10. Bounded Linear Functionals in $L^2$

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

**Definition 78** We call subsequence of a sequence $(x_n)_{n \geq 1}$, any sequence of the form $(x_{\phi(n)})_{n \geq 1}$ where $\phi : \mathbb{N}^* \to \mathbb{N}^*$ is a strictly increasing map.

**Exercise 1.** Let $(E, d)$ be a metric space, with metric topology $\mathcal{T}$. Let $(x_n)_{n \geq 1}$ be a sequence in $E$. For all $n \geq 1$, let $F_n$ be the closure of the set $\{x_k : k \geq n\}$.

1. Show that for all $x \in E$, $x_n \overset{\mathcal{T}}{\to} x$ is equivalent to:
   \[ \forall \epsilon > 0 \, \exists n_0 \geq 1 \, \forall n \geq n_0 \Rightarrow d(x_n, x) \leq \epsilon \]

2. Show that $(F_n)_{n \geq 1}$ is a decreasing sequence of closed sets in $E$.

3. Show that if $F_n \downarrow \emptyset$, then $(F_n^c)_{n \geq 1}$ is an open covering of $E$.
4. Show that if \((E, T)\) is compact then \(\bigcap_{n=1}^{+\infty} F_n \neq \emptyset\).

5. Show that if \((E, T)\) is compact, there exists \(x \in E\) such that for all \(n \geq 1\) and \(\epsilon > 0\), we have \(B(x, \epsilon) \cap \{x_k \mid k \geq n\} \neq \emptyset\).

6. By induction, construct a subsequence \((x_{n_p})_{p \geq 1}\) of \((x_n)_{n \geq 1}\) such that \(x_{n_p} \in B(x, 1/p)\) for all \(p \geq 1\).

7. Conclude that if \((E, T)\) is compact, any sequence \((x_n)_{n \geq 1}\) in \(E\) has a convergent subsequence.

**Exercise 2.** Let \((E, d)\) be a metric space, with metric topology \(T\). We assume that any sequence \((x_n)_{n \geq 1}\) in \(E\) has a convergent subsequence. Let \((V_i)_{i \in I}\) be an open covering of \(E\). For \(x \in E\), let:

\[
r(x) \overset{\Delta}{=} \sup\{r > 0 : B(x, r) \subseteq V_i, \text{ for some } i \in I\}
\]

1. Show that \(\forall x \in E, \exists i \in I, \exists r > 0, \text{ such that } B(x, r) \subseteq V_i.\)
2. Show that \( \forall x \in E, r(x) > 0 \).

**Exercise 3.** Further to ex. (2), suppose \( \inf_{x \in E} r(x) = 0 \).

1. Show that for all \( n \geq 1 \), there is \( x_n \in E \) such that \( r(x_n) < 1/n \).

2. Extract a subsequence \( (x_{n_k})_{k \geq 1} \) of \( (x_n)_{n \geq 1} \) converging to some \( x^* \in E \). Let \( r^* > 0 \) and \( i \in I \) be such that \( B(x^*, r^*) \subseteq V_i \). Show that we can find some \( k_0 \geq 1 \), such that \( d(x^*, x_{n_{k_0}}) < r^*/2 \) and \( r(x_{n_{k_0}}) \leq r^*/4 \).

3. Show that \( d(x^*, x_{n_{k_0}}) < r^*/2 \) implies that \( B(x_{n_{k_0}}, r^*/2) \subseteq V_i \). Show that this contradicts \( r(x_{n_{k_0}}) \leq r^*/4 \), and conclude that \( \inf_{x \in E} r(x) > 0 \).

**Exercise 4.** Further to ex. (3), let \( r_0 \) with \( 0 < r_0 < \inf_{x \in E} r(x) \). Suppose that \( E \) cannot be covered by a finite number of open balls with radius \( r_0 \).

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1. Show the existence of a sequence \((x_n)_{n \geq 1}\) in \(E\), such that for all \(n \geq 1\), \(x_{n+1} \notin B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)\).

2. Show that for all \(n > m\) we have \(d(x_n, x_m) \geq r_0\).

3. Show that \((x_n)_{n \geq 1}\) cannot have a convergent subsequence.

4. Conclude that there exists a finite subset \(\{x_1, \ldots, x_n\}\) of \(E\) such that \(E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)\).

5. Show that for all \(x \in E\), we have \(B(x, r_0) \subseteq V_i\) for some \(i \in I\).

6. Conclude that \((E, T)\) is compact.

7. Prove the following:

**Theorem 47** A metrizable topological space \((E, T)\) is compact, if and only if for every sequence \((x_n)_{n \geq 1}\) in \(E\), there exists a subsequence \((x_{n_k})_{k \geq 1}\) of \((x_n)_{n \geq 1}\) and some \(x \in E\), such that \(x_{n_k} \xrightarrow{T} x\).
**Exercise 5.** Let $a, b \in \mathbb{R}$, $a < b$ and $(x_n)_{n \geq 1}$ be a sequence in $[a, b]$.

1. Show that $(x_n)_{n \geq 1}$ has a convergent subsequence.

2. Can we conclude that $[a, b]$ is a compact subset of $\mathbb{R}$?

**Exercise 6.** Let $E = [-M, M] \times \ldots \times [-M, M] \subseteq \mathbb{R}^n$, where $n \geq 1$ and $M \in \mathbb{R}^+$. Let $\mathcal{T}_{\mathbb{R}^n}$ be the usual product topology on $\mathbb{R}^n$, and $\mathcal{T}_E = (\mathcal{T}_{\mathbb{R}^n})_E$ be the induced topology on $E$.

1. Let $(x_p)_{p \geq 1}$ be a sequence in $E$. Let $x \in E$. Show that $x_p \xrightarrow{\mathcal{T}_E} x$ is equivalent to $x_p \xrightarrow{\mathcal{T}_{\mathbb{R}^n}} x$.

2. Propose a metric on $\mathbb{R}^n$, inducing the topology $\mathcal{T}_{\mathbb{R}^n}$.

3. Let $(x_p)_{p \geq 1}$ be a sequence in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Show that $x_p \xrightarrow{\mathcal{T}_{\mathbb{R}^n}} x$ if and only if, $x^i_p \xrightarrow{\mathcal{T}_{\mathbb{R}}} x^i$ for all $i \in \mathbb{N}_n$. 

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**Exercise 7.** Further to ex. (6), suppose \((x_p)_{p \geq 1}\) is a sequence in \(E\).

1. Show the existence of a subsequence \((x_{\phi(p)})_{p \geq 1}\) of \((x_p)_{p \geq 1}\), such that \(x_{\phi(p)}^1 \overset{T_{[-M,M]}}{\longrightarrow} x^1\) for some \(x^1 \in [-M,M]\).

2. Explain why the above convergence is equivalent to \(x_{\phi(p)}^1 \overset{TR}{\longrightarrow} x^1\).

3. Suppose that \(1 \leq k \leq n - 1\) and \((y_p)_{p \geq 1} = (x_{\phi(p)})_{p \geq 1}\) is a subsequence of \((x_p)_{p \geq 1}\) such that:
   \[
   \forall j = 1, \ldots, k \ , \ x_j^j \overset{TR}{\longrightarrow} x_j^1 \text{ for some } x_j^1 \in [-M,M]
   \]
   Show the existence of a subsequence \((y_{\psi(p)})_{p \geq 1}\) of \((y_p)_{p \geq 1}\) such that \(y_{\psi(p)}^{k+1} \overset{TR}{\longrightarrow} x_k^{k+1}\) for some \(x_{k+1} \in [-M,M]\).

4. Show that \(\phi \circ \psi : \mathbb{N}^* \rightarrow \mathbb{N}^*\) is strictly increasing.

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5. Show that \((x_{\phi \circ \psi(p)})_{p \geq 1}\) is a subsequence of \((x_p)_{p \geq 1}\) such that:
   \[
   \forall j = 1, \ldots, k + 1, \ x^j_{\phi \circ \psi(p)} \xrightarrow{\text{Tr}} x^j \in [-M, M]
   \]

6. Show the existence of a subsequence \((x_{\phi(p)})_{p \geq 1}\) of \((x_p)_{p \geq 1}\), and \(x \in E\), such that \(x_{\phi(p)} \xrightarrow{\mathcal{T}_E} x\)

7. Show that \((E, \mathcal{T}_E)\) is a compact topological space.

**Exercise 8.** Let \(A\) be a closed subset of \(\mathbb{R}^n\), \(n \geq 1\), which is bounded with respect to the usual metric of \(\mathbb{R}^n\).

1. Show that \(A \subseteq E = [-M, M] \times \ldots \times [-M, M]\), for some \(M \in \mathbb{R}^+\).
2. Show from \(E \setminus A = E \cap A^c\) that \(A\) is closed in \(E\).
3. Show \((A, (\mathcal{T}_{\mathbb{R}^n})|_A)\) is a compact topological space.

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4. Conversely, let \( A \) is a compact subset of \( \mathbb{R}^n \). Show that \( A \) is closed and bounded.

**Theorem 48**   A subset of \( \mathbb{R}^n \) is compact if and only if it is closed and bounded with respect to its usual metric.

**Exercise 9.** Let \( n \geq 1 \). Consider the map:

\[
\phi : \begin{cases}
\mathbb{C}^n & \to \mathbb{R}^{2n} \\
(a_1 + ib_1, \ldots, a_n + ib_n) & \to (a_1, b_1, \ldots, a_n, b_n)
\end{cases}
\]

1. Recall the expressions of the usual metrics \( d_{\mathbb{C}^n} \) and \( d_{\mathbb{R}^{2n}} \) of \( \mathbb{C}^n \) and \( \mathbb{R}^{2n} \) respectively.

2. Show that for all \( z, z' \in \mathbb{C}^n \), \( d_{\mathbb{C}^n}(z, z') = d_{\mathbb{R}^{2n}}(\phi(z), \phi(z')) \).

3. Show that \( \phi \) is a homeomorphism from \( \mathbb{C}^n \) to \( \mathbb{R}^{2n} \).
4. Show that a subset $K$ of $C^n$ is compact, if and only if $\phi(K)$ is a compact subset of $\mathbb{R}^{2n}$.

5. Show that $K$ is closed, if and only if $\phi(K)$ is closed.

6. Show that $K$ is bounded, if and only if $\phi(K)$ is bounded.

7. Show that a subset $K$ of $C^n$ is compact, if and only if it is closed and bounded with respect to its usual metric.

**Definition 79** Let $(E, d)$ be a metric space. A sequence $(x_n)_{n \geq 1}$ in $E$ is said to be a Cauchy sequence with respect to the metric $d$, if and only if for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:

$$n, m \geq n_0 \Rightarrow d(x_n, x_m) \leq \epsilon$$

**Definition 80** We say that a metric space $(E, d)$ is complete, if and only if for any Cauchy sequence $(x_n)_{n \geq 1}$ in $E$, there exists $x \in E$ such that $(x_n)_{n \geq 1}$ converges to $x$. 

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Exercise 10.

1. Explain why strictly speaking, given $p \in [1, +\infty]$, definition (77) of Cauchy sequences in $L^p_\mathbb{C}(\Omega, \mathcal{F}, \mu)$ is not a covered by definition (79).

2. Explain why $L^p_\mathbb{C}(\Omega, \mathcal{F}, \mu)$ is not a complete metric space, despite theorem (46) and definition (80).

Exercise 11. Let $\{z_k\}_{k \geq 1}$ be a Cauchy sequence in $\mathbb{C}^n$, $n \geq 1$, with respect to the usual metric $d(z, z') = \|z - z'\|$, where:

$$\|z\| = \sqrt[n]{\sum_{i=1}^{n} |z_i|^2}$$

1. Show that the sequence $\{z_k\}_{k \geq 1}$ is bounded, i.e. that there exists $M \in \mathbb{R}^+$ such that $\|z_k\| \leq M$, for all $k \geq 1$. 
Tutorial 10: Bounded Linear Functionals in $L^2$

2. Define $B = \{ z \in \mathbb{C}^n , \| z \| \leq M \}$. Show that $\delta(B) < +\infty$, and that $B$ is closed in $\mathbb{C}^n$.

3. Show the existence of a subsequence $(z_{kp})_{p \geq 1}$ of $(z_k)_{k \geq 1}$ such that $z_{kp} \overset{\mathbb{C}^n}{\rightarrow} z$ for some $z \in B$.

4. Show that for all $\epsilon > 0$, there exists $p_0 \geq 1$ and $n_0 \geq 1$ such that $d(z, z_{kp_0}) \leq \epsilon/2$ and:
   
   $$ k \geq n_0 \implies d(z_k, z_{kp_0}) \leq \epsilon/2 $$

5. Show that $z_k \overset{\mathbb{C}^n}{\rightarrow} z$.

6. Conclude that $\mathbb{C}^n$ is complete with respect to its usual metric.

7. For which theorem of Tutorial 9 was the completeness of $\mathbb{C}$ used?

**Exercise 12.** Let $(x_k)_{k \geq 1}$ be a sequence in $\mathbb{R}^n$ such that $x_k \overset{\mathbb{C}^n}{\rightarrow} z$, for some $z \in \mathbb{C}^n$.

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1. Show that $z \in \mathbb{R}^n$.

2. Show that $\mathbb{R}^n$ is complete with respect to its usual metric.

**Theorem 49**  
$C^n$ and $\mathbb{R}^n$ are complete w.r. to their usual metrics.

**Exercise 13.** Let $(E, d)$ be a metric space, with metric topology $T$. Let $F \subseteq E$, and $\overline{F}$ denote the closure of $F$.

1. Explain why, for all $x \in \overline{F}$ and $n \geq 1$, we have $F \cap B(x, 1/n) \neq \emptyset$.

2. Show that for all $x \in \overline{F}$, there exists a sequence $(x_n)_{n \geq 1}$ in $F$, such that $x_n \xrightarrow{T} x$.

3. Show conversely that if there is a sequence $(x_n)_{n \geq 1}$ in $F$ with $x_n \xrightarrow{T} x$, then $x \in \overline{F}$.
4. Show that $F$ is closed if and only if for all sequence $(x_n)_{n \geq 1}$ in $F$ such that $x_n \xrightarrow{T} x$ for some $x \in E$, we have $x \in F$.

5. Explain why $(F, T_{|F})$ is metrizable.

6. Show that if $F$ is complete with respect to the metric $d_{|F \times F}$, then $F$ is closed in $E$.

7. Let $d_{\bar{R}}$ be a metric on $\bar{R}$, inducing the usual topology $T_{\bar{R}}$. Show that $d' = (d_{\bar{R}})_{|R \times R}$ is a metric on $R$, inducing the topology $T_R$.

8. Find a metric on $[-1, 1]$ which induces its usual topology.

9. Show that $\{-1, 1\}$ is not open in $[-1, 1]$.

10. Show that $\{-\infty, +\infty\}$ is not open in $\bar{R}$.

11. Show that $R$ is not closed in $\bar{R}$.

12. Let $d_R$ be the usual metric of $\bar{R}$. Show that $d' = (d_R)_{|R \times R}$ and $d_R$ induce the same topology on $R$, but that however, $\bar{R}$
is complete with respect to $d_R$, whereas it cannot be complete with respect to $d'$.

**Definition 81** Let $\mathcal{H}$ be a $K$-vector space, where $K = \mathbb{R}$ or $\mathbb{C}$. We call inner-product on $\mathcal{H}$, any map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to K$ with the following properties:

(i) $\forall x, y \in \mathcal{H}, \langle x, y \rangle = \overline{\langle y, x \rangle}$

(ii) $\forall x, y, z \in \mathcal{H}, \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(iii) $\forall x, y \in \mathcal{H}, \forall \alpha \in K, \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(iv) $\forall x \in \mathcal{H}, \langle x, x \rangle \geq 0$

(v) $\forall x \in \mathcal{H}, \langle x, x \rangle = 0 \iff x = 0$

where for all $z \in \mathbb{C}$, $\overline{z}$ denotes the complex conjugate of $z$. For all $x \in \mathcal{H}$, we call norm of $x$, denoted $\|x\|$, the number defined by:

$$\|x\| \triangleq \sqrt{\langle x, x \rangle}$$

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**Exercise 14.** Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $K$-vector space $\mathcal{H}$.

1. Show that for all $y \in \mathcal{H}$, the map $x \to \langle x, y \rangle$ is linear.

2. Show that for all $x \in \mathcal{H}$, the map $y \to \langle x, y \rangle$ is linear if $K = \mathbb{R}$, and conjugate-linear if $K = \mathbb{C}$.

**Exercise 15.** Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $K$-vector space $\mathcal{H}$.

Let $x, y \in \mathcal{H}$. Let $A = \|x\|^2$, $B = |\langle x, y \rangle|$ and $C = \|y\|^2$. Let $\alpha \in K$ be such that $|\alpha| = 1$ and:

$$B = \alpha \langle x, y \rangle$$

1. Show that $A, B, C \in \mathbb{R}^+$.

2. For all $t \in \mathbb{R}$, show that $\langle x - t\alpha y, x - t\alpha y \rangle = A - 2tB + t^2C$.

3. Show that if $C = 0$ then $B^2 \leq AC$. 

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4. Suppose that $C \neq 0$. Show that $P(t) = A - 2tB + t^2C$ has a minimal value which is in $\mathbb{R}^+$, and conclude that $B^2 \leq AC$.

5. Conclude with the following:

**Theorem 50 (Cauchy-Schwarz’s inequality:second)**  
Let $\mathcal{H}$ be a $K$-vector space, where $K = \mathbb{R}$ or $\mathbb{C}$, and $\langle \cdot , \cdot \rangle$ be an inner-product on $\mathcal{H}$. Then, for all $x, y \in \mathcal{H}$, we have:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

**Exercise 16.** For all $f, g \in L^2_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$, we define:

$$\langle f, g \rangle \triangleq \int_{\Omega} f \overline{g} \, d\mu$$

1. Use the first Cauchy-Schwarz inequality (42) to prove that for all $f, g \in L^2_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$, we have $fg \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$. Conclude that $\langle f, g \rangle$ is a well-defined complex number.
2. Show that for all $f \in L^2_C(\Omega, \mathcal{F}, \mu)$, we have $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

3. Make another use of the first Cauchy-Schwarz inequality to show that for all $f, g \in L^2_C(\Omega, \mathcal{F}, \mu)$, we have:

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2 \tag{1}$$

4. Go through definition (81), and indicate which of the properties $(i) - (v)$ fails to be satisfied by $\langle \cdot, \cdot \rangle$. Conclude that $\langle \cdot, \cdot \rangle$ is not an inner-product on $L^2_C(\Omega, \mathcal{F}, \mu)$, and therefore that inequality (*) is not a particular case of the second Cauchy-Schwarz inequality (50).

5. Let $f, g \in L^2_C(\Omega, \mathcal{F}, \mu)$. By considering $\int (|f| + t|g|)^2 d\mu$ for $t \in \mathbb{R}$, imitate the proof of the second Cauchy-Schwarz inequality to show that:

$$\int_{\Omega} |fg| d\mu \leq \left( \int_{\Omega} |f|^2 d\mu \right)^{\frac{1}{2}} \left( \int_{\Omega} |g|^2 d\mu \right)^{\frac{1}{2}}$$
6. Let \( f, g : (\Omega, \mathcal{F}) \to [0, +\infty] \) non-negative and measurable. Show that if \( \int f^2 \, d\mu \) and \( \int g^2 \, d\mu \) are finite, then \( f \) and \( g \) are \( \mu \)-almost surely equal to elements of \( L^2_{\mathcal{F}}(\Omega, \mathcal{F}, \mu) \). Deduce from 5. a new proof of the first Cauchy-Schwarz inequality:

\[
\int_\Omega fg \, d\mu \leq \left( \int_\Omega f^2 \, d\mu \right)^{\frac{1}{2}} \left( \int_\Omega g^2 \, d\mu \right)^{\frac{1}{2}}
\]

**Exercise 17.** Let \( \langle \cdot, \cdot \rangle \) be an inner-product on a \( K \)-vector space \( \mathcal{H} \).

1. Show that for all \( x, y \in \mathcal{H} \), we have:

\[
\|x + y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \bar{\langle x, y \rangle}
\]

2. Using the second Cauchy-Schwarz inequality (50), show that:

\[
\|x + y\| \leq \|x\| + \|y\|
\]

3. Show that \( d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\| \) defines a metric on \( \mathcal{H} \).
**Definition 82** Let $\mathcal{H}$ be a $K$-vector space, where $K = \mathbb{R}$ or $\mathbb{C}$, and $\langle \cdot, \cdot \rangle$ be an inner-product on $\mathcal{H}$. We call **norm topology** on $\mathcal{H}$, denoted $T_{\langle \cdot, \cdot \rangle}$, the metric topology associated with $d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\|$. 

**Definition 83** We call **Hilbert space** over $K$ where $K = \mathbb{R}$ or $\mathbb{C}$, any ordered pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is an inner-product on a $K$-vector space $\mathcal{H}$, which is complete w.r. to $d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\|$. 

**Exercise 18.** Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$ and let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, (closed with respect to the norm topology $T_{\langle \cdot, \cdot \rangle}$). Define $[\cdot, \cdot] = \langle \cdot, \cdot \rangle|_{\mathcal{M} \times \mathcal{M}}$. 

1. Show that $[\cdot, \cdot]$ is an inner-product on the $K$-vector space $\mathcal{M}$.
2. With obvious notations, show that $d_{[\cdot, \cdot]} = (d_{\langle \cdot, \cdot \rangle})|_{\mathcal{M} \times \mathcal{M}}$.
3. Deduce that $T_{[\cdot, \cdot]} = (T_{\langle \cdot, \cdot \rangle})|_{\mathcal{M}}$. 

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Exercise 19. Further to ex. (18), Let \((x_n)_{n \geq 1}\) be a Cauchy sequence in \(\mathcal{M}\), with respect to the metric \(d_{[\cdot,\cdot]}\).

1. Show that \((x_n)_{n \geq 1}\) is a Cauchy sequence in \(\mathcal{H}\).

2. Explain why there exists \(x \in \mathcal{H}\) such that \(x_n \xrightarrow{\tau_{[\cdot,\cdot]}} x\).

3. Explain why \(x \in \mathcal{M}\).

4. Explain why we also have \(x_n \xrightarrow{\tau_{[\cdot,\cdot]}} x\).

5. Explain why \((\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M} \times \mathcal{M}})\) is a Hilbert space over \(K\).

Exercise 20. For all \(z, z' \in \mathbb{C}^n, n \geq 1\), we define:

\[
\langle z, z' \rangle \triangleq \sum_{i=1}^{n} z_i \overline{z_i'}
\]
1. Show that $\langle \cdot, \cdot \rangle$ is an inner-product on $\mathbb{C}^n$.
2. Show that the metric $d_{\langle \cdot, \cdot \rangle}$ is equal to the usual metric of $\mathbb{C}^n$.
3. Conclude that $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space over $\mathbb{C}$.
4. Show that $\mathbb{R}^n$ is a closed subset of $\mathbb{C}^n$.
5. Show however that $\mathbb{R}^n$ is not a linear subspace of $\mathbb{C}^n$.
6. Show that $(\mathbb{R}^n, \langle \cdot, \cdot \rangle\mid_{\mathbb{R}^n \times \mathbb{R}^n})$ is a Hilbert space over $\mathbb{R}$.

**Definition 84** We call usual inner-product in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the inner-product denoted $\langle \cdot, \cdot \rangle$ and defined by:

$$\forall x, y \in \mathbb{K}^n, \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

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Theorem 51  \( C^n \) and \( \mathbb{R}^n \) together with their usual inner-products, are Hilbert spaces over \( \mathbb{C} \) and \( \mathbb{R} \) respectively.

Definition 85  Let \( \mathcal{H} \) be a \( K \)-vector space, where \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( \mathcal{C} \subseteq \mathcal{H} \). We say that \( \mathcal{C} \) is a convex subset of \( \mathcal{H} \), if and only if, for all \( x, y \in \mathcal{C} \) and \( t \in [0,1] \), we have \( tx + (1-t)y \in \mathcal{C} \).

Exercise 21. Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) be a Hilbert space over \( K \). Let \( \mathcal{C} \subseteq \mathcal{H} \) be a non-empty closed convex subset of \( \mathcal{H} \). Let \( x_0 \in \mathcal{H} \). Define:

\[
\delta_{\min} \triangleq \inf\{ \|x - x_0\| : x \in \mathcal{C} \}
\]

1. Show the existence of a sequence \( (x_n)_{n \geq 1} \) in \( \mathcal{C} \) such that \( \|x_n - x_0\| \to \delta_{\min} \).

2. Show that for all \( x, y \in \mathcal{H} \), we have:

\[
\|x - y\| = 2\|x\|^2 + 2\|y\|^2 - 4 \left\| \frac{x + y}{2} \right\|^2
\]
3. Explain why for all $n, m \geq 1$, we have:
\[
\delta_{\text{min}} \leq \left\| \frac{x_n + x_m}{2} - x_0 \right\|
\]

4. Show that for all $n, m \geq 1$, we have:
\[
\|x_n - x_m\|^2 \leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\delta^2_{\text{min}}
\]

5. Show the existence of some $x^* \in \mathcal{H}$, such that $x_n \xrightarrow{\mathcal{H}} x^*$.

6. Explain why $x^* \in \mathcal{C}$

7. Show that for all $x, y \in \mathcal{H}$, we have $\|x\| - \|y\| \leq \|x - y\|$.

8. Show that $\|x_n - x_0\| \to \|x^* - x_0\|$.

9. Conclude that we have found $x^* \in \mathcal{C}$ such that:
\[
\|x^* - x_0\| = \inf\{\|x - x_0\| : x \in \mathcal{C}\}
\]

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10. Let $y^*$ be another element of $C$ with such property. Show that:
$$
\|x^* - y^*\|^2 \leq 2\|x^* - x_0\|^2 + 2\|y^* - x_0\|^2 - 4\delta_{\min}^2
$$

11. Conclude that $x^* = y^*$.

Theorem 52 Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$, where $K = \mathbb{R}$ or $\mathbb{C}$. Let $C$ be a non-empty, closed and convex subset of $H$. For all $x_0 \in H$, there exists a unique $x^* \in C$ such that:
$$
\|x^* - x_0\| = \inf\{|\|x - x_0\| : x \in C}\}
$$

Definition 86 Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$, where $K = \mathbb{R}$ or $\mathbb{C}$. Let $G \subseteq H$. We call orthogonal of $G$, the subset of $H$ denoted $G^\perp$ and defined by:
$$
G^\perp \triangleq \{ x \in H : \langle x, y \rangle = 0, \forall y \in G \}
$$

Exercise 22. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$ and $G \subseteq H$. 

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1. Show that $G^\perp$ is a linear subspace of $\mathcal{H}$, even if $G$ isn’t.

2. Show that $\phi_y : \mathcal{H} \to K$ defined by $\phi_y(x) = \langle x, y \rangle$ is continuous.

3. Show that $G^\perp = \bigcap_{y \in G} \phi_y^{-1}(\{0\})$.

4. Show that $G^\perp$ is a closed subset of $\mathcal{H}$, even if $G$ isn’t.

5. Show that $\emptyset^\perp = \{0\}^\perp = \mathcal{H}$.

6. Show that $H^\perp = \{0\}$.

**Exercise 23.** Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $K$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, and $x_0 \in \mathcal{H}$.

1. Explain why there exists $x^* \in \mathcal{M}$ such that:

$$\|x^* - x_0\| = \inf \{ \|x - x_0\| : x \in \mathcal{M}\}$$
2. Define \( y^* = x_0 - x^* \in \mathcal{H} \). Show that for all \( y \in \mathcal{M} \) and \( \alpha \in K \):
\[
\|y^*\|^2 \leq \|y^* - \alpha y\|^2
\]

3. Show that for all \( y \in \mathcal{M} \) and \( \alpha \in K \), we have:
\[
0 \leq -\alpha \langle y, y^* \rangle - \frac{\alpha \langle y, y^* \rangle}{\|y\|^2} + |\alpha|^2 \|y\|^2
\]

4. For all \( y \in \mathcal{M} \setminus \{0\} \), taking \( \alpha = \frac{\langle y, y^* \rangle}{\|y\|^2} \), show that:
\[
0 \leq -\frac{\|y, y^*\|^2}{\|y\|^2}
\]

5. Conclude that \( x^* \in \mathcal{M}, y^* \in \mathcal{M}^\perp \) and \( x_0 = x^* + y^* \).

6. Show that \( \mathcal{M} \cap \mathcal{M}^\perp = \{0\} \)

7. Show that \( x^* \in \mathcal{M} \) and \( y^* \in \mathcal{M}^\perp \) with \( x_0 = x^* + y^* \), are unique.
Theorem 53  Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a Hilbert space over \(K\), where \(K = \mathbb{R}\) or \(C\). Let \(M\) be a closed linear subspace of \(\mathcal{H}\). Then, for all \(x_0 \in \mathcal{H}\), there is a unique decomposition:

\[ x_0 = x^* + y^* \]

where \(x^* \in M\) and \(y^* \in M^\perp\).

Definition 87  Let \(\mathcal{H}\) be a \(K\)-vector space, where \(K = \mathbb{R}\) or \(C\). We call linear functional, any map \(\lambda : \mathcal{H} \to K\), such that for all \(x, y \in \mathcal{H}\) and \(\alpha \in K\):

\[ \lambda(x + \alpha y) = \lambda(x) + \alpha \lambda(y) \]

Exercise 24. Let \(\lambda\) be a linear functional on a \(K\)-Hilbert\(^1\) space \(\mathcal{H}\).

1. Suppose that \(\lambda\) is continuous at some point \(x_0 \in \mathcal{H}\). Show the existence of \(\eta > 0\) such that:

\[ \forall x \in \mathcal{H}, \|x - x_0\| \leq \eta \implies |\lambda(x) - \lambda(x_0)| \leq 1 \]

\(^1\)Norm vector spaces are introduced later in these tutorials.
Show that for all $x \in \mathcal{H}$ with $x \neq 0$, we have $|\lambda(\eta x/\|x\|)| \leq 1$.

2. Show that if $\lambda$ is continuous at $x_0$, there exits $M \in \mathbb{R}^+$, with:
   \[ \forall x \in \mathcal{H}, \quad |\lambda(x)| \leq M\|x\| \]  
   (2)

3. Show conversely that if (2) holds, $\lambda$ is continuous everywhere.

**Definition 88** Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert² space over $K = \mathbb{R}$ or $\mathbb{C}$. Let $\lambda$ be a linear functional on $\mathcal{H}$. Then, the following are equivalent:

(i) $\lambda : (\mathcal{H}, \langle \cdot, \cdot \rangle) \rightarrow (K, T_K)$ is continuous
(ii) $\exists M \in \mathbb{R}^+, \forall x \in \mathcal{H}, \quad |\lambda(x)| \leq M\|x\|

In which case, we say that $\lambda$ is a bounded linear functional.

²Norm vector spaces are introduced later in these tutorials.
Exercise 25. Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a Hilbert space over \(\mathbf{K}\). Let \(\lambda\) be a bounded linear functional on \(\mathcal{H}\), such that \(\lambda(x) \neq 0\) for some \(x \in \mathcal{H}\), and define \(\mathcal{M} = \lambda^{-1}(\{0\})\).

1. Show the existence of \(x_0 \in \mathcal{H}\), such that \(x_0 \notin \mathcal{M}\).
2. Show the existence of \(x^* \in \mathcal{M}\) and \(y^* \in \mathcal{M}^\perp\) with \(x_0 = x^* + y^*\).
3. Deduce the existence of some \(z \in \mathcal{M}^\perp\) such that \(\|z\| = 1\).
4. Show that for all \(\alpha \in \mathbf{K} \setminus \{0\}\) and \(x \in \mathcal{H}\), we have:
   \[
   \frac{\lambda(x)}{\alpha} \langle z, \alpha z \rangle = \lambda(x)
   \]
5. Show that in order to have:
   \[
   \forall x \in \mathcal{H} , \lambda(x) = \langle x, \alpha z \rangle
   \]
   it is sufficient to choose \(\alpha \in \mathbf{K} \setminus \{0\}\) such that:
   \[
   \forall x \in \mathcal{H} , \frac{\lambda(x)z}{\alpha} - x \in \mathcal{M}
   \]
6. Show the existence of \( y \in \mathcal{H} \) such that:
\[
\forall x \in \mathcal{H}, \lambda(x) = \langle x, y \rangle
\]

7. Show the uniqueness of such \( y \in \mathcal{H} \).

**Theorem 54** Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) be a Hilbert space over \( K \), where \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( \lambda \) be a bounded linear functional on \( \mathcal{H} \). Then, there exists a unique \( y \in \mathcal{H} \) such that: \( \forall x \in \mathcal{H}, \lambda(x) = \langle x, y \rangle \).

**Definition 89** Let \( K = \mathbb{R} \) or \( \mathbb{C} \). We call a **K-vector space**, any set \( \mathcal{H} \), together with operators \( \oplus \) and \( \otimes \) for which there exists an element \( 0_\mathcal{H} \in \mathcal{H} \) such that for all \( x, y, z \in \mathcal{H} \) and \( \alpha, \beta \in K \), we have:

1. \( 0_\mathcal{H} \oplus x = x \)
2. \( \exists (-x) \in \mathcal{H}, (-x) \oplus x = 0_\mathcal{H} \)
3. \( x \oplus (y \oplus z) = (x \oplus y) \oplus z \)
(iv) \( x \oplus y = y \oplus x \)
(v) \( 1 \otimes x = x \)
(vi) \( \alpha \otimes (\beta \otimes x) = (\alpha \beta) \otimes x \)
(vii) \( (\alpha + \beta) \otimes x = (\alpha \otimes x) \oplus (\beta \otimes x) \)
(viii) \( \alpha \otimes (x \oplus y) = (\alpha \otimes x) \oplus (\alpha \otimes y) \)

**Exercise 26.** For all \( f \in L^2_K(\Omega, \mathcal{F}, \mu) \), define:

\[
\mathcal{H} \triangleq \{ [f] : f \in L^2_K(\Omega, \mathcal{F}, \mu) \}
\]

where \([f] = \{ g \in L^2_K(\Omega, \mathcal{F}, \mu) : g = f, \mu\text{-a.s.} \}\). Let \(0_{\mathcal{H}} = [0] \), and for all \([f], [g] \in \mathcal{H}\), and \(\alpha \in K\), we define:

\[
[f] \oplus [g] \triangleq [f + g] \\
\alpha \otimes [f] \triangleq [\alpha f]
\]

We assume \(f, f', g, g'\) are elements of \(L^2_K(\Omega, \mathcal{F}, \mu)\).
1. Show that for \( f = g \mu \)-a.s. is equivalent to \([f] = [g]\).

2. Show that if \([f] = [f']\) and \([g] = [g']\), then \([f + g] = [f' + g']\).

3. Conclude that \( \oplus \) is well-defined.

4. Show that \( \otimes \) is also well-defined.

5. Show that \((\mathcal{H}, \oplus, \otimes)\) is a \(K\)-vector space.

**Exercise 27.** Further to ex. (26), we define for all \([f], [g] \in \mathcal{H}\):

\[
\langle [f], [g] \rangle_{\mathcal{H}} \triangleq \int_{\Omega} f \bar{g} d\mu
\]

1. Show that \(\langle \cdot, \cdot \rangle_{\mathcal{H}}\) is well-defined.

2. Show that \(\langle \cdot, \cdot \rangle_{\mathcal{H}}\) is an inner-product on \(\mathcal{H}\).

3. Show that \((\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})\) is a Hilbert space over \(K\).
Tutorial 10: Bounded Linear Functionals in $L^2$

4. Why is $(f, g) \triangleq \int_{\Omega} f \bar{g} d\mu$ not an inner-product on $L^2_K(\Omega, \mathcal{F}, \mu)$?

**Exercise 28.** Further to ex. (27), Let $\lambda : L^2_K(\Omega, \mathcal{F}, \mu) \to K$ be a continuous linear functional.\(^3\) Define $\Lambda : \mathcal{H} \to K$ by $\Lambda([f]) = \lambda(f)$.

1. Show the existence of $M \in \mathbb{R}^+$ such that:
   \[
   \forall f \in L^2_K(\Omega, \mathcal{F}, \mu) \ , \ |\lambda(f)| \leq M \|f\|_2
   \]

2. Show that if $[f] = [g]$ then $\lambda(f) = \lambda(g)$.

3. Show that $\Lambda$ is a well defined bounded linear functional on $\mathcal{H}$.

4. Conclude with the following:

\(^3\)As defined in these tutorials, $L^2_K(\Omega, \mathcal{F}, \mu)$ is not a Hilbert space (not even a norm vector space). However, both $L^2_K(\Omega, \mathcal{F}, \mu)$ and $K$ have natural topologies and it is therefore meaningful to speak of continuous linear functional. Note however that we are slightly outside the framework of definition (88).

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Theorem 55  Let $\lambda : L^2_K(\Omega, \mathcal{F}, \mu) \to K$ be a continuous linear functional, where $K = \mathbb{R}$ or $\mathbb{C}$. There exists $g \in L^2_K(\Omega, \mathcal{F}, \mu)$ such that:

$$\forall f \in L^2_K(\Omega, \mathcal{F}, \mu), \lambda(f) = \int_\Omega f \bar{g} d\mu$$
Solutions to Exercises

Exercise 1.

1. Let \((x_n)_{n \geq 1}\) be a sequence in \(E\), and \(x \in E\). Suppose that \(x_n \xrightarrow{T} x\). Let \(\epsilon > 0\). The open ball \(B(x, \epsilon)\) being open in \(E\), there exists \(n_0 \geq 1\), such that \(n \geq n_0 \Rightarrow x_n \in B(x, \epsilon)\). In other words, we have:

\[
    n \geq n_0 \Rightarrow d(x_n, x) \leq \epsilon
\]

(3)

Conversely, suppose that for all \(\epsilon > 0\), there exists \(n_0 \geq 1\) such that (3) holds. Let \(U\) be open in \(E\), with \(x \in U\). By definition (30) of the metric topology, there exists \(\epsilon > 0\) such that \(B(x, \epsilon) \subseteq U\). Since, there exists \(n_0 \geq 1\) such that (3) holds, we have found \(n_0 \geq 1\) such that:

\[
    n \geq n_0 \Rightarrow x_n \in U
\]

This proves that \(x_n \xrightarrow{T} x\).
2. \( F_n = \{ x_k : k \geq n \} \). So \( F_{n+1} \subseteq F_n \) for all \( n \geq 1 \). Being the closure of some subset of \( E \), for all \( n \geq 1 \), \( F_n \) is a closed subset of \( E \) (see definition (37) and following exercise). It follows that \( (F_n)_{n \geq 1} \) is a decreasing sequence of closed subsets of \( E \).

3. Suppose \( F_n \downarrow \emptyset \), i.e. \( F_{n+1} \subseteq F_n \) with \( \cap_{n \geq 1} F_n = \emptyset \). Then:

\[
E = \emptyset^c = \left( \bigcap_{n=1}^{+\infty} F_n \right)^c = \bigcup_{n=1}^{+\infty} F_n^c
\]

Since each \( F_n \) is closed in \( E \), each \( F_n^c \) is an open subset of \( E \). We conclude that \( (F_n^c)_{n \geq 1} \) is an open covering of \( E \).

4. Suppose \( (E, T) \) is compact. If \( \cap_{n \geq 1} F_n = \emptyset \), then from 3. \( (F_n^c)_{n \geq 1} \) is an open covering of \( E \), of which we can extract a finite sub-covering (see definition (65)). There exists a finite subset \( \{n_1, \ldots, n_p\} \) of \( \mathbb{N}^* \) such that:

\[
E = F_{n_1}^c \cup \ldots \cup F_{n_p}^c
\]

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and therefore $F_{n_1} \cap \ldots \cap F_{n_p} = \emptyset$. However, without loss of
generality, we can assume that $n_p \geq n_k$ for all $k = 1, \ldots, p$.
Since $F_{n+1} \subseteq F_n$ for all $n \geq 1$, it follows that:

$$F_{n_p} = F_{n_1} \cap \ldots \cap F_{n_p} = \emptyset$$

This is a contradiction since $F_{n_p}$ contains all $x_k$'s for $k \geq n_p$.

We conclude that if $(E, T)$ is a compact, then $\cap_{n \geq 1} F_n \neq \emptyset$.

5. Suppose $(E, T)$ is compact. From 4., there exists $x \in \cap_{n \geq 1} F_n$.
Then, for all $n \geq 1$, we have $x \in F_n = \{x_k : k \geq n\}$, i.e. $x$ lies
in the closure of $\{x_k : k \geq n\}$. It follows that for all $\epsilon > 0$:

$$\{x_k : k \geq n\} \cap B(x, \epsilon) \neq \emptyset$$

We have proved the existence of $x \in E$, such that (4) holds for
all $n \geq 1$ and $\epsilon > 0$.

6. Let $x \in E$ be as in 5. Take $n = 1$ and $\epsilon = 1$. Then, we have
$\{x_k : k \geq 1\} \cap B(x, 1) \neq \emptyset$. There exists $n_1 \geq 1$, such that
$x_{n_1} \in B(x, 1)$. Suppose we have found $n_1 < \ldots < n_p$ ($p \geq 1$),
such that $x_{nk} \in B(x, 1/k)$ for all $k \in \mathbb{N}_p$. Take $n = n_p + 1$ and $\epsilon = 1/(p+1)$ in 5. We have:
\[
\{x_k : k \geq n_p + 1\} \cap B(x, 1/(p+1)) \neq \emptyset
\]
So there exists $n_{p+1} > n_p$, such that $x_{n_{p+1}} \in B(x, 1/(p+1))$. Following this induction argument, we can construct a subsequence $(x_{n_p})_{p \geq 1}$ of $(x_n)_{n \geq 1}$, such that $x_{n_p} \in B(x, 1/p)$ for all $p \geq 1$.

7. If $(E, T)$ is compact, then from 5. and 6., given a sequence $(x_n)_{n \geq 1}$ in $E$, there exists $x \in E$ and a subsequence $(x_{n_p})_{p \geq 1}$ such that $d(x, x_{n_p}) < 1/p$ for all $p \geq 1$. From 1., it follows that $x_{n_p} \xrightarrow{T} x$ as $p \to +\infty$, and we have proved that any sequence in a compact metric space, has a convergent subsequence.

Exercise 1
Exercise 2.

1. Let $x \in E$. By assumption, $(V_i)_{i \in I}$ is an open covering of $E$, so in particular $E = \bigcup_{i \in I} V_i$. There exists $i \in I$, such that $x \in V_i$. Furthermore, $V_i$ is open with respect to the metric topology on $E$. There exists $r > 0$, such that $B(x, r) \subseteq V_i$. We have proved that for all $x \in E$, there exists $i \in I$ and $r > 0$, such that $B(x, r) \subseteq V_i$.

2. Let $x \in E$. Then $r(x) = \sup A(x)$, where:

$$A(x) \triangleq \{ r > 0 : \exists i \in I \ , \ B(x, r) \subseteq V_i \}$$

From 1., the set $A(x)$ is non-empty. There exists $r > 0$ such that $r \in A(x)$. $r(x)$ being an upper-bound of $A(x)$, we have $r \leq r(x)$. In particular, $r(x) > 0$. We have proved that for all $x \in E$, $r(x) > 0$. 

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Exercise 3.

1. Let $\alpha = \inf_{x \in E} r(x)$. We assume that $\alpha = 0$. Let $n \geq 1$. Then $\alpha < 1/n$. $\alpha$ being the greatest lower bound of all $r(x)$’s for $x \in E$, $1/n$ cannot be such lower bound. There exists $x_n \in E$, such that $r(x_n) < 1/n$.

2. From 1., we have constructed a sequence $(x_n)_{n \geq 1}$ in $E$, such that $r(x_n) < 1/n$ for all $n \geq 1$. By assumption (see previous exercise (2)), the metric space $(E, d)$ is such that any sequence has a convergent sub-sequence. Let $(x_{n_k})_{k \geq 1}$ be a sub-sequence of $(x_n)_{n \geq 1}$ and let $x^* \in E$, be such that $x_{n_k} \xrightarrow{r} x^*$. From exercise (2), there exists $r^* > 0$ and $i \in I$, with $B(x^*, r^*) \subseteq V_i$. Since $r^* > 0$ and $x_{n_k} \xrightarrow{r} x^*$, there exists $k'_0 \geq 1$, such that:

$$k \geq k'_0 \Rightarrow d(x^*, x_{n_k}) < r^*/2$$
Since \( n_k \uparrow +\infty \) as \( k \to +\infty \), there exists \( k_0'' \geq 1 \), such that:

\[
k \geq k_0'' \Rightarrow \frac{1}{n_k} \leq \frac{r^*}{4}
\]

It follows that for all \( k \geq k_0'' \), we have \( r(x_{n_k}) \leq \frac{1}{n_k} \leq \frac{r^*}{4} \).

Take \( k_0 = \max(k_0', k_0'') \). We have both \( d(x^*, x_{n_{k_0}}) < \frac{r^*}{2} \) and \( r(x_{n_{k_0}}) \leq \frac{r^*}{4} \).

3. From 2., we have found \( k_0 \geq 1 \), such that \( d(x^*, x_{n_{k_0}}) < \frac{r^*}{2} \).

Let \( y \in B(x_{n_{k_0}}, \frac{r^*}{2}) \). Then, from the triangle inequality:

\[
d(x^*, y) \leq d(x^*, x_{n_{k_0}}) + d(x_{n_{k_0}}, y) < \frac{r^*}{2} + \frac{r^*}{2} = r^*
\]

So \( y \in B(x^*, r^*) \). Hence, we see that \( B(x_{n_{k_0}}, \frac{r^*}{2}) \subseteq B(x^*, r^*) \).

However, from 2., \( B(x^*, r^*) \subseteq V_i \). So \( B(x_{n_{k_0}}, \frac{r^*}{2}) \subseteq V_i \). It follows that \( r^*/2 \) belongs to the set:

\[
A(x_{n_{k_0}}) = \{ r > 0 : \exists i \in I , \ B(x_{n_{k_0}}, r) \subseteq V_i \}
\]
and consequently, \( r^*/2 \leq r(x_{n_k0}) = \sup A(x_{n_k0}) \). This contradicts the fact that \( r(x_{n_k0}) \leq r^*/4 \), as obtained in 2. We conclude that our initial hypothesis of \( \alpha = \inf_{x \in E} r(x) = 0 \) is absurd, and we have proved that \( \inf_{x \in E} r(x) > 0 \).

Exercise 3
Exercise 4.

1. Let \( r_0 > 0 \) be such that \( 0 < r_0 < \inf_{x \in E} r(x) \). We assume that \( E \) cannot be covered by a finite number of open balls with radius \( r_0 \). Let \( x_1 \) be an arbitrary element of \( E \). Then, by assumption, \( B(x_1, r_0) \) cannot cover the whole of \( E \). There exists \( x_2 \in E \), such that \( x_2 \notin B(x_1, r_0) \). By assumption, \( B(x_1, r_0) \cup B(x_2, r_0) \) cannot cover the whole of \( E \). There exists \( x_3 \in E \), such that \( x_3 \notin B(x_1, r_0) \cup B(x_2, r_0) \). By induction, we can construct a sequence \( (x_n)_{n \geq 1} \) in \( E \), such that for all \( n \geq 1 \):

\[
x_{n+1} \notin B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)
\]

2. Let \( n > m \). Then \( x_n \notin B(x_m, r_0) \). So \( d(x_n, x_m) \geq r_0 \).

3. Suppose \( (x_n)_{n \geq 1} \) has a convergent sub-sequence, There exists \( x^* \in E \), and a sub-sequence \( (x_{n_k})_{k \geq 1} \) such that \( x_{n_k} \xrightarrow{\text{f}} x^* \). Take \( \epsilon = r_0/4 > 0 \). There exists \( k_0 \geq 1 \), such that:

\[
k \geq k_0 \Rightarrow d(x^*, x_{n_k}) < r_0/4
\]
It follows that for all $k, k' \geq k_0$, we have:

$$d(x_n^k, x_n^{k'}) \leq d(x^*, x_n^k) + d(x^*, x_n^{k'}) < r_0/2$$

This contradicts 2., since $d(x_n^k, x_n^{k'}) \geq r_0$ for $k \neq k'$. So $(x_n)_{n \geq 1}$ cannot have a convergent sub-sequence.

4. From 3., $(x_n)_{n \geq 1}$ cannot have a convergent sub-sequence. This is a contradiction to our initial assumption (see exercise (2)), that any sequence in $E$ should have a convergent sub-sequence. It follows that the hypothesis in 1. is absurd, and we conclude that $E$ can indeed be covered by a finite number of open balls of radius $r_0$. In other words, there exists a finite subset $\{x_1, \ldots, x_n\}$ of $E$, such that $E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.

5. Let $x \in E$. By assumption, $r_0 < \inf_{x \in E} r(x)$. In particular, we have $r_0 < r(x) = \sup A(x)$, where:

$$A(x) = \{ r > 0 : \exists i \in I , B(x, r) \subseteq V_i \}$$
$r(x)$ being the smallest upper-bound of $A(x)$, it follows that $r_0$ cannot be such upper bound. There exists $r > 0$, $r \in A(x)$, such that $r_0 < r$. Since $r \in A(x)$, there exists $i \in I$, such that $B(x, r) \subseteq V_i$. But from $r_0 < r$, we conclude that $B(x, r_0) \subseteq V_i$.

We have proved that for all $x \in E$, there exists $i \in I$, such that $B(x, r_0) \subseteq V_i$.

6. From 4., we have $E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$. However, from 5., for all $k \in \mathbb{N}_n$, there exists $i_k \in I$, such that $B(x_k, r_0) \subseteq V_{i_k}$.

It follows that:

$$E = V_{i_1} \cup \ldots \cup V_{i_n} \quad (5)$$

Given a family of open sets $(V_i)_{i \in I}$ such that $E = \cup_{i \in I} V_i$ (see exercise (2)), we have been able to find a finite subset $\{i_1, \ldots, i_n\}$ of $I$, such that (5) holds. We conclude that the metrizable space $(E, T)$ is a compact topological space.

7. Let $(E, T)$ be a metrizable topological space. If $(E, T)$ is compact, then from exercise (1), any sequence in $E$ has a convergent
sub-sequence. Conversely, if $E$ is such that any sequence in $E$
has a convergent sub-sequence, then as proved in 6., $(E, T)$ is
a compact topological space. This proves the difficult and very
important theorem (47).

Exercise 4
Exercise 5.

1. Let \( a, b \in \mathbb{R} \), \( a < b \). Let \( (x_n)_{n \geq 1} \) be a sequence in \( ]a, b[ \). In particular, \( (x_n)_{n \geq 1} \) is a sequence in \( [a, b] \). From theorem (34), \( [a, b] \) is a compact subset of \( \mathbb{R} \). Applying theorem (47), there exists a subsequence \( (x_{n_k})_{k \geq 1} \) of \( (x_n)_{n \geq 1} \), and \( x \in [a, b] \), such that \( x_{n_k} \to x \). So \( (x_n)_{n \geq 1} \) has a convergent subsequence.

2. No. One cannot conclude that \( [a, b[ \) is compact. In fact, \( \mathbb{R} \) being Hausdorff, from theorem (35), if \( [a, b[ \) was compact, it would be closed, and \( ]-\infty, a[ \cup [b, \infty[ \) would be open in \( \mathbb{R} \) . . . One has to be careful with the phrase having a convergent subsequence. As proved in 1., any sequence in \( ]a, b[ \) has a convergent subsequence, but the limit of such subsequence may not lie in \( ]a, b[ \) itself (we only know for sure it lies in \( [a, b] \)). This is why, one cannot apply theorem (47) to conclude that \( [a, b[ \) is compact.

---

In a clear context, we shall omit notations such as \( x_{n_k} \xrightarrow{\mathbb{R}} x \) or \( x_{n_k} \xrightarrow{[a, b]} x \).

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Exercise 6.

1. The equivalence between \( x_p \xrightarrow{T_E} x \) and \( x_p \xrightarrow{T_{\mathbb{R}^n}} x \) has already been proved in exercise (7) of the previous tutorial. Since the topology \( T_E \) is induced by the topology \( T_{\mathbb{R}^n} \) on \( E \), whether we regard \( (x_p)_{p \geq 1} \) and \( x \) as belonging to \( E \) or \( \mathbb{R}^n \), is irrelevant as far as the convergence \( x_p \to x \) is concerned. Note however that it is important to have \( x_p \in E \) for all \( p \geq 1 \), and \( x \in E \).

2. As seen in exercise (14) of Tutorial 6, various metrics will induce the product topology \( T_{\mathbb{R}^n} \) on \( \mathbb{R}^n \). The most common, viewed as the usual metric on \( \mathbb{R}^n \), is:

\[
d_2(x, y) \triangleq \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}
\]
Other possible metrics are:

\[ d_1(x, y) \triangleq \sum_{i=1}^{n} |x^i - y^i| \]

or:

\[ d_\infty(x, y) \triangleq \max_{i \in \mathbb{N}^n} |x^i - y^i| \]

3. Let \((x_p)_{p \geq 1}\) be a sequence in \(\mathbb{R}^n\) and \(x \in \mathbb{R}^n\). Suppose that \(x_p \to x\). Then for all \(\epsilon > 0\), there exists \(p_0 \geq 1\), such that:

\[ p \geq p_0 \implies d_1(x, x_p) = \sum_{i=1}^{n} |x^i - x^i_p| \leq \epsilon \]

In particular, for all \(i \in \mathbb{N}_n\), we have:

\[ p \geq p_0 \implies |x^i - x^i_p| \leq \epsilon \]

\(^5\text{i.e. } x_p \xrightarrow{T_{\mathbb{R}^n}} x, \text{ as should be clear from context.}\)
So $x^i_p \to x^i$ for all $i \in \mathbb{N}_n$. Conversely, suppose $x^i_p \to x^i$ for all $i$’s. Given $\epsilon > 0$, for all $i \in \mathbb{N}_n$, there exists $p_i \geq 1$, such that:
\[ p \geq p_i \implies |x^i - x^i_p| \leq \epsilon/n \]

Taking $p_0 = \max(p_1, \ldots, p_n)$, we obtain:
\[ p \geq p_0 \implies d_1(x, x_p) = \sum_{i=1}^{n} |x^i - x^i_p| \leq \epsilon \]

So $x_p \to x$, which is equivalent to $[x^i_p \to x^i$ for all $i \in \mathbb{N}_n]$. Note that although we used the metric structure of $\mathbb{R}$ and $\mathbb{R}^n$ to prove this equivalence, we had no need to do so. In fact, any sequence with values in an arbitrary product, even uncountable, of topological spaces, even non-metrizable, will converge if and only if each coordinate sequence itself converges. For those interested in this small digression, here is a quick proof: let $(x_p)_{p \geq 1}$ be a sequence in the product $\prod_{i \in I} \Omega_i$. Let $x$ be an element of

\footnote{i.e. $x^i_p \overset{\text{R}}{\to} x^i$, as should be clear from context.}
Πᵢ∈ΙΩᵢ. Suppose \( x_p \to x \), with respect to the product topology. Let \( i \in I \) and \( U \) be an arbitrary open set in \( Ω_i \) containing \( x^i \). Then \( U \times Π_{j\neq i} Ω_j \) is an open set in \( Π_{i∈I} Ω_i \) containing \( x \). Since \( x_p \to x \), \( x_p \) is eventually\(^7\) in \( U \times Π_{j\neq i} Ω_j \). It follows that \( x^i_p \) is eventually in \( U \), and we see that \( x^i_p \to x^i \). Conversely, suppose \( x^i_p \to x^i \) for all \( i \in I \). Let \( U \) be an open set in \( Π_{i∈I} Ω_i \) containing \( x \). There exists a rectangle \( V = Π_{i∈I} A_i \) such that \( x \in V \subseteq U \). The set \( J = \{ i \in I : A_i \neq Ω_i \} \) is finite, and for all \( j \in J \), \( A_j \) is an open set in \( Ω_j \) containing \( x^j \). From \( x^j_p \to x^j \) we see that \( x^j_p \) is eventually in \( A_j \). \( J \) being finite, it follows that \( x_p \) is eventually in \( (Π_{j∈J} A_j) \times (Π_{i\notin J} Ω_i) = V \). Since \( V \subseteq U \), \( x_p \) is eventually in \( U \), and we have proved that \( x_p \to x \).

**Exercise 6**

\(^7\)there exists \( p_0 \geq 1 \) such that \( p \geq p_0 \Rightarrow x_p \in U \times Π_{j\neq i} Ω_j \).
Exercise 7.

1. Let \((x_p)_{p \geq 1}\) be a sequence in \(E\). Then \((x^1_p)_{p \geq 1}\) is a sequence in \([-M,M]\), which is a compact subset of \(\mathbb{R}\). From theorem (47), we can extract a subsequence of \((x^1_p)_{p \geq 1}\), converging to some \(x^1 \in [-M,M]\). In other words, from definition (78), there exists a strictly increasing map \(\phi : \mathbb{N}^* \to \mathbb{N}^*,\) and \(x^1 \in [-M,M]\) such that\(^8\) \(x^1_{\phi(p)} \to x^1\). Hence, we have found a subsequence \((x_{\phi(p)})_{p \geq 1}\) such that \(x^1_{\phi(p)} \to x^1\), for some \(x^1 \in [-M,M]\).

2. The topology on \([-M,M]\) being induced by the topology on \(\mathbb{R}\), the convergence \(x^1_{\phi(p)} \to x^1\) is independent of the particular topology (that of \(\mathbb{R}\) or \([-M,M]\)) with respect to which, it is being considered.

3. Let \(1 \leq k \leq n-1\). Let \((y_p)_{p \geq 1} = (x_{\phi(p)})_{p \geq 1}\) be a subsequence of \((x_p)_{p \geq 1}\), with the property that for all \(j \in \mathbb{N}_k\), we have \(y^j_p \to x^j\)

\(^8\)i.e. \(x^1_{\phi(p)} \xrightarrow{\tau_{[-M,M]}} x^1\), which is the same as \(x^1_{\phi(p)} \xrightarrow{\tau_\mathbb{R}} x^1\).
for some \( x^j \in [-M, M] \). Then, \((y^{k+1}_p)_{p \geq 1}\) is a sequence in the compact interval \([-M, M]\). From theorem (47), there exists a strictly increasing map \( \psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \) such that \( y^{k+1}_{\psi(p)} \rightarrow x^{k+1} \), for some \( x^{k+1} \in [-M, M] \).

4. If both \( \phi, \psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \) are strictly increasing, so is \( \phi \circ \psi \).

5. Since \( \phi \circ \psi \) is strictly increasing, \( (x_{\phi \circ \psi(p)})_{p \geq 1} \) is indeed a subsequence of \( (x_p)_{p \geq 1} \), which furthermore coincides with \( (y_{\psi(p)})_{p \geq 1} \), as defined in 3. It follows that \( x^{k+1}_{\phi \circ \psi(p)} \rightarrow x^{k+1} \). Furthermore, from 3, the subsequence \( (y_p)_{p \geq 1} \) is assumed to be such that \( y^j_p \rightarrow x^j \) for all \( j \in \mathbb{N}_k \). Hence, we also have \( y^j_{\psi(p)} \rightarrow x^j \), i.e. \( x^j_{\phi \circ \psi(p)} \rightarrow x^j \) for all \( j \in \mathbb{N}_k \). We conclude that \( (x_{\phi \circ \psi(p)})_{p \geq 1} \) is a subsequence of \( (x_p)_{p \geq 1} \) such that \( x^j_{\phi \circ \psi(p)} \rightarrow x^j \) for all \( j \in \mathbb{N}_{k+1} \).

6. From 1., given a sequence \( (x_p)_{p \geq 1} \) in \( E \), we can extract a subsequence \( (x_{\phi(p)})_{p \geq 1} \) of \( (x_p)_{p \geq 1} \) such that \( x^1_{\phi(p)} \rightarrow x^1 \) for some \( x^1 \in [-M, M] \). Given \( 1 \leq k \leq n - 1 \), and a subsequence
\[(x_{\phi(p)})_{p \geq 1} \text{ of } (x_p)_{p \geq 1}, \text{ such that for all } j \in \mathbb{N}_k, x_{\phi(p)}^j \to x^j \]

for some \(x^j \in [-M, M]\), we showed in 5. that we could extract a further subsequence \((x_{\phi\phi\phi(p)})_{p \geq 1}\) having a similar property for all \(j \in \mathbb{N}_{k+1}\). By induction, it follows that there exists a subsequence \((x_{\phi(p)})_{p \geq 1}\) of \((x_p)_{p \geq 1}\), such that for all \(j \in \mathbb{N}_n\), we have \(x_{\phi(p)}^j \to x^j\) for some \(x^j \in [-M, M]\). Hence, taking \(x = (x^1, \ldots, x^n)\), we see that \(x_{\phi(p)} \to x^9\).

7. Let \((x_p)_{p \geq 1}\) be a sequence in \(E\). From 6., there exists \(x \in E\), and a subsequence \((x_{\phi(p)})_{p \geq 1}\) of \((x_p)_{p \geq 1}\), with \(x_{\phi(p)} \to x\). From theorem (47), we conclude that \((E, T_E)\) is a compact topological space, or equivalently, that \(E\) is a compact subset of \(\mathbb{R}^n\).

The purpose of this exercise is to prove that the \(n\)-dimensional product \([-M, M] \times \ldots \times [-M, M]\) is compact\(^{10}\).

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\(^9\)Both with respect to \(T_E\) and \(T_{\mathbb{R}^n}\).

\(^{10}\)Tychonoff theorem will hopefully be the subject of some future tutorial :-)

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Exercise 8.

1. If $A = \emptyset$ then $A \subseteq [-M, M] \times \ldots \times [-M, M]$, for all $M \in \mathbb{R}^+$. We assume that $A \neq \emptyset$. Let $\delta(A)$ be the diameter of $A$ (see definition (68)) with respect to the usual metric:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}$$

i.e. $\delta(A) = \sup\{d(x, y) : x, y \in A\}$. Since $A \neq \emptyset$, $\delta(A) \geq 0$. Furthermore, $A$ being assumed to be bounded with respect to the usual metric of $\mathbb{R}^n$, we have $\delta(A) < +\infty$. So $\delta(A) \in \mathbb{R}^+$. Let $y$ be an arbitrary element of $A$. For all $x \in A$, we have:

$$|x^i - y^i| \leq d(x, y) \leq \delta(A)$$

So $|x^i| \leq |y^i| + \delta(A)$, and taking $M = \max(|y^1|, \ldots, |y^n|) + \delta(A)$, we conclude that $A \subseteq [-M, M] \times \ldots \times [-M, M]$, with $M \in \mathbb{R}^+$. 

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2. By assumption, \( A \) is a closed subset of \( \mathbb{R}^n \). So \( A^c \) is open. It follows that \( E \setminus A = E \cap A^c \) is an element of the topology induced on \( E \), by the topology on \( \mathbb{R}^n \). In other words, \( E \setminus A \) is an open subset of \( E \). We conclude that \( A \) is a closed subset of \( E \).

3. From ex. (7), \((E, T_E)\) is a compact topological space. From 2., \( A \) is a closed subset of \( E \). Using exercise (2)[6.] of Tutorial 8, we conclude that \( A \) is a compact subset of \( E \). In other words, \((A,(T_E)|_A)\) is a compact topological space. However, the topology \( T_E \) is induced by \( T_{\mathbb{R}^n} \), i.e. \( T_E = (T_{\mathbb{R}^n})|_E \). It follows that \((T_E)|_A = (T_{\mathbb{R}^n})|_A \). So \((A,(T_{\mathbb{R}^n})|_A)\) is a compact topological space, or equivalently, \( A \) is a compact subset of \( \mathbb{R}^n \).

4. Let \( A \) be a compact subset of \( \mathbb{R}^n \). From theorem (35), \( \mathbb{R}^n \) being Hausdorff, \( A \) is closed in \( \mathbb{R}^n \). From exercise (6)[4.] of Tutorial 8, \( A \) is bounded with respect to any metric inducing the usual topology of \( \mathbb{R}^n \). This proves theorem (48).
Exercise 9.

1. \( d_{C^n} \) and \( d_{R^{2n}} \) are defined by:

\[
d_{C^n}(z, z') = \sqrt{\sum_{i=1}^{n} |z_i - z'_i|^2}
\]

\[
d_{R^{2n}}(x, x') = \sqrt{\sum_{i=1}^{2n} (x_i - x'_i)^2}
\]

for all \( z, z' \in C^n \) and \( x, x' \in R^{2n} \).

2. Given \( z, z' \in C^n \), we have:

\[
d_{C^n}(z, z') = \sqrt{\sum_{i=1}^{n} (Re(z_i) - Re(z'_i))^2 + \sum_{i=1}^{n} (Im(z_i) - Im(z'_i))^2}
\]

It follows that \( d_{C^n}(z, z') = d_{R^{2n}}(\phi(z), \phi(z')) \).

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3. \( \phi \) is clearly a bijection between \( \mathbb{C}^n \) and \( \mathbb{R}^{2n} \). From 2., we see that \( \phi \) is continuous, and furthermore that:

\[
d_{\mathbb{C}^n}(\phi^{-1}(x), \phi^{-1}(x')) = d_{\mathbb{R}^{2n}}(x, x')
\]

for all \( x, x' \in \mathbb{R}^{2n} \). So \( \phi^{-1} \) is also continuous. From definition (31), \( \phi \) is a homeomorphism from \( \mathbb{C}^n \) to \( \mathbb{R}^{2n} \).

4. Let \( K \subseteq \mathbb{C}^n \). Suppose \( K \) is a compact subset of \( \mathbb{C}^n \). Then, \( (K, (\mathcal{T}_{\mathbb{C}^n})|_K) \) is a compact topological space. \( \phi \) being continuous, its restriction \( \phi|_K \) is also continuous.\(^{11}\) Using exercise (8) of Tutorial 8., the direct image \( \phi(K) \) is a compact subset of \( \mathbb{R}^{2n} \). In other words, \( \phi(K) \) is a compact subset of \( \mathbb{R}^{2n} \). Conversely, suppose \( \phi(K) \) is a compact subset of \( \mathbb{R}^{2n} \). Since \( K \) can be written as the direct image \( K = \phi^{-1}(\phi(K)) \) of \( \phi(K) \) by \( \phi^{-1} \), and \( \phi^{-1} \) is continuous, we conclude similarly that \( K \) is a compact subset of \( \mathbb{C}^n \). We have proved that for all \( K \subseteq \mathbb{C}^n \), \( K \) is compact if and only if \( \phi(K) \) is compact.

\(^{11}\) If uneasy with \( K = \emptyset \) and \( \phi|_K = \emptyset \), consider the case separately.
5. Let $K \subseteq \mathbb{C}^n$. Suppose $K$ is a closed subset of $\mathbb{C}^n$. Since $\phi(K)$ can be written as the inverse image $\phi(K) = (\phi^{-1})^{-1}(K)$ of $K$ by $\phi^{-1}$, and $\phi^{-1}$ is continuous, we see that $\phi(K)$ is a closed subset of $\mathbb{R}^{2n}$. Conversely, suppose $\phi(K)$ is a closed subset of $\mathbb{R}^{2n}$. Since $K$ can be written as the inverse image $K = \phi^{-1}(\phi(K))$ of $\phi(K)$ by $\phi$, and $\phi$ is continuous, we see that $K$ is a closed subset of $\mathbb{C}^n$. We have proved that for all $K \subseteq \mathbb{C}^n$, $K$ is closed if and only if $\phi(K)$ is closed.

6. Let $K \subseteq \mathbb{C}^n$ and $\delta(\phi(K))$ be the diameter of $\phi(K)$ in $\mathbb{R}^{2n}$:

$$
\delta(\phi(K)) = \sup \{ d_{\mathbb{R}^{2n}}(x, x') : x, x' \in \phi(K) \}
= \sup \{ d_{\mathbb{R}^{2n}}(\phi(z), \phi(z')) : z, z' \in K \}
= \sup \{ d_{\mathbb{C}^n}(z, z') : z, z' \in K \}
$$

i.e. $\delta(\phi(K)) = \delta(K)$, where $\delta(K)$ is the diameter of $K$ in $\mathbb{C}^n$. It follows that $\delta(K) < +\infty$ is equivalent to $\delta(\phi(K)) < +\infty$. We have proved that for all $K \subseteq \mathbb{C}^n$, $K$ is bounded if and only if $\phi(K)$ is bounded.
7. Let $K \subseteq \mathbb{C}^n$. From 4., $K$ is compact, if and only if $\phi(K)$ is compact. From theorem (48), $\phi(K)$ being a subset of $\mathbb{R}^{2n}$, it is compact if and only if, it is closed and bounded. From 5. and 6., this in turn is equivalent to $K$ being itself closed and bounded. We have proved that for all $K \subseteq \mathbb{C}^n$, $K$ is compact if and only if $K$ is closed and bounded.

Exercise 9
Exercise 10.

1. Definition (79) defines the notion of Cauchy sequences in a metric space. In contrast, definition (77) defines the notion of Cauchy sequences in \( L^p_C(\Omega, \mathcal{F}, \mu) \). Since that latter was defined in (73) as a set of functions, as opposed to a set of \( \mu \)-almost sure equivalence classes, strictly speaking \( L^p_C(\Omega, \mathcal{F}, \mu) \) is not a metric space. So definition (77) is not a particular case of definition (79).

2. Definition (80) defines the notion of complete metric space, as a metric space where all Cauchy sequences converge.\(^{12}\) Theorem (46) does state that all Cauchy sequences in \( L^p_C(\Omega, \mathcal{F}, \mu) \) converge. However, since \( L^p_C(\Omega, \mathcal{F}, \mu) \) is not strictly speaking a metric space, it cannot be said to be a complete metric space.

Exercise 10

\(^{12}\)to a limit belonging to that same metric space...
Exercise 11.

1. Let \((z_k)_{k \geq 1}\) be a Cauchy sequence in \(\mathbb{C}^n\). Taking \(\epsilon = 1\), there exists \(k_0 \geq 1\), such that:

\[ k, k' \geq k_0 \Rightarrow \|z_k - z_{k'}\| \leq 1 \]

Since \(\|z\| - \|z'\| \leq \|z - z'\|\) for all \(z, z' \in \mathbb{C}^n\), we have:

\[ k \geq k_0 \Rightarrow \|z_k\| \leq 1 + \|z_{k_0}\| \]

Taking \(M = \max(1 + \|z_{k_0}\|, \|z_1\|, \ldots, \|z_{k_0-1}\|)\), we see that \(\|z_k\| \leq M\) for all \(k \geq 1\). We have proved that \((z_k)_{k \geq 1}\) is a bounded sequence in \(\mathbb{C}^n\).

2. Let \(B = \{z \in \mathbb{C}^n : \|z\| \leq M\}\). For all \(z, z' \in B\), we have \(\|z - z'\| \leq \|z\| + \|z'\| \leq 2M\). It follows that \(\delta(B) \leq 2M\), where \(\delta(B)\) is the diameter of \(B\) in \(\mathbb{C}^n\). So \(\delta(B) < +\infty\), i.e. \(B\) is a bounded subset of \(\mathbb{C}^n\). Let \(z_0 \in B^c\). Then \(M < \|z_0\|\). Let \(\epsilon = \|z_0\| - M > 0\), and \(z \in \mathbb{C}^n\) with \(\|z - z_0\| < \epsilon\). Then, we have \(\|z_0\| - \|z\| \leq \|z - z_0\| < \epsilon = \|z_0\| - M\), and consequently
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$M < \|z\|$, i.e. $z \in B^c$. So $B(z_0, \epsilon) \subseteq B^c$. For all $z_0 \in B^c$, we have found $\epsilon > 0$, such that $B(z_0, \epsilon) \subseteq B^c$. This proves that $B^c$ is open with respect to the (metric) topology of $\mathbb{C}^n$. So $B$ is a closed subset of $\mathbb{C}^n$.

3. From 2., $B$ is a closed and bounded subset of $\mathbb{C}^n$. From exercise (9), it follows that $B$ is a compact subset of $\mathbb{C}^n$. In other words, $(B, (\mathcal{T}_{\mathbb{C}^n})_B)$ is a compact topological space. However, from 1., $(z_k)_{k \geq 1}$ is a sequence of elements of $B$. Using theorem (47), $(z_k)_{k \geq 1}$ has a convergent subsequence, i.e. there exists $z \in B$, and a subsequence $(z_{k_p})_{p \geq 1}$, such that $z_{k_p} \to z$.\(^{13}\)

4. $(z_k)_{k \geq 1}$ being Cauchy, given $\epsilon > 0$, there exist $n_0 \geq 1$, such that:

$$k, k' \geq n_0 \Rightarrow d(z_k, z_{k'}) \leq \epsilon/2$$

Furthermore, since $z_{k_p} \to z$, there exists $p'_0 \geq 1$, such that:

$$p \geq p'_0 \Rightarrow d(z_{k_p}, z) \leq \epsilon/2$$

\(^{13}\)Both with respect to $\mathcal{T}_{\mathbb{C}^n}$ and the induced topology $(\mathcal{T}_{\mathbb{C}^n})_B$.
Moreover, since \( k_p \uparrow +\infty \) as \( p \to +\infty \), there exists \( p''_0 \geq 1 \), such that \( p \geq p''_0 \Rightarrow k_p \geq n_0 \). Take \( p_0 = \max(p'_0, p''_0) \). Then, \( d(z, z_{k_{p_0}}) \leq \epsilon/2 \), and we have:

\[
k \geq n_0 \Rightarrow d(z, z_{k_{p_0}}) \leq \epsilon/2
\]

5. From 4., we have found \( n_0 \geq 1 \), such that:

\[
k \geq n_0 \Rightarrow d(z, z_k) \leq \epsilon
\]

It follows that \( z_k \to z \).

6. From 5., we see that every Cauchy sequence \((z_k)_{k \geq 1}\) in \( C^n \), converges to some limit \( z \in C^n \). From definition (80), we conclude that \( C^n \) is complete metric space.

7. The completeness of \( C \) was used in exercise (12)[6.] of Tutorial 9, leading to theorem (44) where we proved that any sequence
(fn)n≥1 in LpC(Ω, ℱ, μ) such that:

\[ \sum_{k=1}^{+\infty} \| f_{k+1} - f_k \|_p < +\infty \]

converges to some \( f \in L_p^C(\Omega, \mathcal{F}, \mu) \). This, in turn, was crucially important in proving theorem (46), where \( L_p^C(\Omega, \mathcal{F}, \mu) \) is shown to be complete.

Exercise 11
Exercise 12.

1. Let \((x_k)_{k \geq 1}\) be a sequence in \(\mathbb{R}^n\), such that \(x_k \to z\), for some \(z \in \mathbb{C}^n\). For all \(k \geq 1\) and \(i \in \mathbb{N}_n\), we have:

\[
|\text{Im}(z_i)| = |\text{Im}(z_i) - \text{Im}(x_k^i)| \leq \|z - x_k\|
\]

Taking the limit as \(k \to +\infty\), we obtain \(\text{Im}(z_i) = 0\). This being true for all \(i \in \mathbb{N}_n\), we have proved that \(z \in \mathbb{R}^n\).

2. Let \((x_k)_{k \geq 1}\) be a Cauchy sequence in \(\mathbb{R}^n\). In particular, it is a Cauchy sequence in \(\mathbb{C}^n\). From exercise (11), \(\mathbb{C}^n\) is a complete metric space. Hence, there exists \(z \in \mathbb{C}^n\), such that \(x_k \to z\). From 1., \(z\) is in fact an element of \(\mathbb{R}^n\). We have proved that any Cauchy sequence \((x_k)_{k \geq 1}\) in \(\mathbb{R}^n\), converges to some \(z \in \mathbb{R}^n\). From definition (80), we conclude that \(\mathbb{R}^n\) is a complete metric space. This, together with exercise (11), proves theorem (49).

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Exercise 13.

1. Let $x \in \bar{F}$. From definition (37), if $U$ is an open set with $x \in U$, then $F \cap U \neq \emptyset$. Given $n \geq 1$, the open ball $B(x, 1/n)$ is an open set with $x \in B(x, 1/n)$. So $F \cap B(x, 1/n) \neq \emptyset$.

2. Let $x \in \bar{F}$. From 1., for all $n \geq 1$, we can choose an arbitrary element $x_n \in F \cap B(x, 1/n)$. This defines a sequence $(x_n)_{n \geq 1}$ of elements of $F$, such that $d(x, x_n) < 1/n$ for all $n \geq 1$. So $x_n \to x$.

3. Let $x \in E$. We assume that there exists a sequence $(x_n)_{n \geq 1}$ of elements of $F$, with $x_n \to x$. Let $U$ be an open set containing $x$. Since $x_n \to x$, there exists $n_0 \geq 1$, such that:

$$n \geq n_0 \Rightarrow x_n \in U$$

In particular, $x_{n_0} \in U$. But $x_{n_0}$ is also an element of $F$. So $x_{n_0} \in F \cap U$. We have proved that for all open set $U$ containing $x$, we have $F \cap U \neq \emptyset$. From definition (37), we conclude that $x \in \bar{F}$. 

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4. Suppose that $F$ is closed, and let $(x_n)_{n \geq 1}$ be a sequence in $F$ such that $x_n \to x$ for some $x \in E$. From 3, we have $x \in \bar{F}$. However from exercise (21) of Tutorial 4, we have $F = \bar{F}$. So $x \in F$. Conversely, suppose that for any sequence $(x_n)_{n \geq 1}$ in $F$ such that $x_n \to x$ for some $x \in E$, we have $x \in F$. We claim that $F$ is closed. From exercise (21) of Tutorial 4, it is sufficient to show that $\bar{F} = F$, or equivalently that $\bar{F} \subseteq F$. So let $x \in \bar{F}$. From 2, there exists a sequence $(x_n)_{n \geq 1}$ in $F$ such that $x_n \to x$. By assumption, this implies that $x \in F$. It follows that $\bar{F} \subseteq F$.

5. The fact that the induced topological space $(F, \mathcal{T}_F)$ is metrizable, is a consequence of theorem (12). The induced topology $\mathcal{T}_F$ is nothing but the metric topology associated with the induced metric $d|_{F} = d|_{F \times F}$.

6. Suppose $F$ is complete with respect to the induced metric $d|_{F}$. Let $x \in E$ and $(x_n)_{n \geq 1}$ be a sequence of elements of $F$, with $x_n \to x$. In particular, $(x_n)_{n \geq 1}$ is a Cauchy sequence with respect to the metric $d$. $(x_n)_{n \geq 1}$ being a sequence of elements
of $F$, it is also a Cauchy sequence with respect to the induced metric $d_{|F}$. $F$ being complete, there exists $y \in F$, such that $x_n \to y$. This convergence, with respect to $T_{|F}$, is also valid with respect $T$. Since we also have $x_n \to x$, we see that $x = y$. It follows that $x \in F$. Given $x \in E$, and a sequence $(x_n)_{n \geq 1}$ of elements of $F$ such that $x_n \to x$, we have proved that $x \in F$. From 4., this shows that $F$ is a closed subset of $E$. We conclude that if $F$ is complete (with respect to its natural metric $d_{|F}$), then it is a closed subset of $E$.

7. From theorem (12), the induced metric $d' = (d_{R})_{|R}$ induces the induced topology $(T_{R})_{|R}$. Such topology is nothing but the usual topology on $R$. It follows that $d'$ induces $T_{R}$.

8. Let $d_{R}$ be the usual metric on $R$. From theorem (12), the induced metric $(d_{R})_{|[-1, 1]}$ induces the induced topology on $[-1, 1]$. Such topology is nothing but the usual topology on $[-1, 1]$.

9. From 8., if $\{-1, 1\}$ was open in $[-1, 1]$, there would exists $\epsilon > 0$,
such that \([1 - \epsilon, 1] \subseteq \{-1, 1\}\), which is absurd.

10. If \((-\infty, +\infty)\) was open in \(\bar{\mathbb{R}}\), then \([-1, 1]\) would be open in \([-1, 1]\), since one is the inverse image of the other, by a strictly increasing homeomorphism.

11. If \(\mathbb{R}\) was closed in \(\bar{\mathbb{R}}\), then \((-\infty, +\infty)\) would be open in \(\bar{\mathbb{R}}\).

12. Let \(d_{\mathbb{R}}\) be the usual metric on \(\mathbb{R}\). Then \(d_{\mathbb{R}}\) induces the usual topology on \(\mathbb{R}\). However, from 7., the metric \(d'\) also induces the usual topology on \(\mathbb{R}\). It follows that \(d_{\mathbb{R}}\) and \(d'\) both induce the same topology. From theorem (49), \(\mathbb{R}\) is complete with respect to its usual metric \(d_{\mathbb{R}}\). If \(\mathbb{R}\) was complete with respect to \(d' = (d_{\mathbb{R}})_{\vert \mathbb{R}}\), then from 6., \(\mathbb{R}\) would be a closed subset of \(\bar{\mathbb{R}}\), contradicting 11. So \(\mathbb{R}\) is not complete with respect to \(d'\). We conclude that although the two metric spaces \((\mathbb{R}, d_{\mathbb{R}})\) and \((\mathbb{R}, d')\) are identical in the topological sense, one is complete whereas the other is not.

Exercise 13

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Exercise 14.

1. Let \( y \in \mathcal{H} \). For all \( x, x' \in \mathcal{H} \) and \( \alpha \in \mathbb{K} \), using \((ii)\) and \((iii)\) of definition \((81)\), we obtain:

\[
\langle x + \alpha x', y \rangle = \langle x, y \rangle + \alpha \langle x', y \rangle
\]

We conclude that \( x \to \langle x, y \rangle \) is linear for all \( y \in \mathcal{H} \).

2. Let \( x \in \mathcal{H} \). For all \( y, y' \in \mathcal{H} \) and \( \alpha \in \mathbb{K} \), using \((i)\), \((ii)\) and \((iii)\) of definition \((81)\), we obtain:

\[
\langle x, y + \alpha y' \rangle = \langle x, y \rangle + \bar{\alpha} \langle x, y' \rangle
\]

where \( \bar{\alpha} \) is the complex conjugate of \( \alpha \). Hence, \( y \to \langle x, y \rangle \) is conjugate-linear for all \( x \in \mathcal{H} \). In the case when \( \mathbb{K} = \mathbb{R} \), it is in fact linear.

Exercise 14
Exercise 15.

1. The inner-product $\langle \cdot, \cdot \rangle$ has values in $K$. From $(iv)$ of definition $(81)$, $\langle x, x \rangle \geq 0$ for all $x \in H$. It follows that $\|x\| = \sqrt{\langle x, x \rangle}$ is a well-defined element of $R^+$, for all $x \in H$. Hence, we see that $A = \|x\|^2$ and $C = \|y\|^2$ are both well-defined elements of $R^+$. Furthermore, $B = |\langle x, y \rangle|$ being the modulus of an element of $K$, is a well-defined element of $R^+$.

2. Let $t \in R$. Using the linearity properties of exercise $(14)$:
   \[
   \langle x - t\alpha y, x - t\alpha y \rangle = \langle x, x \rangle - t\alpha \overline{\langle x, y \rangle} - t\alpha \langle x, y \rangle + t^2 \alpha \overline{\alpha} \langle y, y \rangle
   \]
   Since $B = \overline{B} = \alpha \langle x, y \rangle$ and $\alpha \overline{\alpha} = 1$, we conclude that:
   \[
   \langle x - t\alpha y, x - t\alpha y \rangle = A - 2tB + t^2C
   \]

3. Suppose $C = 0$. Then $\langle y, y \rangle = 0$. From $(v)$ of definition $(81)$, we see that $y = 0$. From the conjugate linearity of $y' \to \langle x, y' \rangle$, we have $\langle x, 0 \rangle = 0$ for all $x \in H$, and consequently $\langle x, y \rangle = 0$. So $B = 0$, and finally $B^2 \leq AC$. 

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4. Suppose $C \neq 0$. Let $P(t) = A - 2tB + t^2C$ for all $t \in \mathbb{R}$. Since $C > 0$ and $P'(t) = 2tC - 2B$, the second degree polynomial $P$ has a minimum value at $t = B/C$. From 2., for all $t \in \mathbb{R}$:

$$P(t) = \langle x - t\alpha y, x - t\alpha y \rangle \geq 0$$

In particular, $P(B/C) \geq 0$. It follows that $B^2 \leq AC$.

5. From $B^2 \leq AC$, since $A, B, C \in \mathbb{R}^+$, we obtain $B \leq \sqrt{AC}$, i.e.

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

This proves theorem (50).
Exercise 16.

1. Let \( f, g \in L^2_C(\Omega, \mathcal{F}, \mu) \). Then, \( f \bar{g} \) is a complex-valued and measurable map. Furthermore, from theorem (42):

\[
\int |f| \overline{|g|} \, d\mu \leq \left( \int |f|^2 \, d\mu \right)^{\frac{1}{2}} \left( \int |g|^2 \, d\mu \right)^{\frac{1}{2}}
\]

So \( \int |f \bar{g}| \, d\mu < +\infty \) and \( f \bar{g} \in L^1_C(\Omega, \mathcal{F}, \mu) \). It follows that \( \langle f, g \rangle = \int f \bar{g} \, d\mu \) is a well-defined complex number.

2. Let \( f \in L^2_C(\Omega, \mathcal{F}, \mu) \). From definition (73), \( \| f \|_2 \) is defined as

\[
\| f \|_2 = \left( \int |f|^2 \, d\mu \right)^{\frac{1}{2}} = \sqrt{\langle f, f \rangle}
\]

3. Let \( f, g \in L^2_C(\Omega, \mathcal{F}, \mu) \). From theorems (24) and (42), we have:

\[
|\langle f, g \rangle| = \left| \int f \bar{g} \, d\mu \right| \leq \int |f| |g| \, d\mu \leq \| f \|_2 \cdot \| g \|_2
\]
4. Among properties $(i) - (v)$ of definition (81), only $(v)$ fails to be satisfied. Indeed, although $f = 0$ does imply that $\langle f, f \rangle = \int |f|^2 d\mu = 0$, the converse is not true. Having $\int |f|^2 d\mu = 0$ only guarantees that $f = 0$ $\mu$-almost surely, and not necessarily everywhere. We conclude that $\langle \cdot, \cdot \rangle$ is not strictly speaking an inner-product on $L^2(\Omega, \mathcal{F}, \mu)$, as defined by definition (81). It follows that equation (1) which we proved in 3., cannot be viewed as a consequence of theorem (50).

5. Let $f, g \in L^2(\Omega, \mathcal{F}, \mu)$. Let $P(t) = \int (||f|+t||g||)^2 d\mu$ for all $t \in \mathbb{R}$. Then, $P(t) \geq 0$ for all $t \in \mathbb{R}$, and furthermore:

$$P(t) = A + 2tB + t^2C$$

where $A = \int |f|^2 d\mu$, $B = \int |f||g| d\mu$ and $C = \int |g|^2 d\mu$. All three numbers $A, B$ and $C$ are elements of $\mathbb{R}^+$.\(^{14}\) If $C = 0$, then $g = 0$ $\mu$-a.s. and consequently $B = 0$. In particular, the inequality $B^2 \leq AC$ holds. If $C \neq 0$, from $P(-B/C) \geq 0$ we

\(^{14}\)B can be shown to be finite from $|fg| \leq (|f|^2 + |g|^2)/2$.  

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obtain $B^2 \leq AC$, and consequently:

$$\int |fg|d\mu \leq \left( \int |f|^2 d\mu \right)^{\frac{1}{2}} \left( \int |g|^2 d\mu \right)^{\frac{1}{2}}$$

6. Let $f, g : (\Omega, \mathcal{F}) \rightarrow [0, +\infty]$ be non-negative and measurable. Suppose both integrals $\int f^2 d\mu$ and $\int g^2 d\mu$ are finite. Then $f$ and $g$ are $\mu$-almost surely finite, and therefore $\mu$-almost surely equal to $f1_{\{f<+\infty\}}$ and $g1_{\{g<+\infty\}}$ respectively. It follows that $f$ and $g$ are $\mu$-almost surely equal to elements of $L^2(\Omega, \mathcal{F}, \mu)$. Applying 5. to $f1_{\{f<+\infty\}}$ and $g1_{\{g<+\infty\}}$, we obtain:

$$\int fg d\mu \leq \left( \int f^2 d\mu \right)^{\frac{1}{2}} \left( \int g^2 d\mu \right)^{\frac{1}{2}}$$

If $\int f^2 d\mu = +\infty$ or $\int g^2 d\mu = +\infty$, such inequality still holds. We have effectively proved theorem (42), without using holder’s inequality (41).

Exercise 16

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Exercise 17.

1. Let \( x, y \in \mathcal{H} \). Using (ii) of definition (81), we have:

\[
\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x + y \rangle + \langle y, x + y \rangle
\]

Furthermore, using (i) and (ii):

\[
\langle x, x + y \rangle = \langle x + y, x \rangle = \langle x, x \rangle + \langle y, x \rangle = \|x\|^2 + \langle x, y \rangle
\]

and also:

\[
\langle y, x + y \rangle = \langle x + y, y \rangle = \|y\|^2 + \langle x, y \rangle
\]

We conclude that:

\[
\|x + y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle x, y \rangle
\]

2. From the Cauchy-Schwarz inequality of theorem (50):

\[
|\langle x, y \rangle| = |\langle x, y \rangle| \leq \|x\| \cdot \|y\|
\]
Consequently, using 1., we have:

\[ \|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \]

We conclude that for all \( x, y \in \mathcal{H} \), we have:

\[ \|x + y\| \leq \|x\| + \|y\| \]

3. Let \( \langle \cdot, \cdot \rangle \) be the map defined by \( d(x, y) = \|x - y\| \). Note that from (iv) of definition (81):

\[ d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle} \]

is well-defined, and non-negative. So \( d \) is indeed a map from \( \mathcal{H} \times \mathcal{H} \), with values in \( \mathbb{R}^+ \). Let \( x, y, z \in \mathcal{H} \). \( d(x, y) = 0 \) is equivalent to \( \langle x - y, x - y \rangle = 0 \), which from (v) of definition (81), is itself equivalent to \( x = y \). So (i) of definition (28) is satisfied by \( d \). Furthermore, we have:

\[ \| - x \|^2 = \langle -x, -x \rangle = -\langle -x, x \rangle = \| x \|^2 \]
and consequently, \( d(x, y) = \|x - y\| = \|y - x\| = d(y, x) \). So (\emph{ii}) of definition (28) is satisfied by \( d \). Finally, using 2.: 

\[
\|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\|
\]

and we see that \( d(x, y) \leq d(x, z) + d(z, y) \). So (\emph{iii}) of definition (28) is also satisfied by \( d \). Having checked conditions (\emph{i}), (\emph{ii}) and (\emph{iii}) of definition (28), we conclude that \( d \) is indeed a metric on \( \mathcal{H} \).

**Exercise 17**
Exercise 18.

1. \( M \) being a linear subspace of the \( K \)-vector space \( \mathcal{H} \), is itself a \( K \)-vector space. \([\cdot,\cdot]\) being the restriction of \( \langle \cdot,\cdot \rangle \) to \( M \times M \), is indeed a map \([\cdot,\cdot] : M \times M \to K \). For all \( x, y \in M \), we have:

\[
[x, y] = \langle x, y \rangle = \langle y, x \rangle = [y, x]
\]

So (i) of definition (81) is satisfied by \([\cdot,\cdot]\). Similarly, it is clear that all properties (ii) - (v) of definition (81) are also satisfied by \([\cdot,\cdot]\). We conclude that \([\cdot,\cdot]\) is indeed an inner-product on the \( K \)-vector space \( M \).

2. Recall that from definition (83), the metric \( d_{[\cdot,\cdot]} \) is defined by:

\[
d_{[\cdot,\cdot]}(x, y) = \sqrt{[x - y, x - y]}
\]

\([\cdot,\cdot]\) being the restriction of \( \langle \cdot,\cdot \rangle \) to \( M \times M \), we have:

\[
d_{[\cdot,\cdot]}(x, y) = \sqrt{\langle x - y, x - y \rangle} = d_{\langle \cdot,\cdot \rangle}(x, y)
\]
We conclude that the metric $d_{[\cdot,\cdot]}$ is nothing but the restriction of the metric $d_{(\cdot,\cdot)}$ to $\mathcal{M} \times \mathcal{M}$, i.e. $d_{[\cdot,\cdot]} = (d_{(\cdot,\cdot)})_{|\mathcal{M} \times \mathcal{M}}$.

3. From theorem (12), the topology induced on $\mathcal{M}$ by the norm topology $\mathcal{T}_{(\cdot,\cdot)}$ (the latter being the metric topology associated with $d_{(\cdot,\cdot)}$, by definition (82)), is nothing but the metric topology associated with $(d_{(\cdot,\cdot)})_{|\mathcal{M} \times \mathcal{M}} = d_{[\cdot,\cdot]}$ (which by definition (82), is the norm topology on $\mathcal{M}$, i.e. $\mathcal{T}_{[\cdot,\cdot]}$). So $(\mathcal{T}_{(\cdot,\cdot)})_{|\mathcal{M}} = \mathcal{T}_{[\cdot,\cdot]}$. 

Exercise 18
Exercise 19.

1. Since \((x_n)_{n \geq 1}\) is a Cauchy sequence in \(M\), with respect to the metric \(d_{[,]}\), from definition (79), for all \(\epsilon > 0\), there exists an integer \(n_0 \geq 1\), such that:

\[
n, m \geq n_0 \Rightarrow d_{[,]}(x_n, x_m) \leq \epsilon
\]

However, since \(d_{[,]}\) is the restriction of \(d_{\langle \cdot, \cdot \rangle}\) to \(M \times M\), we have \(d_{[,]}(x, y) = d_{\langle \cdot, \cdot \rangle}(x, y)\) for all \(x, y \in M\). It follows that \((x_n)_{n \geq 1}\) is also a Cauchy sequence in \(H\), with respect to the metric \(d_{\langle \cdot, \cdot \rangle}\).

2. \((H, \langle \cdot, \cdot \rangle)\) being a Hilbert space, from definition (83), \(H\) is also a complete metric space. From definition (80), \((x_n)_{n \geq 1}\) being a Cauchy sequence in \(H\), there exists \(x \in H\) such that \(x_n \rightarrow x\).

3. \(M\) is a closed subset of \(H\), and \((x_n)_{n \geq 1}\) is a sequence of elements of \(M\) converging to \(x \in H\). From exercise (13) [4.], we conclude that \(x \in M\).
4. As seen in the previous exercise, the norm topology $T_{\langle \cdot, \cdot \rangle}$ on $\mathcal{M}$ is induced by the norm topology $T_{\langle \cdot, \cdot \rangle}$ on $\mathcal{H}$. Since $(x_n)_{n \geq 1}$ is a sequence in $\mathcal{M}$ and $x \in \mathcal{M}$, the convergence $x_n \to x$ relative to the topology $T_{\langle \cdot, \cdot \rangle}$, is equivalent to the convergence $x_n \to x$ relative to the topology $T_{\langle \cdot, \cdot \rangle}$.

5. Given a Cauchy sequence $(x_n)_{n \geq 1}$ in $\mathcal{M}$, we have found an element $x \in \mathcal{M}$, such that $x_n \to x$. From definition (80), this shows that $(\mathcal{M}, d_{\langle \cdot, \cdot \rangle})$ is a complete metric space. It follows that $\mathcal{M}$ is a $K$-vector space, that $\langle \cdot, \cdot \rangle$ is an inner-product on $\mathcal{M}$, under which $\mathcal{M}$ is complete. From definition (83), we conclude that $(\mathcal{M}, \langle \cdot, \cdot \rangle) = (\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M} \times \mathcal{M}})$ is a Hilbert space over $K$. The purpose of this exercise is to show that any closed linear subspace of a Hilbert space, is itself a Hilbert space, together with its restricted inner-product.

Exercise 19
Exercise 20.

1. Let \( z, z', z'' \in \mathbb{C}^n \) and \( \alpha \in \mathbb{C} \). We have:

\[
\langle z, z' \rangle = \sum_{i=1}^{n} z_i \bar{z}'_i = \sum_{i=1}^{n} \bar{z}_i z'_i = \langle z', z \rangle
\]

\[
\langle z + z', z'' \rangle = \sum_{i=1}^{n} (z_i + z'_i) \bar{z}''_i = \langle z, z'' \rangle + \langle z', z'' \rangle
\]

\[
\langle \alpha z, z' \rangle = \sum_{i=1}^{n} (\alpha z_i) \bar{z}'_i = \alpha \langle z, z' \rangle
\]

\[
\langle z, z \rangle = \sum_{i=1}^{n} z_i \bar{z}_i = \sum_{i=1}^{n} |z_i|^2 \geq 0
\]

and finally, \( \langle z, z \rangle = 0 \) is equivalent to \( z_i = 0 \) for all \( i \in \mathbb{N}_n \), itself equivalent to \( z = 0 \). Hence, we see that all five conditions (i) – (v) of definition (81) are satisfied by \( \langle \cdot, \cdot \rangle \). So \( \langle \cdot, \cdot \rangle \) is indeed an inner-product on \( \mathbb{C}^n \).
2. The metric $d_{\langle \cdot, \cdot \rangle}$ is defined by:

$$d_{\langle \cdot, \cdot \rangle}(z, z') = \sqrt{\langle z - z', z - z' \rangle} = \sqrt{\sum_{i=1}^{n} |z_i - z'_i|^2}$$

It therefore coincides with the usual metric on $\mathbb{C}^n$.

3. From theorem (49), $\mathbb{C}^n$ is a complete metric space, with respect to its usual metric. The latter being the same as the metric $d_{\langle \cdot, \cdot \rangle}$, we conclude from definition (83) that $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space over $\mathbb{C}$.

4. For all $i \in \mathbb{N}_n$, let $\phi_i : \mathbb{C}^n \to \mathbb{R}$ be defined by $\phi_i(z) = Im(z_i)$. For all $z, z' \in \mathbb{C}^n$, we have:

$$|\phi_i(z) - \phi_i(z')| = |Im(z_i - z'_i)| \leq \|z - z'| = d_{\mathbb{C}^n}(z, z')$$

So each $\phi_i$ is a continuous map. The set $\{0\}$ being a closed subset of $\mathbb{R}$, the inverse image $\phi_i^{-1}(\{0\})$ is a closed subset of $\mathbb{C}^n$. 

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It follows that $\mathbb{R}^n = \cap_{i=1}^{n} \phi_i^{-1}(\{0\})$ as an intersection of closed subsets of $\mathbb{C}^n$, is itself a closed subset of $\mathbb{C}^n$.

5. Given $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{C}$, $\alpha \cdot x$ is not in general an element of $\mathbb{R}^n$. So $\mathbb{R}^n$ is not a linear subspace of $\mathbb{C}^n$. It is of course an $\mathbb{R}$-vector space.

6. Since $\mathbb{R}^n$ is not a linear subspace of $\mathbb{C}^n$, we cannot rely on exercise (19) to argue that $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space. In fact, we want to show that $\mathbb{R}^n$ is a Hilbert space over $\mathbb{R}$, (not $\mathbb{C}$), so exercise (19) is no good to us... However, the restriction of $\langle \cdot, \cdot \rangle$ to $\mathbb{R}^n \times \mathbb{R}^n$ also satisfies conditions (i) – (v) of definition (81), and is therefore an inner-product on $\mathbb{R}^n$, which furthermore induces the usual metric on $\mathbb{R}^n$. Since from theorem (49), $\mathbb{R}^n$ is complete with respect to its usual metric, we conclude from definition (83) that it is a Hilbert space over $\mathbb{R}$.

Exercise 20

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Exercise 21.

1. Since $\mathcal{C} \neq \emptyset$, there exists $y \in \mathcal{C}$. From $\delta_{\text{min}} \leq \|y - x_0\|$, we obtain $\delta_{\text{min}} < +\infty$. In particular, $\delta_{\text{min}} < \delta_{\text{min}} + 1/n$ for all $n \geq 1$. $\delta_{\text{min}}$ being the greatest of all lower-bound of $\|x - x_0\|$ for $x \in \mathcal{C}$, it follows that $\delta_{\text{min}} + 1/n$ cannot be such lower-bound. There exists $x_n \in \mathcal{C}$, such that $\|x_n - x_0\| < \delta_{\text{min}} + 1/n$. This being true for all $n \geq 1$, we have found a sequence $(x_n)_{n \geq 1}$ in $\mathcal{C}$, such that $\delta_{\text{min}} \leq \|x_n - x_0\| < \delta_{\text{min}} + 1/n$, for all $n \geq 1$. In particular, $\|x_n - x_0\| \to \delta_{\text{min}}$.

2. For all $x, y \in \mathcal{H}$:

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 + \|y\|^2 - \langle x, y \rangle - \langle x, y \rangle$$

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle x, y \rangle$$

and therefore:

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$$
or equivalently:

$$\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4 \left| \frac{x + y}{2} \right|^2$$  \hfill (6)

3. Let $n, m \geq 1$. $x_n$ and $x_m$ are both elements of $\mathcal{C}$. Since we have $1/2 \in [0, 1]$, from definition (85), $\mathcal{C}$ being convex, $(x_n + x_m)/2$ is also an element of $\mathcal{C}$. Since $\delta_{\text{min}}$ is a lower-bound of $\|x - x_0\|$ for $x \in \mathcal{C}$, we conclude that:

$$\delta_{\text{min}} \leq \left\| \frac{x_n + x_m}{2} - x_0 \right\|$$  \hfill (7)

4. Let $n, m \geq 1$. Applying (6) to $x = x_n - x_0$ and $y = x_m - x_0$:

$$\|x_n - x_m\|^2 = 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4 \left| \frac{x_n + x_m}{2} - x_0 \right|^2$$

and therefore, from (7):

$$\|x_n - x_m\|^2 \leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\delta_{\text{min}}^2$$  \hfill (8)
5. Let $\epsilon > 0$. Since $(x_n)_{n \geq 1}$ is such that $\|x_n - x_0\| \to \delta_{\text{min}}$, in particular, there exists $N \geq 1$ such that:

$$n \geq N \Rightarrow 2\|x_n - x_0\|^2 \leq 2\delta_{\text{min}}^2 + \epsilon^2/2$$

Using (8), we have:

$$n, m \geq N \Rightarrow \|x_n - x_m\|^2 \leq \epsilon^2$$

It follows from definition (79) that $(x_n)_{n \geq 1}$ is a Cauchy sequence in $\mathcal{H}$. Since $\mathcal{H}$ is a Hilbert space, it is also a complete metric space. So $(x_n)_{n \geq 1}$ has a limit in $\mathcal{H}$. There exists $x^* \in \mathcal{H}$, such that $x_n \to x^*$\textsuperscript{15}.

6. From 5., we have $x_n \to x^*$, while $(x_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{C}$. Since by assumption, $\mathcal{C}$ is a closed subset of $\mathcal{H}$, using exercise (13) [4.], we conclude that $x^* \in \mathcal{C}$.

\textsuperscript{15}Convergence relative to the norm topology, so $x_n \overset{\mathcal{T}_{\|\cdot\|}}{\to} x^*$.
7. Let \( x, y \in \mathcal{H} \). From exercise (17), we have:

\[
\|x\| \leq \|x - y\| + \|y\|
\]

\[
\|y\| \leq \|x - y\| + \|x\|
\]

where we have used the fact that \( \|x - y\| = \|y - x\| \). Hence:

\[
-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|
\]

or equivalently \( |\|x\| - \|y\|| \leq \|x - y\| \).

8. For all \( n \geq 1 \), from 7.6, we have:

\[
|\|x_n - x_0\| - \|x^* - x_0\|| \leq \|x^* - x_n\|
\]

Since \( x_n \to x^* \), \( \|x^* - x_n\| \to 0 \), and so \( \|x_n - x_0\| \to \|x^* - x_0\| \).

9. By construction, \( (x_n)_{n \geq 1} \) is such that \( \|x_n - x_0\| \to \delta_{\min} \). However, from 8., \( \|x_n - x_0\| \to \|x^* - x_0\| \). So \( \|x^* - x_0\| = \delta_{\min} \).

Since \( x^* \in \mathcal{C} \), we have found \( x^* \in \mathcal{C} \), such that:

\[
\|x^* - x_0\| = \inf\{\|x - x_0\| : x \in \mathcal{C}\}
\]
10. Suppose \( y^* \) is another element of \( C \), such that:

\[
\| y^* - x_0 \| = \inf \{ \| x - x_0 \| : x \in C \}
\]

Applying (6) to \( x = x^* - x_0 \) and \( y = y^* - x_0 \), we obtain:

\[
\| x^* - y^* \|^2 = 2\| x^* - x_0 \|^2 + 2\| y^* - x_0 \|^2 - 4\left\| \frac{x^* + y^*}{2} - x_0 \right\|^2
\]

Since \( C \) is convex and \( x^*, y^* \) are elements of \( C \), \( \frac{x^* + y^*}{2} \) is also an element of \( C \). It follows that:

\[
\delta_{\text{min}} \leq \left\| \frac{x^* + y^*}{2} - x_0 \right\|
\]

and finally \( \| x^* - y^* \|^2 \leq 2\| x^* - x_0 \|^2 + 2\| y^* - x_0 \|^2 - 4\delta_{\text{min}}^2 \).

11. Since \( \delta_{\text{min}} = \| x^* - x_0 \| = \| y^* - x_0 \| \), we see from 10. that \( \| x^* - y^* \| = 0 \), and finally \( x^* = y^* \). This proves theorem (52).

Exercise 21

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Exercise 22.

1. For all \( y \in \mathcal{G} \), \( \langle 0, y \rangle = 0 \). So \( 0 \in \mathcal{G}^\perp \) and in particular \( \mathcal{G}^\perp \neq \emptyset \). Let \( x_1, x_2 \in \mathcal{G}^\perp \) and \( \alpha \in \mathbb{K} \). For all \( y \in \mathcal{G} \), we have \( \langle x_1, y \rangle = 0 \) and \( \langle x_2, y \rangle = 0 \). Hence:

\[
\langle x_1 + \alpha x_2, y \rangle = \langle x_1, y \rangle + \alpha \langle x_2, y \rangle = 0
\]

This being true for all \( y \in \mathcal{G} \), \( x_1 + \alpha x_2 \in \mathcal{G}^\perp \). We conclude that \( \mathcal{G}^\perp \) is a linear sub-space of \( \mathcal{H} \). Note that no assumption was made, as to whether \( \mathcal{G} \) is itself a linear sub-space or not.

2. Given \( y \in \mathcal{H} \), let \( \phi_y : \mathcal{H} \to \mathbb{K} \) be defined by \( \phi_y(x) = \langle x, y \rangle \). From the Cauchy-Schwarz inequality of theorem (50), if \( x_1, x_2 \in \mathcal{H} \), we have \( |\phi_y(x_1) - \phi_y(x_2)| = |\langle x_1 - x_2, y \rangle| \leq \|y\| \|x_1 - x_2\| \) or equivalently \( d_\mathbb{K}(\phi_y(x_1), \phi_y(x_2)) \leq \|y\| d_{\langle \cdot, \cdot \rangle}(x_1, x_2) \), where \( d_\mathbb{K} \) is the usual metric on \( \mathbb{K} \). It follows that \( \phi_y : \mathcal{H} \to \mathbb{K} \) is a continuous map, with respect to the norm topology on \( \mathcal{H} \), and the usual topology on \( \mathbb{K} \).
3. Suppose \( x \in \mathcal{G}^\perp \). For all \( y \in \mathcal{G} \), we have \( \langle x, y \rangle = 0 = \phi_y(x) \). So \( x \in \cap_{y \in \mathcal{G}} \phi_y^{-1}(\{0\}) \). Conversely, if \( x \in \cap_{y \in \mathcal{G}} \phi_y^{-1}(\{0\}) \), then for all \( y \in \mathcal{G} \), we have \( \phi_y(x) = 0 = \langle x, y \rangle \), and therefore \( x \in \mathcal{G}^\perp \). This proves that \( \mathcal{G}^\perp = \cap_{y \in \mathcal{G}} \phi_y^{-1}(\{0\}) \).

4. The set \( \{0\} \) is a closed subset of \( \mathbf{K} \). Since \( \phi_y : \mathcal{H} \rightarrow \mathbf{K} \) is a continuous map for all \( y \in \mathcal{H} \), the inverse image \( \phi_y^{-1}(\{0\}) \) is a closed subset of \( \mathcal{H} \). From 3., \( \mathcal{G}^\perp \) being an arbitrary intersection of closed subsets of \( \mathcal{H} \), we conclude that \( \mathcal{G}^\perp \) is itself a closed subset of \( \mathcal{H} \).

5. \( \emptyset^\perp \subseteq \mathcal{H} \) and \( \{0\}^\perp \subseteq \mathcal{H} \) are obviously true. Furthermore, a statement such that \( \forall y \in \emptyset, \langle x, y \rangle = 0 \) is also true for any \( x \in \mathcal{H} \). So \( \mathcal{H} \subseteq \emptyset^\perp \). Moreover, for all \( x \in \mathcal{H} \), \( \langle x, 0 \rangle = 0 \), i.e. \( x \in \{0\}^\perp \). So \( \mathcal{H} \subseteq \{0\}^\perp \). We have proved that \( \mathcal{H} = \emptyset^\perp = \{0\}^\perp \).

6. For all \( y \in \mathcal{H} \), \( \langle 0, y \rangle = 0 \). So \( \{0\} \subseteq \mathcal{H}^\perp \). Conversely, if \( x \in \mathcal{H}^\perp \), then \( \langle x, x \rangle = 0 \) and therefore \( x = 0 \). So \( \mathcal{H}^\perp \subseteq \{0\} \).

Exercise 22
Exercise 23.

1. $\mathcal{M}$ being a linear sub-space of $\mathcal{H}$, it has at least one element, namely 0. So $\mathcal{M} \neq \emptyset$. Furthermore, for all $x, y \in \mathcal{M}$ and $\alpha, \beta \in \mathbb{K}$, we have $\alpha x + \beta y \in \mathcal{M}$. In particular, for all $t \in [0, 1]$, $tx + (1-t)y \in \mathcal{M}$. From definition (85), it follows that $\mathcal{M}$ is also a convex subset of $\mathcal{H}$. Having assumed $\mathcal{M}$ to be closed, it is therefore a non-empty, closed and convex subset of $\mathcal{H}$. Applying theorem (52), there exists $x^* \in \mathcal{M}$ such that:

$$\|x^* - x_0\| = \inf\{\|x - x_0\| : x \in \mathcal{M}\}$$

2. Let $y^* = x_0 - x^*$. Since $x^* \in \mathcal{M}$, for all $y \in \mathcal{M}$ and $\alpha \in \mathbb{K}$, $x^* + \alpha y$ is also an element of $\mathcal{M}$. It follows that:

$$\|x^* - x_0\| \leq \|x^* + \alpha y - x_0\|$$

or equivalently:

$$\|y^*\|^2 \leq \|y^* - \alpha y\|^2 \quad (9)$$
3. Let $y \in \mathcal{M}$ and $\alpha \in \mathbb{K}$. We have:
\[
\|y^* - \alpha y\|^2 = \|y^*\|^2 - \alpha \langle y, y^* \rangle - \overline{\alpha \langle y, y^* \rangle} + |\alpha|^2 \|y\|^2
\]
Hence, using (9), we obtain:
\[
0 \leq -\alpha \langle y, y^* \rangle - \overline{\alpha \langle y, y^* \rangle} + |\alpha|^2 \|y\|^2
\]
(10)

4. Given $y \in \mathcal{M} \setminus \{0\}$, take $\alpha = \frac{\langle y, y^* \rangle}{\|y\|^2}$ in (10). We obtain:
\[
0 \leq -\left| \frac{\langle y, y^* \rangle}{\|y\|^2} \right|^2
\]

5. It follows from 4. that $|\langle y, y^* \rangle|^2 \leq 0$ for all $y \in \mathcal{M} \setminus \{0\}$. So $\langle y^*, y \rangle = \langle y, y^* \rangle = 0$, for all $y \in \mathcal{M} \setminus \{0\}$. Since $\langle y^*, 0 \rangle = 0$, we in fact have $\langle y^*, y \rangle = 0$ for all $y \in \mathcal{M}$, and we see that $y^* \in \mathcal{M}^\perp$.
So $x^* \in \mathcal{M}$, $y^* \in \mathcal{M}^\perp$, and since $y^* = x_0 - x^*$, we conclude that $x_0 = x^* + y^*$.

6. $\mathcal{M}$ and $\mathcal{M}^\perp$ being linear sub-spaces of $\mathcal{H}$, 0 is an element of both $\mathcal{M}$ and $\mathcal{M}^\perp$. So $\{0\} \subseteq \mathcal{M} \cap \mathcal{M}^\perp$. Conversely, suppose
From $x \in M \cap M^\perp$, we have $\langle x, y \rangle = 0$ for all $y \in M$. From $x \in M$, we see in particular that $\langle x, x \rangle = 0$. From (v) of definition (81), we conclude that $x = 0$. So $M \cap M^\perp = \{0\}$.

7. Suppose there exist $\bar{x} \in M$ and $\bar{y} \in M^\perp$, such that $x_0 = \bar{x} + \bar{y}$. Then $x^* + y^* = \bar{x} + \bar{y}$ and consequently $x^* - \bar{x} = \bar{y} - y^*$, while $x^* - \bar{x} \in M$ and $\bar{y} - y^* \in M^\perp$. Since $M \cap M^\perp = \{0\}$, we conclude that $x^* = \bar{x}$ and $y^* = \bar{y}$. So $x^* \in M$ and $y^* \in M^\perp$ such that $x_0 = x^* + y^*$ are unique. This proves theorem (53).

Exercise 23
Exercise 24.

1. Let $\lambda : \mathcal{H} \to \mathbf{K}$ be a linear functional, which is continuous at $x_0 \in \mathcal{H}^{16}$. Given an open set $V$ in $\mathbf{K}$ containing $\lambda(x_0)$, there exists an open set $U$ in $\mathcal{H}$ containing $x_0$, such that $f(U) \subseteq V$. Since the two topologies on $\mathcal{H}$ and $\mathbf{K}$ are metric, this is easily shown to be equivalent to the property that for all $\epsilon > 0$, there exists $\delta > 0$, such that:

$$\forall x \in \mathcal{H} \ , \ ||x - x_0|| < \delta \Rightarrow |\lambda(x) - \lambda(x_0)| < \epsilon$$

In particular, taking $\epsilon = 1$ and some $\eta > 0$ strictly smaller than the associated $\delta$, we have:

$$\forall x \in \mathcal{H} \ , \ ||x - x_0|| \leq \eta \Rightarrow |\lambda(x) - \lambda(x_0)| \leq 1$$

Hence, given $x \in \mathcal{H}, x \neq 0$, we have:

$$|\lambda(\eta x/||x||)| = |\lambda(x_0 + \eta x/||x||) - \lambda(x_0)| \leq 1$$

$^{16}$Continuity at a given point is defined in what follows.
2. If \( \lambda \) is continuous at some \( x_0 \in \mathcal{H} \), from 1., there exists \( \eta > 0 \) such that \( |\lambda(\eta x/\|x\|)| \leq 1 \) for all \( x \in \mathcal{H} \setminus \{0\} \). So \( |\lambda(x)| \leq \|x\|/\eta \) for all \( x \in \mathcal{H} \setminus \{0\} \), which is obviously still valid if \( x = 0 \). We have found \( M = 1/\eta \in \mathbb{R}^+ \), such that:

\[
\forall x \in \mathcal{H} , \ |\lambda(x)| \leq M \|x\| \quad (11)
\]

3. Suppose \( \lambda : \mathcal{H} \to K \) is a linear functional, such that (11) holds for some \( M \in \mathbb{R}^+ \). Then for all \( x_1, x_2 \in \mathcal{H} \), we have:

\[
|\lambda(x_1) - \lambda(x_2)| = |\lambda(x_1 - x_2)| \leq M \|x_1 - x_2\|
\]

So \( \lambda \) is continuous (everywhere).

Exercise 24
Exercise 25.

1. Let \( x_0 \in \mathcal{H} \) such that \( \lambda(x_0) \neq 0 \). Then \( x_0 \notin \mathcal{M} = \lambda^{-1}(\{0\}) \).

2. \( \mathcal{M} = \lambda^{-1}(\{0\}) \) is a linear sub-space of \( \mathcal{H} \). Indeed, it is not empty (\( \lambda(0) = 0 \)), and if \( \lambda(x_1) = \lambda(x_2) = 0 \) and \( \alpha \in \mathbb{K} \), then:
   
   \[ \lambda(x_1 + \alpha x_2) = \lambda(x_1) + \alpha \lambda(x_2) = 0 \]

   Furthermore, \( \lambda \) being a bounded linear functional, is continuous, and \( \mathcal{M} = \lambda^{-1}(\{0\}) \) is therefore a closed subset of \( \mathcal{H} \). So \( \mathcal{M} \) is a closed linear sub-space of \( \mathcal{H} \). From theorem (53), there exists \( x^* \in \mathcal{M}, y^* \in \mathcal{M}^\perp \), such that \( x_0 = x^* + y^* \).

3. Since \( x^* \in \mathcal{M} \), \( \lambda(y^*) = \lambda(x_0) \) and therefore \( \lambda(y^*) \neq 0 \). In particular, \( y^* \neq 0 \). Taking \( z = y^*/\|y^*\| \), we have found \( z \in \mathcal{M}^\perp \), such that \( \|z\| = 1 \).

4. Let \( \alpha \in \mathbb{K} \setminus \{0\} \). We have \( \langle z, \alpha z \rangle / \alpha = \langle z, (\alpha z) / \alpha \rangle = \langle z, z \rangle = 1 \).

   It follows that \( \lambda(x) \langle z, \alpha z \rangle / \alpha = \lambda(x) \) for all \( x \in \mathcal{H} \).
5. In order to have \( \lambda(x) = \langle x, \alpha z \rangle \) for all \( x \in H \), we need:

\[
0 = \lambda(x) - \langle x, \alpha z \rangle = \lambda(x)z//\lambda - \langle x, \alpha z \rangle = \langle \lambda(x)z/\lambda - x, \alpha z \rangle
\]

Since \( z \in M^\perp \), it is sufficient to choose \( \alpha \in K \setminus \{0\} \), with:

\[
\forall x \in H, \quad \frac{\lambda(x)z}{\lambda} - x \in M
\]

(12)

6. Since \( M = \lambda^{-1}(\{0\}) \), property (12) is equivalent to:

\[
0 = \lambda \left( \frac{\lambda(x)z}{\lambda} - x \right) = \lambda(x)\lambda(z)/\lambda - \lambda(x)
\]

for all \( x \in H \), which is satisfied for \( \alpha = \lambda(z) \), provided \( \lambda(z) \neq 0 \). But if \( \lambda(z) = 0 \), then \( z \in M \). So \( z \in M \cap M^\perp \) and \( \langle z, z \rangle = 0 \), contradicting the fact that \( \|z\| = 1 \). Hence, if we take \( \alpha = \lambda(z) \), then condition (12) is satisfied, and therefore \( \lambda(x) = \langle x, \alpha z \rangle \) for all \( x \in H \). Taking \( y = \alpha z = \lambda(z)z \), we have found \( y \in H \), with:

\[
\forall x \in H, \quad \lambda(x) = \langle x, y \rangle
\]

(13)
In case one has any doubt about (13), one can quickly check:

\[
\lambda(x) - \langle x, \lambda(z)z \rangle = \lambda(x)\langle z, z \rangle - \lambda(z)\langle x, z \rangle = \langle \lambda(x)z - \lambda(z)x, z \rangle = 0
\]

the last equality arising from \( \lambda(x)z - \lambda(z)x \in M \) and \( z \in M^\perp \).

7. Suppose \( \bar{y} \in H \) is such that \( \lambda(x) = \langle x, \bar{y} \rangle \) for all \( x \in H \). Then \( \langle x, y - \bar{y} \rangle = 0 \) for all \( x \in H \), and in particular \( \|y - \bar{y}\|^2 = 0 \), i.e. \( \bar{y} = y \). So \( y \in H \) satisfying (13) is unique. This proves theorem (54) \(^{17}\).

Exercise 25

\(^{17}\)The case \( \lambda = 0 \) is easy to handle.
Exercise 26.

1. Suppose $f = g \mu$-a.s. For all $h \in [f]$, we have $h = f \mu$-a.s. and therefore $h = g \mu$-a.s., i.e. $h \in [g]$. So $[f] \subseteq [g]$, and similarly $[g] \subseteq [f]$. Conversely, if $[f] = [g]$, then in particular $f \in [g]$ and therefore $f = g \mu$-a.s. We have proved that $f = g \mu$-a.s. is equivalent to $[f] = [g]$.

2. Suppose $[f] = [f']$ and $[g] = [g']$. Then $f = f' \mu$-a.s. and $g = g' \mu$-a.s. So $f + g = f' + g' \mu$-a.s. and $[f + g] = [f' + g']$.

3. $\oplus$ is defined as $[f] \oplus [g] = [f + g]$. This definition may not be legitimate, as $[f] \oplus [g]$ is defined in terms of particular representatives $f$ and $g$ of the equivalence classes $[f]$ and $[g]$. Since such representative are normally far from being unique, this may lead to different values of $[f + g]$, as $f$ and $g$ range over all possible choices. However, as shown in 2., $[f + g]$ is in fact independent of the particular choice of $f \in [f]$ and $g \in [g]$. So $[f] \oplus [g]$ is unambiguously defined, i.e. the operator $\oplus$ is well-defined.
4. Let $\alpha \in \mathbb{K}$. If $[f] = [f']$, then $f = f'$ $\mu$-a.s. and $\alpha f = \alpha f'$ $\mu$-a.s. So $[\alpha f] = [\alpha f']$. It follows that $[\alpha f]$ is independent of the particular choice of $f \in [f]$. So $\alpha \otimes [f]$ is unambiguously defined, i.e. the operator $\otimes$ is well-defined.

5. For all $[f], [g], [h] \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{K}$, we have:

   \begin{enumerate}[(i)]
   
   \item $[0] \oplus [f] = [0 + f] = [f]$
   
   \item $[-f] \oplus [f] = [-f + f] = [0]$
   
   \item $[f] \oplus ([g] \oplus [h]) = [f + g + h] = ([f] \oplus [g]) \oplus [h]$
   
   \item $[f] \oplus [g] = [f + g] = [g] \oplus [f]$
   
   \item $1 \otimes [f] = [1.f] = [f]$
   
   \item $\alpha \otimes (\beta \otimes [f]) = [\alpha \beta f] = (\alpha \beta) \otimes [f]$
   
   \item $(\alpha + \beta) \otimes [f] = [\alpha f + \beta f] = (\alpha \otimes [f]) \oplus (\beta \otimes [f])$
   
   \item $\alpha \otimes ([f] \oplus [g]) = [\alpha f + \alpha g] = (\alpha \otimes [f]) \oplus (\alpha \otimes [g])$
   
   \end{enumerate}
Exercise 27.

1. Suppose \([f] = [f']\) and \([g] = [g']\). Then \(f = f'\) \(\mu\)-a.s. and \(g = g'\) \(\mu\)-a.s. So \(fg = f'g'\) \(\mu\)-a.s. and therefore:

\[
\int f g d\mu = \int f' g' d\mu \tag{14}
\]

It follows that (14) is independent of the order of choice of \(f \in [f]\) and \(g \in [g]\). We conclude that \(\langle [f], [g] \rangle_\mathcal{H}\) is unambiguously defined, i.e. \(\langle \cdot, \cdot \rangle_\mathcal{H}\) is well-defined.

2. Let \([f], [g] \in \mathcal{H}\), \(\alpha \in \mathbb{K}\) and \(\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\mathcal{H}\). We have:

(i) \(\langle [f], [g] \rangle = \int f g d\mu = \langle [g], [f] \rangle\)

(ii) \(\langle [f] \oplus [g], [h] \rangle = \int (f + g) h d\mu = \langle [f], [h] \rangle + \langle [g], [h] \rangle\)

(iii) \(\langle \alpha \otimes [f], [g] \rangle = \int (\alpha f) g d\mu = \alpha \langle [f], [g] \rangle\)

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(iv) \( \langle [f], [f] \rangle = \int |f|^2 d\mu \in \mathbb{R}^+ \)

and finally, \( \langle [f], [f] \rangle = 0 \) is equivalent to \( \int |f|^2 d\mu = 0 \), which is in turn equivalent to \( f = 0 \) \( \mu \)-a.s., i.e. \( [f] = [0] \). From definition (81), we conclude that \( \langle \cdot, \cdot \rangle \) is an inner-product on \( \mathcal{H} \).

3. \( \mathcal{H} \) is a \( K \)-vector space, and \( \langle \cdot, \cdot \rangle_\mathcal{H} \) is an inner-product on \( \mathcal{H} \). From definition (83), to show that \( (\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}) \) is a Hilbert space over \( K \), we need to prove that \( \mathcal{H} \) is in fact complete with respect to the metric induced by the inner-product. Let \( ([f_n])_{n\geq1} \) be a Cauchy sequence in \( \mathcal{H} \). For all \( \epsilon > 0 \), there exists \( n_0 \geq 1 \) with:

\[
 n, m \geq n_0 \Rightarrow \| [f_n] - [f_m] \|_\mathcal{H} \leq \epsilon^{18}
\]

However, for all \( f \in L^2_K(\Omega, \mathcal{F}, \mu) \), we have:

\[
 \| [f] \|_\mathcal{H} = (\langle [f], [f] \rangle_\mathcal{H})^{\frac{1}{2}} = \left( \int |f|^2 d\mu \right)^{\frac{1}{2}} = \| f \|_2
\]

\( [f_n] - [f_m] \) is a light notation to indicate \( [f_n] \oplus [-f_m] \).

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Solutions to Exercises 106

It follows that $(f_n)_{n \geq 1}$ is a Cauchy sequence in $L^2_K(\Omega, \mathcal{F}, \mu)$. From theorem (46), there exists $f \in L^2_K(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ in $L^2$. In other words, for all $\epsilon > 0$, there exists $n_0 \geq 1$, such that:

$$n \geq n_0 \Rightarrow \|f_n - f\|_2 \leq \epsilon$$

Since $\|f_n - f\|_2 = \|[f_n] - [f]\|_{\mathcal{H}}$, we conclude that $[f_n] \to [f]$ with respect to the norm topology on $\mathcal{H}$. Having found a limit for the Cauchy sequence $([f_n])_{n \geq 1}$, we have proved that $\mathcal{H}$ is complete, and $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is finally a Hilbert space over $K$.

4. $\langle f, g \rangle = \int fg d\mu$ is not an inner-product on $L^2_K(\Omega, \mathcal{F}, \mu)$, as property (v) of definition (81) fails to be satisfied. If $\langle f, f \rangle = 0$, then we know for sure that $f = 0$ $\mu$-a.s. There is no reason why $f$ should be 0 everywhere. This is the very reason why in this exercise, we go through so much trouble considering the quotient set $\mathcal{H} = (L^2_K(\Omega, \mathcal{F}, \mu))_{\mathcal{R}}$, where $\mathcal{R}$ is the $\mu$-a.s. equivalence relation on $L^2_K(\Omega, \mathcal{F}, \mu)$.

Exercise 27
1. Since $L^2_K(\Omega, \mathcal{F}, \mu)$ is not a Hilbert space, we cannot use exercise (24) in its literal form. However, most of what we did then, can be reproduced here. Let $\lambda : L^2_K(\Omega, \mathcal{F}, \mu) \to K$ be a continuous linear functional. The open ball $B(0, 1) = \{z \in K : |z| < 1\}$ being open in $K$, the inverse image $\lambda^{-1}(B(0, 1))$ is an open subset of $L^2_K(\Omega, \mathcal{F}, \mu)$. Since $0 \in \lambda^{-1}(B(0, 1))$, there exists $\delta > 0$, such that $B(0, \delta) \subseteq \lambda^{-1}(B(0, 1))$, where $B(0, \delta)$ is the open ball in $L^2_K(\Omega, \mathcal{F}, \mu)$. Taking an arbitrary $\eta > 0$, strictly smaller than $\delta$, for all $f \in L^2_K(\Omega, \mathcal{F}, \mu)$, we have:

$$
\|f\|_2 \leq \eta \Rightarrow |\lambda(f)| \leq 1
$$

It follows that $|\lambda(\eta f/\|f\|_2)| \leq 1$ for all $f \in L^2_K(\Omega, \mathcal{F}, \mu)$, $f \neq 0$, and finally:

$$
\forall f \in L^2_K(\Omega, \mathcal{F}, \mu) \ , \ |\lambda(f)| \leq \frac{1}{\eta}\|f\|_2 \quad (15)
$$
2. If \([f] = [g]\), then \(f - g = 0\) \(\mu\)-a.s. and \(\|f - g\|_2 = 0\). It follows from (15) that \(\lambda(f) = \lambda(g)\).

3. \(\Lambda : \mathcal{H} \rightarrow \mathbb{K}\) is defined by \(\Lambda([f]) = \lambda(f)\). Since \(\lambda(f)\) is independent of the particular choice of \(f \in [f]\), \(\Lambda([f])\) is unambiguously defined, i.e. \(\Lambda\) is well-defined. For all \([f], [g] \in \mathcal{H}\) and \(\alpha \in \mathbb{K}\):

\[
\Lambda([f] \oplus (\alpha \otimes [g])) = \Lambda([f + \alpha g]) = \lambda(f) + \alpha \lambda(g) = \Lambda([f]) + \alpha \Lambda([g])
\]

So \(\Lambda\) is a linear functional on \(\mathcal{H}\). Furthermore, since we have \(\|f\|_{\mathcal{H}} = \|f\|_2\) for all \(f \in L^2_K(\Omega, \mathcal{F}, \mu)\), we obtain immediately from (15) that:

\[
\forall [f] \in \mathcal{H}, \quad |\Lambda([f])| \leq \frac{1}{\eta} \|f\|_{\mathcal{H}}
\]

and we conclude from definition (88) that \(\Lambda\) is a well-defined bounded linear functional on \(\mathcal{H}\).

4. Let \(\lambda : L^2_K(\Omega, \mathcal{F}, \mu) \rightarrow \mathbb{K}\) be a continuous linear functional. Then from 3., \(\Lambda : \mathcal{H} \rightarrow \mathbb{K}\) defined by \(\Lambda([f]) = \lambda(f)\) is a
bounded linear functional on the Hilbert space $\mathcal{H}$. Applying theorem (54), there exists $[g] \in \mathcal{H}$, such that:

$$\forall [f] \in \mathcal{H}, \Lambda([f]) = \langle [f], [g] \rangle_{\mathcal{H}}$$

It follows that:

$$\forall f \in L^2_K(\Omega, \mathcal{F}, \mu), \lambda(f) = \int f \bar{g} \, d\mu$$

This proves theorem (55).

Exercise 28