \( B_i \)'s, \( i \in S \), are pairwise disjoint. We have effectively obtained a strict inequality, when only a large inequality was required.

3. Let \( \|\mu\| = |\mu|(\mathbb{R}^n) < +\infty \) be the total mass of \( |\mu| \). From 2:

\[ dx(K) \leq 3^n \lambda^{-1} |\mu| \left( \bigcup_{i \in S} B(x_i, \epsilon_i) \right) \leq 3^n \lambda^{-1} \|\mu\| \]

4. Having considered a complex measure \( \mu \) on \( \mathbb{R}^n \), with maximal function \( M\mu \), given \( \lambda \in \mathbb{R}^+ \setminus \{0\} \), for all \( K \) non-empty compact subset of \( \{ \lambda < M\mu \} \), we have proved that:

\[ dx(K) \leq 3^n \lambda^{-1} \|\mu\| \]

Note that this inequality is still valid if \( K = \emptyset \). The Lebesgue measure on \( \mathbb{R}^n \) being locally finite, from theorem (74) it is inner-
regular. In particular, we have:
\[ dx(\{ \lambda < M \mu \}) = \sup \{ dx(K) : K \subseteq \{ \lambda < M \mu \}, K \text{ compact} \} \]
In other words, \( dx(\{ \lambda < M \mu \}) \) is the smallest upper-bound of all \( dx(K) \)'s, as \( K \) runs through the set of all compact subsets of \( \{ \lambda < M \mu \} \). Having proved that \( 3^n \lambda^{-1} \| \mu \| \) is one of those upper-bounds, we conclude that:
\[ dx(\{ \lambda < M \mu \}) \leq 3^n \lambda^{-1} \| \mu \| \]
This completes the proof of theorem (100).

Exercise 15
Exercise 16.

1. Let \( f \in L^1_C(\mathbb{R}^n, B(\mathbb{R}^n), dx) \), \( n \geq 1 \). From theorem (63), \( \mu = \int f dx \) is a well-defined complex measure on \( \mathbb{R}^n \), and its total variation \( |\mu| \) is given by \( |\mu| = \int |f| dx \). From definition (120), the maximal function \( Mf \) of \( f \) is exactly the maximal function \( M\mu \) of \( \mu \). Hence, for all \( x \in \mathbb{R}^n \):

\[
(Mf)(x) = (M\mu)(x) = \sup_{\epsilon > 0} \frac{|\mu|(B(x, \epsilon))}{dx(B(x, \epsilon))} = \sup_{\epsilon > 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f| dx
\]

2. If \( \mu = \int f dx \) then \( |\mu| = \int |f| dx \) and consequently:

\[
\|\mu\| = |\mu|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |f| dx = \|f\|_1
\]
Applying theorem (100) to $\mu$, for all $\lambda > 0$ we obtain:

$$dx(\{\lambda < Mf\}) = dx(\{\lambda < M\mu\})$$

$$\leq 3^n \lambda^{-1} \|\mu\|$$

$$= 3^n \lambda^{-1} \|f\|_1$$

Exercise 16
Exercise 17.

1. Let $f \in L^1_C(\mathbb{R}^n, B(\mathbb{R}^n), dx)$, $n \geq 1$. Let $x \in \mathbb{R}^n$. We assume that $f$ is continuous at $x$. Let $\eta > 0$. There is $\delta > 0$ such that:

$$\forall y \in \mathbb{R}^n, \|x - y\| \leq \delta \Rightarrow |f(x) - f(y)| \leq \eta$$

Suppose $\epsilon > 0$ is such that $0 < \epsilon < \delta$. Then:

$$\frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| dy \leq \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} \eta dy = \eta$$

We conclude that:

$$\lim_{\epsilon \downarrow 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| dy = 0$$

and $x$ is therefore a Lebesgue point of $f$.

2. Let $x \in \mathbb{R}^n$. We assume that $x$ is a Lebesgue point of $f$. For
all $\epsilon > 0$, denoting $B_\epsilon = B(x, \epsilon)$ we have:

\[
\left| \frac{1}{d\mathcal{X}(B_\epsilon)} \int_{B_\epsilon} f(y) dy - f(x) \right| = \left| \frac{1}{d\mathcal{X}(B_\epsilon)} \int_{B_\epsilon} (f(y) - f(x)) dy \right| \\
\leq \frac{1}{d\mathcal{X}(B_\epsilon)} \int_{B_\epsilon} |f(y) - f(x)| dy
\]

Hence, from:

\[
\lim_{\epsilon \to 0} \frac{1}{d\mathcal{X}(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| dy = 0
\]

we conclude that:

\[
f(x) = \lim_{\epsilon \to 0} \frac{1}{d\mathcal{X}(B(x, \epsilon))} \int_{B(x, \epsilon)} f(y) dy
\]

**Exercise 17**
Exercise 18.

1. Given \( f \in L^1_{C}(\mathbb{R}^n, B(\mathbb{R}^n), dx) \), for all \( \epsilon > 0 \) and \( x \in \mathbb{R}^n \), let:

\[
(T_\epsilon f)(x) = \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| dy
\]

and:

\[
(T f)(x) = \inf_{\epsilon > 0} \sup_{u \in [0, \epsilon]} (T_u f)(x)
\]

From theorem (79), the space \( C^c_0(\mathbb{R}^n) \) of continuous \( C \)-valued functions defined on \( \mathbb{R}^n \) with compact support, is dense in \( L^1 \).

Given \( \eta > 0 \), there exists \( g \in C^c_0(\mathbb{R}^n) \) such that \( \|f - g\|_1 \leq \eta \).

2. Let \( h = f - g \). For all \( \epsilon > 0 \) and \( x \in \mathbb{R}^n \) we have:

\[
(T, h)(x) = \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |h(y) - h(x)| dy
\]

\[
\leq \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} (|h(y)| + |h(x)|) dy
\]
\[
= \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |h(y)| dy + |h(x)| \\
= \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |h| dx + |h(x)|
\]

3. Let \( x \in \mathbb{R}^n \). From exercise (16) we have:

\[
(Mh)(x) = \sup_{\epsilon > 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |h| dx
\]

In particular, for all \( \epsilon > 0 \), from 2. we obtain:

\[
(T_{\epsilon} h)(x) \leq (Mh)(x) + |h(x)|
\]

Hence, if \( \epsilon > 0 \) is given, \((Mh)(x) + |h(x)|\) is an upper-bound of all \((T_u h)(x)\) as \( u \in [0, \epsilon] \). It follows that:

\[
\sup_{u \in [0, \epsilon]} (T_u h)(x) \leq (Mh)(x) + |h(x)|
\]
and we have:

\[(Th)(x) = \inf_{\epsilon'>0} \sup_{u \in [0,\epsilon']} (Tu)h(x)\]

\[\leq \sup_{u \in [0,\epsilon']} (Tu)h(x)\]

\[\leq (Mh)(x) + |h(x)|\]

This being true for all \(x \in \mathbb{R}^n\), \(Th \leq Mh + |h|\).

4. Let \(x \in \mathbb{R}^n\) and \(\epsilon > 0\). Let \(B_\epsilon = B(x, \epsilon)\). Then:

\[(T_\epsilon f)(x) = \frac{1}{dx(\epsilon)} \int_{B_\epsilon} |f(y) - f(x)| dy\]

\[= \frac{1}{dx(\epsilon)} \int_{B_\epsilon} |g(y) - g(x) + h(y) - h(x)| dy\]

\[\leq \frac{1}{dx(\epsilon)} \left( \int_{B_\epsilon} |g(y) - g(x)| dy + \int_{B_\epsilon} |h(y) - h(x)| dy \right)\]

\[= (T_\epsilon g)(x) + (T_\epsilon h)(x)\]
This being true for all \( x \in \mathbb{R}^n \), \( T_\epsilon f \leq T_\epsilon g + T_\epsilon h \).

5. Let \( x \in \mathbb{R}^n \). Let \( \epsilon_1, \epsilon_2 > 0 \) be given and \( \epsilon = \min(\epsilon_1, \epsilon_2) \). For all \( u \in ]0, \epsilon] \), using 4. we have:

\[
(T_u f)(x) \leq (T_u g)(x) + (T_u h)(x) \\
\leq \sup_{u \in ]0, \epsilon_1]} (T_u g)(x) + \sup_{u \in ]0, \epsilon_2]} (T_u h)(x)
\]

Hence, the right-hand-side of this inequality is an upper-bound of all \((T_u f)(x)\)'s as \( u \in ]0, \epsilon] \). It follows that:

\[
(T f)(x) = \inf_{\epsilon' > 0} \sup_{u \in ]0, \epsilon']} (T_u f)(x) \\
\leq \sup_{u \in ]0, \epsilon]} (T_u f)(x) \\
\leq \sup_{u \in ]0, \epsilon_1]} (T_u g)(x) + \sup_{u \in ]0, \epsilon_2]} (T_u h)(x)
\]

Suppose \( \sup_{u \in ]0, \epsilon_1]} (T_u g)(x) < +\infty \). Then this quantity can be safely subtracted from both sides of the previous inequality, to
obtain:

$$(Tf)(x) - \sup_{u \in [0, \epsilon_1]} (T_u g)(x) \leq \sup_{u \in [0, \epsilon_2]} (T_u h)(x)$$

Hence, $\epsilon_1 > 0$ being given, we see that the left-hand-side of this inequality is a lower-bound of all $\sup_{u \in [0, \epsilon_2]} (T_u h)(x)$'s, as $\epsilon_2 > 0$. Since $(Th)(x)$ is the greatest of such lower-bounds, we obtain:

$$(Tf)(x) - \sup_{u \in [0, \epsilon_1]} (T_u g)(x) \leq (Th)(x)$$

or equivalently:

$$(Tf)(x) \leq \sup_{u \in [0, \epsilon_1]} (T_u g)(x) + (Th)(x)$$

which is still valid when $\sup_{u \in [0, \epsilon_1]} (T_u g)(x) = +\infty$. Suppose now that $(Th)(x) < +\infty$. Then $(Th)(x)$ can be safely subtracted from both sides of the previous inequality, to obtain:

$$(Tf)(x) - (Th)(x) \leq \sup_{u \in [0, \epsilon_1]} (T_u g)(x)$$
This being established for all $\epsilon_1 > 0$, $(Tf)(x) - (Th)(x)$ is a lower-bound of all $\sup_{u \in [0, \epsilon_1]} (Tu,g)(x)$’s, as $\epsilon_1 > 0$. Since $(Tg)(x)$ is the greatest of such lower-bounds, we obtain:

$$(Tf)(x) - (Th)(x) \leq (Tg)(x)$$

or equivalently:

$$(Tf)(x) \leq (Tg)(x) + (Th)(x)$$

This being true for all $x \in \mathbb{R}^n$, $Tf \leq Tg + Th$.

6. Let $x \in \mathbb{R}^n$. Since $g \in C_c^\infty(\mathbb{R}^n)$, $g$ is a continuous element of $L^1$. From exercise (17), $x$ is therefore a Lebesgue point of $g$. Hence, from definition (121):

$$\lim_{\epsilon \downarrow 0} (T_\epsilon g)(x) = \lim_{\epsilon \downarrow 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |g(y) - g(x)| dy = 0$$

Let $\delta > 0$. There exists $\epsilon > 0$ such that:

$$u \in [0, \epsilon[ \Rightarrow (T_u g)(x) \leq \delta$$
So $\delta$ is an upper-bound of all $(T_u g)(x)$’s as $u \in ]0, \epsilon[$, and consequently $\sup_{u \in ]0, \epsilon[}(T_u g)(x) \leq \delta$. Hence:

$$
(Tg)(x) = \inf_{\epsilon' > 0} \sup_{u \in ]0, \epsilon'[} (T_u g)(x) \\
\leq \sup_{u \in ]0, \epsilon[}(T_u g)(x) \\
\leq \delta
$$

This being true for all $\delta > 0$, we conclude that $(Tg)(x) = 0$. This being true for all $x \in \mathbb{R}^n$, we have proved that $Tg = 0$.

7. Using 3. and 5. together with $Tg = 0$, we obtain:

$$
Tf \leq Tg + Th = Th \leq Mh + |h|
$$

8. Let $\alpha > 0$. Let $x \in \mathbb{R}^n$ and suppose that $(Mh)(x) \leq \alpha$ together with $|h|(x) \leq \alpha$. Using 7. we obtain:

$$
(Tf)(x) \leq (Mh)(x) + |h|(x) \leq 2\alpha
$$
Hence, we have shown the inclusion:
\[ \{ Mh \leq \alpha \} \cap \{ |h| \leq \alpha \} \subseteq \{ Tf \leq 2\alpha \} \]
from which we conclude that:
\[ \{ 2\alpha < Tf \} \subseteq \{ \alpha < Mh \} \cup \{ \alpha < |h| \} \]

9. We have:
\[
\begin{align*}
dx(\{ \alpha < |h| \}) &= \alpha^{-1} \int \alpha 1_{(\alpha < |h|)} dx \\
& \leq \alpha^{-1} \int |h| 1_{(\alpha < |h|)} dx \\
& \leq \alpha^{-1} \int |h| dx \\
& = \alpha^{-1} \| h \|_1
\end{align*}
\]

10. Let \( \alpha > 0 \) and \( \eta > 0 \). From 1. we have the existence of \( g \in C_0^\infty(\mathbb{R}^n) \) such that \( \| h \|_1 \leq \eta \) where \( h = f - g \). Define \( M_{\alpha,\eta} = \)

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\{\alpha < Mh\} \cup \{\alpha < |h|\}. From exercise (13) applied to the complex measure \( \mu = \int hdx \), \( Mh \) is a Borel measurable map. Since \(|h|\) is also Borel measurable, we see that \( M_{\alpha,\eta} \in B(\mathbb{R}^n) \). Furthermore from 8. we have \( \{2\alpha < Tf\} \subseteq M_{\alpha,\eta} \). Finally, using 9. and exercise (16), we obtain:

\[
\begin{align*}
    dx(M_{\alpha,\eta}) &= dx(\{\alpha < Mh\} \cup \{\alpha < |h|\}) \\
    &\leq dx(\{\alpha < Mh\}) + dx(\{\alpha < |h|\}) \\
    &\leq 3^n \alpha^{-1} ||h||_1 + \alpha^{-1} ||h||_1 \\
    &= (3^n + 1)\alpha^{-1} ||h||_1 \\
    &\leq (3^n + 1)\alpha^{-1} \eta
\end{align*}
\]

Hence, given \( \alpha > 0 \) and \( \eta > 0 \), we have found \( M_{\alpha,\eta} \in B(\mathbb{R}^n) \) such that \( \{2\alpha < Tf\} \subseteq M_{\alpha,\eta} \) and \( dx(M_{\alpha,\eta}) \leq (3^n + 1)\alpha^{-1} \eta \). Take \( N_{\alpha,\eta} = M_{\alpha,\eta} \) where \( \eta'' = (3^n + 1)^{-1} \alpha \eta \). Then \( N_{\alpha,\eta} \in B(\mathbb{R}^n) \), \( \{2\alpha < Tf\} \subseteq N_{\alpha,\eta} \) and \( dx(N_{\alpha,\eta}) \leq \eta \), which is exactly what we want.
11. Let $\alpha > 0$. With an obvious change of notation, given $n \geq 1$, from 10. there exists $N_{\alpha,n} \in \mathcal{B}(\mathbb{R}^n)$ such that we have $\{2\alpha < Tf\} \subseteq N_{\alpha,n}$ and $dx(N_{\alpha,n}) \leq 1/n$. Let $N_\alpha = \cap_{n \geq 1} N_{\alpha,n}$. Then $N_\alpha \in \mathcal{B}(\mathbb{R}^n)$, $\{2\alpha < Tf\} \subseteq N_\alpha$ and furthermore for all $n \geq 1$:

$$dx(N_\alpha) = dx(\cap_{n \geq 1} N_{\alpha,n}) \leq dx(N_{\alpha,n}) \leq \frac{1}{n}$$

So $dx(N_\alpha) = 0$.

12. Let $n \geq 1$. With an obvious change of notation, from 11. there exists $N_n \in \mathcal{B}(\mathbb{R}^n)$ such that $\{2/n < Tf\} \subseteq N_n$ together with $dx(N_n) = 0$. Define $N = \cup_{n \geq 1} N_n$. Then $N \in \mathcal{B}(\mathbb{R}^n)$ and $dx(N) = 0$. Furthermore:

$$\{Tf > 0\} = \bigcup_{n \geq 1} \{2/n < Tf\} \subseteq \bigcup_{n \geq 1} N_n = N$$

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13. From 12, there exists $N \in \mathcal{B}(\mathbb{R}^n)$ with $dx(N) = 0$ such that

$\{Tf > 0\} \subseteq N$. Hence, for all $x \in \mathbb{R}^n$, we have $x \in N^c \Rightarrow

(Tf)(x) = 0$. We conclude that $Tf = 0 \ dx$-a.s.

14. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)$. Let $x \in \mathbb{R}^n$ and suppose that $(Tf)(x) = 0$. Let $\delta > 0$. Then $(Tf)(x) < \delta$. Since $(Tf)(x)$ is the greatest lower-bound of all $\sup_{u \in [0, \epsilon']} (T_u f)(x)$’s as $\epsilon' > 0$, $\delta$ cannot be such a lower-bound. There exists $\epsilon' > 0$ such that $\sup_{u \in [0, \epsilon']} (T_u f)(x) < \delta$. Hence for all $\epsilon \in [0, \epsilon']$, we have:

$$
\frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| \, dy = (T_{\epsilon} f)(x)
\leq \sup_{u \in [0, \epsilon']} (T_u f)(x) < \delta
$$

We have proved that:

$$
\lim_{\epsilon \downarrow 0} \frac{1}{dx(B(x, \epsilon))} \int_{B(x, \epsilon)} |f(y) - f(x)| \, dy = 0
$$
i.e. that $x$ is a Lebesgue point of $f$. So every $x \in \mathbb{R}^n$ such that $(Tf)(x) = 0$ is a Lebesgue point of $f$. Since $Tf = 0$ $dx$-almost surely, we conclude that $dx$-almost all $x \in \mathbb{R}^n$ are Lebesgue points of $f$. This completes the proof of theorem (101).

Exercise 18
Exercise 19.

1. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and \(\Omega' \in \mathcal{F}\). Let \(\mathcal{F}' = \mathcal{F}|_{\Omega'}\) and \(\mu' = \mu|_{\mathcal{F}'}\). Let \(A \in \mathcal{F}'\). Since \(\mathcal{F}'\) is the trace of \(\mathcal{F}\) on \(\Omega'\), from definition (22) there exists \(A \in \mathcal{F}\) such that \(A' = A \cap \Omega'\). Since \(\Omega' \in \mathcal{F}\), we see that \(A' \in \mathcal{F}\). This shows that \(\mathcal{F}' \subseteq \mathcal{F}\) and the restriction \(\mu' = \mu|_{\mathcal{F}'}\) is a well-defined measure on \((\Omega', \mathcal{F}')\).

2. For all maps \(f\) defined on \(\Omega'\) with values in \(\mathbb{C}\) or \([0, +\infty]\), we define an extension of \(f\) on \(\Omega\), denoted \(\tilde{f}\), by setting \(\tilde{f}(\omega) = 0\) for all \(\omega \in \Omega \setminus \Omega'\). Let \(A \in \mathcal{F}'\) and \(1'_A\) be the indicator function of \(A\) on \(\Omega'\). \(A\) is also a subset of \(\Omega\), and we denote \(1_A\) its indicator function on \(\Omega\). Let \(\omega \in \Omega\). If \(\omega \in A \subseteq \Omega'\), then:

\[
\tilde{1}'_A(\omega) \overset{\Delta}{=} 1'_A(\omega) = 1 = 1_A(\omega)
\]

If \(\omega \in \Omega' \setminus A\), then:

\[
\tilde{1}'_A(\omega) \overset{\Delta}{=} 1'_A(\omega) = 0 = 1_A(\omega)
\]
if $\omega \in \Omega \setminus \Omega'$, then:

$$\tilde{1}_A'(\omega) \triangleq 0 = 1_A(\omega)$$

In any case we have $\tilde{1}_A(\omega) = 1_A(\omega)$. So $\tilde{1}_A = 1_A$.

3. Let $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ be a non-negative and measurable map. For all $B \in \mathcal{B}([0, +\infty])$ we have:

$$\{\tilde{f} \in B\} = \left(\left\{ f \in B \cap \Omega' \right\} \cup \left\{ f \in B \cap (\Omega \setminus \Omega') \right\}\right)$$

$$= \{ f \in B \} \cup \left\{ \{0 \in B \cap (\Omega \setminus \Omega') \right\}$$

where $\{0 \in B\}$ denotes $\Omega$ if $0 \in B$ and $\emptyset$ if $0 \notin B$. Since $f$ is measurable, we have $\{f \in B\} \in \mathcal{F}' \subseteq \mathcal{F}$. Since $\Omega' \in \mathcal{F}$, it is clear that $\{0 \in B \} \cap (\Omega \setminus \Omega') \in \mathcal{F}$. It follows that $\{\tilde{f} \in B\} \in \mathcal{F}$, and we have proved that $\tilde{f}$ is a non-negative and measurable map. Suppose $f$ is of the form $1_A'$ for some $A \in \mathcal{F}'$. Then:

$$\int_{\Omega'} 1_A' d\mu' = \mu'(A) = \int_{\Omega} 1_A d\mu = \int_{\Omega} \tilde{1}_A' d\mu$$

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Suppose now that \( f = \sum_{i=1}^{n} \alpha_i 1'_{A_i} \) is a simple function on \((\Omega', \mathcal{F}')\). To make our proof clearer, let us denote \( \tilde{\phi}(g) \) the extension \( \tilde{g} \) of any map \( g \) defined on \( \Omega' \). Then:

\[
\int_{\Omega'} f \, d\mu' = \int_{\Omega'} \left( \sum_{i=1}^{n} \alpha_i 1'_{A_i} \right) \, d\mu' = \sum_{i=1}^{n} \alpha_i \int_{\Omega'} 1'_{A_i} \, d\mu' = \sum_{i=1}^{n} \alpha_i \int_{\Omega} \phi(1'_{A_i}) \, d\mu = \int_{\Omega} \left( \sum_{i=1}^{n} \alpha_i \phi(1'_{A_i}) \right) \, d\mu = \int_{\Omega} \phi \left( \sum_{i=1}^{n} \alpha_i 1'_{A_i} \right) \, d\mu
\]
\[ \int_{\Omega} \phi(f) d\mu = \int_{\Omega} \tilde{f} d\mu \]

Finally, if \( f : (\Omega', \mathcal{F}') \to [0, +\infty] \) is an arbitrary non-negative and measurable map, from theorem (18) there exists a sequence \((s_n)_{n \geq 1}\) of simple functions on \((\Omega', \mathcal{F}')\) such that \( s_n \uparrow f \), i.e. for all \( \omega \in \Omega' \), \( s_n(\omega) \leq s_{n+1}(\omega) \) for all \( n \geq 1 \), and \( s_n(\omega) \to f(\omega) \). It is clear that \( \tilde{s}_n \uparrow \tilde{f} \), and from the monotone convergence theorem (19) we obtain:

\[
\int_{\Omega'} f d\mu' = \lim_{n \to +\infty} \int_{\Omega'} s_n d\mu' \\
= \lim_{n \to +\infty} \int_{\Omega} \tilde{s}_n d\mu \\
= \int_{\Omega} \tilde{f} d\mu
\]

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4. Let \( f \in L^1_\mathbb{C}(\Omega', \mathcal{F}', \mu') \). Let \( u = \text{Re}(f) \) and \( v = \text{Im}(f) \). To make our proof clearer, we shall denote \( \phi(g) \) the extension \( \tilde{g} \) of any map \( g \) defined on \( \Omega' \). From \( f = u^+ - u^- + i(v^+ - v^-) \) we obtain \( \phi(f) = \phi(u^+) - \phi(u^-) + i(\phi(v^+) - \phi(v^-)) \). From 3. each \( \phi(u^\pm) \) and \( \phi(v^\pm) \) is measurable, and consequently \( \phi(f) \) is itself measurable. Note that given \( B \in \mathcal{B}(\mathbb{C}) \), it is not difficult to show directly that \( \{ \tilde{f} \in B \} \in \mathcal{F} \) just like we did in 3. with \( B \in \mathcal{B}([0, +\infty]) \). It is clear that \( |\phi(f)| = \phi(|f|) \), and applying 3. to the non-negative and measurable map \( |f| \) we obtain:

\[
\int_{\Omega} |\phi(f)| \, d\mu = \int_{\Omega} \phi(|f|) \, d\mu = \int_{\Omega'} |f| \, d\mu' < +\infty
\]

Hence, we have proved that \( \tilde{f} = \phi(f) \in L^1_\mathbb{C}(\Omega, \mathcal{F}, \mu) \). Finally, using 3. once more together with the linearity of the integral:

\[
\int_{\Omega'} f \, d\mu' = \int_{\Omega'} u^+ \, d\mu' - \int_{\Omega'} u^- \, d\mu'
\]
\begin{align*}
&+ i \left( \int_{\Omega'} v^+ d\mu' - \int_{\Omega'} v^- d\mu' \right) \\
&= \int_{\Omega} \phi(u^+) d\mu - \int_{\Omega} \phi(u^-) d\mu \\
&+ i \left( \int_{\Omega} \phi(v^+) d\mu - \int_{\Omega} \phi(v^-) d\mu \right) \\
&= \int_{\Omega} [\phi(u^+) - \phi(u^-) + i(\phi(v^+) - \phi(v^-))] d\mu \\
&= \int_{\Omega} \phi(f) d\mu = \int_{\Omega} \tilde{f} d\mu
\end{align*}

Exercise 19
Exercise 20.

1. Let $b : \mathbb{R}^+ \to \mathbb{C}$ be a map. Suppose $b$ is absolutely continuous. From definition (122), $b$ is right-continuous of finite variation, and furthermore it is absolutely continuous with respect to the right-continuous and non-decreasing map $a : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(0) \geq 0$, defined by $a(t) = t$. From theorem (89), there exists $f \in L^1_{\text{loc}}(t)$ such that $b(t) = \int_0^t f(s)ds$ for all $t \in \mathbb{R}^+$. Conversely, suppose such an $f$ exists. From theorem (88), $b = f.a$ is a right-continuous map of finite variation, and from theorem (89), it is in fact absolutely continuous with respect to $a(t) = t$. So $b$ is absolutely continuous. We have proved that $b$ is absolutely continuous, if and only if there exists $f \in L^1_{\text{loc}}(t)$ such that $b(t) = \int_0^t f(s)ds$ for all $t \in \mathbb{R}^+$.

2. Suppose $b$ is absolutely continuous and let $f \in L^1_{\text{loc}}(t)$ be such that $b(t) = \int_0^t f(s)ds$ for all $t \in \mathbb{R}^+$. From theorem (88), we have $\Delta b = f \Delta t = 0$. Since $b$ is right-continuous of finite varia-
tion, in particular it is cadlag. We conclude from exercise (29) (part 1) of Tutorial 14 that \( b \) is in fact continuous with \( b(0) = 0 \).

Exercise 20
Exercise 21.

1. Let \( b : \mathbb{R}^+ \to \mathbb{C} \) be absolutely continuous. Let \( f \in L^1_{\text{loc}}(t) \) be such that \( b(t) = \int_0^t f(s)ds \) for all \( t \in \mathbb{R}^+ \). For all \( n \geq 1 \), we define \( f_n : \mathbb{R} \to \mathbb{C} \) by:

\[
\begin{align*}
    f_n(t) &\triangleq \\
    &\begin{cases}
        f(t)1_{[0,n]}(t) &\text{if } t \in \mathbb{R}^+ \\
        0 &\text{if } t < 0
    \end{cases}
\end{align*}
\]

Applying exercise (19) to \((\Omega, \Omega') = (\mathbb{R}, \mathbb{R}^+)\), bearing in mind that \( \mathcal{B}(\mathbb{R}^+) = \mathcal{B}(\mathbb{R})|_{\mathbb{R}^+} \), we have \( f_n = \phi(f1_{[0,n]}) \) where \( \phi(g) \) denotes the extension \( \tilde{g} \) on \( \mathbb{R} \), of any map \( g \) defined on \( \mathbb{R}^+ \).

Since \( f \in L^1_{\text{loc}}(t) \), we have \( f1_{[0,n]} \in L^1_{\text{loc}}(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dx) \) and consequently \( f_n = \phi(f1_{[0,n]}) \in L^1_{\text{loc}}(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx) \). Note that we are using the same notation \( dx \) to denote successively the Lebesgue measure on \( \mathbb{R}^+ \) and the Lebesgue measure on \( \mathbb{R} \), the former being the restriction of the latter to \( \mathcal{B}(\mathbb{R}^+) \subseteq \mathcal{B}(\mathbb{R}) \).
n \geq 1 \text{ and } t \in [0, n]. \text{ Using exercise (19) once more:}

\begin{align*}
\int_0^t f_n dx &= \int_R f_n 1_{[0,t]} dx \\
&= \int_R \phi(f 1_{[0,n]} 1_{[0,t]} ) dx \\
&= \int_{R^+} f 1_{[0,n]} 1_{[0,t]} dx \\
&= \int_{R^+} f 1_{[0,t]} dx \\
&= \int_0^t f(s) ds = b(t)
\end{align*}

Note that we use the same notations $1_{[0,t]}$ and $1_{[0,n]}$ to denote characteristic functions defined successively on $\mathbb{R}$ and $\mathbb{R}^+$.

2. Since $f_n \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$, from theorem (101), $dx$-almost every $t \in \mathbb{R}$ is a Lebesgue point of $f_n$. Hence, there exists
$N_n \in \mathcal{B}(\mathbb{R})$ with $dx(N_n) = 0$ such that for all $t \in N_n^c$, $t$ is a Lebesgue point of $f_n$.

3. Let $t \in \mathbb{R}$ and $\epsilon > 0$. Since $B(t, \epsilon) = [t - \epsilon, t + \epsilon]$, we have:

$$\frac{1}{\epsilon} \int_{t}^{t+\epsilon} |f_n(s) - f_n(t)|ds = \frac{2}{dx(B(t, \epsilon))} \int_{t}^{t+\epsilon} |f_n(s) - f_n(t)|ds$$

$$\leq \frac{2}{dx(B(t, \epsilon))} \int_{t-\epsilon}^{t+\epsilon} |f_n(s) - f_n(t)|ds$$

$$= \frac{2}{dx(B(t, \epsilon)) \int_{B(t, \epsilon)}} |f_n(s) - f_n(t)|ds$$

4. Let $t \in N_n^c$. Then $t$ is a Lebesgue point of $f_n$. From the inequality obtained in 3, we have:

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{t}^{t+\epsilon} |f_n(s) - f_n(t)|ds = 0$$
Furthermore, since:

\[ \left| \frac{1}{\epsilon} \int_{t}^{t+\epsilon} f_n(s) ds - f_n(t) \right| = \frac{1}{\epsilon} \left| \int_{t}^{t+\epsilon} (f_n(s) - f_n(t)) ds \right| \leq \frac{1}{\epsilon} \int_{t}^{t+\epsilon} |f_n(s) - f_n(t)| ds \]

We conclude that:

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t}^{t+\epsilon} f_n(s) ds = f_n(t) \]

5. Similarly to 3. and 4. we have:

\[ \left| \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s) ds - f_n(t) \right| = \frac{1}{\epsilon} \left| \int_{t-\epsilon}^{t} (f_n(s) - f_n(t)) ds \right| \leq \frac{1}{\epsilon} \int_{t-\epsilon}^{t} |f_n(s) - f_n(t)| ds \]

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\[
\leq \frac{2}{dx(B(t, \epsilon))} \int_{B(t, \epsilon)} |f_n(s) - f_n(t)|ds
\]

Hence for all \( t \in N^{c}_n \), \( t \) being a Lebesgue point of \( f_n \):

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s)ds = f_n(t)
\]

6. Let \( t \in N^{c}_n \cap [0, n] \). From 1. we have \( b(t) = \int_{0}^{t} f_n(s)ds \). Furthermore, for \( \epsilon > 0 \) small enough we have \( t + \epsilon \in [0, n] \), and consequently \( b(t + \epsilon) = \int_{0}^{t+\epsilon} f_n(s)ds \). Hence:

\[
\lim_{\epsilon \downarrow 0} \frac{b(t + \epsilon) - b(t)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{t}^{t+\epsilon} f_n(s)ds = f_n(t)
\]

Moreover, assuming \( t > 0 \), \( t - \epsilon \in [0, n] \) for \( \epsilon > 0 \) small enough, and consequently \( b(t - \epsilon) = \int_{0}^{t-\epsilon} f_n(s)ds \). Hence:

\[
\lim_{\epsilon \downarrow 0} \frac{b(t) - b(t - \epsilon)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s)ds = f_n(t)
\]
We conclude that for all \( t \in N^c \cap [0, n] \), if \( t = 0 \), the right-hand-side derivative \( b'(0) \) exists and is equal to \( f_n(0) \). If \( t > 0 \), the derivative \( b'(t) \) exists and is equal to \( f_n(t) \). However if \( t \in [0, n] \), \( f_n(t) = f(t) \). So for all \( t \in N^c_n \cap [0, n] \), \( b'(t) = f(t) \).

7. Define \( N = (\cup_{n \geq 1} N_n) \cap \mathbb{R}^+ \). Then \( N \in \mathcal{B}(\mathbb{R}^+) \) and \( dx(N) = 0 \). Let \( t \in N^c \). Choosing \( n \geq 1 \) such that \( t \in [0, n] \), from \( t \notin N \) we obtain \( t \notin N_n \) and consequently \( t \in N^c_n \cap [0, n] \). From 6, it follows that \( b'(t) \) exists and is equal to \( f(t) \). We have found \( N \in \mathcal{B}(\mathbb{R}^+) \) with \( dx(N) = 0 \), such that for all \( t \in N^c \), \( b'(t) \) exists and is equal to \( f(t) \).

8. We have shown in exercise (20) that a map \( b \) is absolutely continuous, if and only if there exists \( f \in L^1_{\text{loc}}(t) \) such that \( b = f.t \). Furthermore, it follows from 7, that if \( b \) is absolutely continuous, it is almost surely differentiable with \( b' = f \) \( dx \)-almost surely. This completes the proof of theorem (102).

Exercise 21

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