9. \(L^p\)-spaces, \(p \in [1, +\infty]\)

In the following, \((\Omega, \mathcal{F}, \mu)\) is a measure space.

**Exercise 1.** Let \(f, g : (\Omega, \mathcal{F}) \to [0, +\infty] \) be non-negative and measurable maps. Let \(p, q \in \mathbb{R}^+\), such that \(1/p + 1/q = 1\).

1. Show that \(1 < p < +\infty\) and \(1 < q < +\infty\).
2. For all \(x \in [0, +\infty]\), we define \(\phi^\alpha : [0, +\infty] \to [0, +\infty] \) by:
   \[
   \phi^\alpha(x) = \begin{cases} 
   x^\alpha & \text{if } x \in \mathbb{R}^+ \\
   +\infty & \text{if } x = +\infty
   \end{cases}
   \]
   Show that \(\phi^\alpha\) is a continuous map.
3. Define \(A = (\int f^p d\mu)^{1/p}\), \(B = (\int g^q d\mu)^{1/q}\) and \(C = \int f g d\mu\). Explain why \(A, B\) and \(C\) are well defined elements of \([0, +\infty]\).
4. Show that if \(A = 0\) or \(B = 0\) then \(C \leq A B\).
5. Show that if \(A = +\infty\) or \(B = +\infty\) then \(C \leq A B\).
6. We assume from now on that \(0 < A < +\infty\) and \(0 < B < +\infty\). Define \(F = f/A\) and \(G = g/B\). Show that:
   \[
   \int_\Omega F^p d\mu = \int_\Omega G^p d\mu = 1
   \]
7. Let \(a, b \in [0, +\infty]\). Show that:
   \[
   \ln(a) + \ln(b) \leq \ln\left( \frac{1}{p} a^p + \frac{1}{q} b^q \right)
   \]
   and:
   \[
   ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q
   \]
   Prove this last inequality for all \(a, b \in [0, +\infty]\).
8. Show that for all \(\omega \in \Omega\), we have:
   \[
   F(\omega) G(\omega) \leq \frac{1}{p} (F(\omega))^p + \frac{1}{q} (G(\omega))^q
   \]
9. Show that \(C \leq A B\).

**Theorem 41 (Holder inequality)** Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and \(f, g : (\Omega, \mathcal{F}) \to [0, +\infty] \) be two non-negative and measurable maps. Let \(p, q \in \mathbb{R}^+\) be such that \(1/p + 1/q = 1\). Then:

\[
\int_\Omega f g d\mu \leq \left( \int_\Omega f^p d\mu \right)^{\frac{1}{p}} \left( \int_\Omega g^q d\mu \right)^{\frac{1}{q}}
\]

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Theorem 42 (Cauchy-Schwarz inequality: first)
Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and \(f, g : (\Omega, \mathcal{F}) \to [0, +\infty]\) be two non-negative and measurable maps. Then:
\[
\int f g d\mu \leq \left( \int f^2 d\mu \right)^{\frac{1}{2}} \left( \int g^2 d\mu \right)^{\frac{1}{2}}
\]

Exercise 2. Let \(f, g : (\Omega, \mathcal{F}) \to [0, +\infty]\) be two non-negative and measurable maps. Let \(p \in [1, +\infty]\). Define \(A = (\int f^p d\mu)^{1/p}\) and \(B = (\int g^p d\mu)^{1/p}\) and \(C = (\int (f + g)^p d\mu)^{1/p}\).

1. Explain why \(A, B\) and \(C\) are well defined elements of \([0, +\infty]\).
2. Show that for all \(a, b \in [0, +\infty]\), we have:
   \[(a + b)^p \leq 2^{p-1}(a^p + b^p)\]
   Prove this inequality for all \(a, b \in [0, +\infty]\).
3. Show that if \(A = +\infty\) or \(B = +\infty\) or \(C = 0\) then \(C \leq A + B\).
4. Show that if \(A < +\infty\) and \(B < +\infty\) then \(C < +\infty\).
5. We assume from now that \(A, B \in [0, +\infty]\) and \(C \in [0, +\infty]\). Show the existence of some \(q \in \mathbb{R}^+\) such that \(1/p + 1/q = 1\).
6. Show that for all \(a, b \in [0, +\infty]\), we have:
   \[(a + b)^p = (a + b)(a + b)^{p-1}\]
7. Show that:
   \[
   \int f (f + g)^{p-1} d\mu \leq AC^{\frac{p}{q}}
   \]
   \[
   \int g (f + g)^{p-1} d\mu \leq BC^{\frac{p}{q}}
   \]
8. Show that:
   \[
   \int (f + g)^p d\mu \leq C^{\frac{p}{q}} (A + B)
   \]
9. Show that \(C \leq A + B\).
10. Show that the inequality still holds if we assume that \(p = 1\).

Theorem 43 (Minkowski inequality) Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and \(f, g : (\Omega, \mathcal{F}) \to [0, +\infty]\) be two non-negative and measurable maps. Let \(p \in [1, +\infty]\). Then:
\[
\left( \int (f + g)^p d\mu \right)^{\frac{1}{p}} \leq \left( \int f^p d\mu \right)^{\frac{1}{p}} + \left( \int g^p d\mu \right)^{\frac{1}{p}}
\]
Definition 73  The $L^p$-spaces, $p \in [1, +\infty]$, on $(\Omega, \mathcal{F}, \mu)$, are:

\[
L^p_R(\Omega, \mathcal{F}, \mu) \triangleq \left\{ f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ measurable}, \int_{\Omega} |f|^p d\mu < +\infty \right\}
\]

\[
L^p_C(\Omega, \mathcal{F}, \mu) \triangleq \left\{ f : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \text{ measurable}, \int_{\Omega} |f|^p d\mu < +\infty \right\}
\]

For all $f \in L^p_C(\Omega, \mathcal{F}, \mu)$, we put:

\[
\| f \|_p \triangleq \left( \int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}}
\]

Exercise 3. Let $p \in [1, +\infty]$. Let $f, g \in L^p_C(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^p_R(\Omega, \mathcal{F}, \mu) = \{ f \in L^p_C(\Omega, \mathcal{F}, \mu) \mid f(\Omega) \subseteq \mathbb{R} \}$.
2. Show that $L^p_R(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbb{R}$-linear combinations.
3. Show that $L^p_C(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbb{C}$-linear combinations.
4. Show that $\| f + g \|_p \leq \| f \|_p + \| g \|_p$.
5. Show that $\| f \|_p = 0 \iff f = 0 \mu$-a.s.
6. Show that for all $\alpha \in \mathbb{C}$, $\| \alpha f \|_p = |\alpha| \cdot \| f \|_p$.
7. Explain why $(f, g) \rightarrow \| f - g \|_p$ is not a metric on $L^p_C(\Omega, \mathcal{F}, \mu)$

Definition 74  For all $f : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ measurable, Let:

\[
\| f \|_{\infty} \triangleq \inf\{ M \in \mathbb{R}^+ \mid |f| \leq M \ \mu\text{-a.s.} \}
\]

The $L^\infty$-spaces on a measure space $(\Omega, \mathcal{F}, \mu)$ are:

\[
L^\infty_R(\Omega, \mathcal{F}, \mu) \triangleq \left\{ f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ measurable}, \| f \|_{\infty} < +\infty \right\}
\]

\[
L^\infty_C(\Omega, \mathcal{F}, \mu) \triangleq \left\{ f : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \text{ measurable}, \| f \|_{\infty} < +\infty \right\}
\]

Exercise 4. Let $f, g \in L^\infty_C(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^\infty_R(\Omega, \mathcal{F}, \mu) = \{ f \in L^\infty_C(\Omega, \mathcal{F}, \mu) \mid f(\Omega) \subseteq \mathbb{R} \}$.
2. Show that $\| f \| \leq \| f \|_{\infty} \mu$-a.s.
3. Show that $\| f + g \|_{\infty} \leq \| f \|_{\infty} + \| g \|_{\infty}$.
4. Show that $L^\infty_R(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbb{R}$-linear combinations.
5. Show that $L^\infty_C(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbb{C}$-linear combinations.
6. Show that $\| f \|_{\infty} = 0 \iff f = 0 \mu$-a.s.
7. Show that for all $\alpha \in \mathbb{C}$, $\| \alpha f \|_\infty = |\alpha| \| f \|_\infty$.

8. Explain why $(f, g) \to \| f - g \|_\infty$ is not a metric on $L^\infty(\Omega, \mathcal{F}, \mu)$.

**Definition 75** Let $p \in [1, +\infty]$. Let $K = \mathbb{R}$ or $\mathbb{C}$. For all $\varepsilon > 0$ and $f \in L^p_K(\Omega, \mathcal{F}, \mu)$, we define the so-called open ball in $L^p_K(\Omega, \mathcal{F}, \mu)$:

$$B(f, \varepsilon) \overset{\triangle}{=} \{ g \in L^p_K(\Omega, \mathcal{F}, \mu) : \| f - g \|_p < \varepsilon \}$$

We call **usual topology** in $L^p_K(\Omega, \mathcal{F}, \mu)$, the set $\mathcal{T}$ defined by:

$$\mathcal{T} \overset{\triangle}{=} \{ U : U \subseteq L^p_K(\Omega, \mathcal{F}, \mu), \forall f \in U, \exists \varepsilon > 0, B(f, \varepsilon) \subseteq U \}$$

Note that if $(f, g) \to \| f - g \|_p$ was a metric, the usual topology in $L^p_K(\Omega, \mathcal{F}, \mu)$, would be nothing but the **metric** topology.

**Exercise 5.** Let $p \in [1, +\infty]$. Suppose there exists $N \in \mathcal{F}$ with $\mu(N) = 0$ and $N \neq \emptyset$. Put $f = 1_N$ and $g = 0$

1. Show that $f, g \in L^p_C(\Omega, \mathcal{F}, \mu)$ and $f \neq g$.

2. Show that any open set containing $f$ also contains $g$.

3. Show that $L^p_C(\Omega, \mathcal{F}, \mu)$ and $L^p_R(\Omega, \mathcal{F}, \mu)$ are not Hausdorff.

**Exercise 6.** Show that the usual topology on $L^p_R(\Omega, \mathcal{F}, \mu)$ is induced by the usual topology on $L^p_C(\Omega, \mathcal{F}, \mu)$, where $p \in [1, +\infty]$.

**Definition 76** Let $(E, \mathcal{T})$ be a topological space. A sequence $(x_n)_{n \geq 1}$ in $E$ is said to **converge** to $x \in E$, and we write $x_n \overset{\mathcal{T}}{\rightarrow} x$, if and only if, for all $U \in \mathcal{T}$ such that $x \in U$, there exists $n_0 \geq 1$ such that:

$$n \geq n_0 \Rightarrow x_n \in U$$

When $E = L^p_C(\Omega, \mathcal{F}, \mu)$ or $E = L^p_R(\Omega, \mathcal{F}, \mu)$, we write $x_n \overset{L^p}{\rightarrow} x$.

**Exercise 7.** Let $(E, \mathcal{T})$ be a topological space and $E' \subseteq E$. Let $\mathcal{T}' = \mathcal{T}|_{E'}$ be the induced topology on $E'$. Show that if $(x_n)_{n \geq 1}$ is a sequence in $E'$ and $x \in E'$, then $x_n \overset{\mathcal{T}}{\rightarrow} x$ is equivalent to $x_n \overset{\mathcal{T}'}{\rightarrow} x$.

**Exercise 8.** Let $f, g, (f_n)_{n \geq 1}$ be in $L^p(\Omega, \mathcal{F}, \mu)$ and $p \in [1, +\infty]$.

1. Recall what the notation $f_n \rightarrow f$ means.

2. Show that $f_n \overset{L^p}{\rightarrow} f$ is equivalent to $\| f_n - f \|_p \rightarrow 0$.

3. Show that if $f_n \overset{L^p}{\rightarrow} f$ and $f_n \overset{L^p}{\rightarrow} g$ then $f = g \mu$-a.s.

**Exercise 9.** Let $p \in [1, +\infty]$. Suppose there exists some $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $N \neq \emptyset$. Find a sequence $(f_n)_{n \geq 1}$ in $L^p_C(\Omega, \mathcal{F}, \mu)$ and $f, g$ in $L^p_C(\Omega, \mathcal{F}, \mu)$, $f \neq g$ such that $f_n \overset{L^p}{\rightarrow} f$ and $f_n \overset{L^p}{\rightarrow} g$.

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Definition 77 Let \((f_n)_{n \geq 1}\) be a sequence in \(L^p_C(\Omega, \mathcal{F}, \mu)\), where \((\Omega, \mathcal{F}, \mu)\) is a measure space and \(p \in [1, +\infty]\). We say that \((f_n)_{n \geq 1}\) is a Cauchy sequence, if and only if, for all \(\epsilon > 0\), there exists \(n_0 \geq 1\) such that:
\[
 n, m \geq n_0 \Rightarrow \|f_n - f_m\|_p \leq \epsilon
\]

Exercise 10. Let \(f, (f_n)_{n \geq 1}\) be in \(L^p_C(\Omega, \mathcal{F}, \mu)\) and \(p \in [1, +\infty]\). Show that if \(f_n \overset{L^p}{\to} f\), then \((f_n)_{n \geq 1}\) is a Cauchy sequence.

Exercise 11. Let \((f_n)_{n \geq 1}\) be Cauchy in \(L^p_C(\Omega, \mathcal{F}, \mu)\), \(p \in [1, +\infty]\).

1. Show the existence of \(n_1 \geq 1\) such that:
   \[
   n \geq n_1 \Rightarrow \|f_n - f_{n_1}\|_p \leq \frac{1}{2}
   \]

2. Suppose we have found \(n_1 < n_2 < \ldots < n_k, \ k \geq 1\), such that:
   \[
   \forall j \in \{1, \ldots, k\}, \ n \geq n_j \Rightarrow \|f_n - f_{n_j}\|_p \leq \frac{1}{2^j}
   \]
   Show the existence of \(n_{k+1}, \ n_k < n_{k+1}\) such that:
   \[
   n \geq n_{k+1} \Rightarrow \|f_n - f_{n_{k+1}}\|_p \leq \frac{1}{2^{k+1}}
   \]

3. Show that there exists a subsequence \((f_{n_k})_{k \geq 1}\) of \((f_n)_{n \geq 1}\) with:
   \[
   \sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty
   \]

Exercise 12. Let \(p \in [1, +\infty]\), and \((f_n)_{n \geq 1}\) be in \(L^p_C(\Omega, \mathcal{F}, \mu)\), with:
\[
\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty
\]
We define:
\[
g \overset{\Delta}{=} \sum_{n=1}^{+\infty} |f_{n+1} - f_n|
\]

1. Show that \(g : (\Omega, \mathcal{F}) \to [0, +\infty]\) is non-negative and measurable.
2. If \(p = +\infty\), show that \(g \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty \mu\)-a.s.
3. If \(p \in [1, +\infty]\), show that for all \(N \geq 1\), we have:
\[
\left\| \sum_{n=1}^{N} |f_{n+1} - f_n| \right\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p
\]
4. If \( p \in [1, +\infty[ \), show that:
\[
\left( \int_{\Omega} g^p \, d\mu \right)^{\frac{1}{p}} \leq \sum_{n=1}^{+\infty} \| f_{n+1} - f_n \|_p
\]

5. Show that for \( p \in [1, +\infty[ \), we have \( g < +\infty \) \( \mu \)-a.s.

6. Define \( A = \{ g < +\infty \} \). Show that for all \( \omega \in A \), \( (f_n(\omega))_{n \geq 1} \) is a Cauchy sequence in \( C \). We denote \( z(\omega) \) its limit.

7. Define \( f : (\Omega, \mathcal{F}) \to (\mathcal{C}, \mathcal{B}(\mathcal{C})) \), by:
\[
f(\omega) = \begin{cases} 
z(\omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}
\]
Show that \( f \) is measurable and \( f_n \to f \) \( \mu \)-a.s.

8. if \( p = +\infty \), show that for all \( n \geq 1 \), \( |f_n| \leq |f_1| + g \) and conclude that \( f \in L_\infty(\Omega, \mathcal{F}, \mu) \).

9. If \( p \in [1, +\infty[ \), show the existence of \( n_0 \geq 1 \), such that:
\[
n \geq n_0 \Rightarrow \int_{\Omega} |f_n - f_n| \, d\mu \leq 1
\]
Deduce from Fatou lemma that \( f - f_{n_0} \in L_p(\Omega, \mathcal{F}, \mu) \).

10. Show that for \( p \in [1, +\infty[ \), \( f \in L_p(\Omega, \mathcal{F}, \mu) \).

11. Suppose that \( f_n \in L_p(\Omega, \mathcal{F}, \mu) \), for all \( n \geq 1 \). Show the existence of \( f \in L_p(\Omega, \mathcal{F}, \mu) \), such that \( f_n \to f \) \( \mu \)-a.s.

**Exercise 13.** Let \( p \in [1, +\infty[ \), and \((f_n)_{n \geq 1}\) be in \( L_p(\Omega, \mathcal{F}, \mu) \), with:
\[
\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty
\]

1. Does there exist \( f \in L_p(\Omega, \mathcal{F}, \mu) \) such that \( f_n \to f \) \( \mu \)-a.s.

2. Suppose \( p = +\infty \). Show that for all \( n < m \), we have:
\[
|f_m - f_n| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty \mu \text{-a.s.}
\]

3. Suppose \( p = +\infty \). Show that for all \( n \geq 1 \), we have:
\[
\|f - f_n\|_\infty \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_\infty
\]

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4. Suppose $p \in [1, +\infty]$. Show that for all $n < m$, we have:
\[
\left( \int_{\Omega} |f_{m+1} - f_n|^p \, d\mu \right)^{\frac{1}{p}} \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_p
\]

5. Suppose $p \in [1, +\infty]$. Show that for all $n \geq 1$, we have:
\[
\|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p
\]

6. Show that for $p \in [1, +\infty]$, we also have $f_n \overset{L^p}{\to} f$.

7. Suppose conversely that $g \in L^p_C(\Omega, \mathcal{F}, \mu)$ is such that $f_n \overset{L^p}{\to} g$. Show that $f = g$ $\mu$-a.s.. Conclude that $f_n \overset{\text{a.s.}}{\to} g$.

**Theorem 44** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $p \in [1, +\infty]$, and $(f_n)_{n \geq 1}$ be a sequence in $L^p_C(\Omega, \mathcal{F}, \mu)$ such that:
\[
\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty
\]
Then, there exists $f \in L^p_C(\Omega, \mathcal{F}, \mu)$ such that $f_n \overset{\text{a.s.}}{\to} f$. Moreover, for all $g \in L^p_C(\Omega, \mathcal{F}, \mu)$, the convergence $f_n \overset{\text{a.s.}}{\to} g$ and $f_n \overset{L^p}{\to} g$ are equivalent.

**Exercise 14.** Let $f, (f_n)_{n \geq 1}$ be in $L^p_C(\Omega, \mathcal{F}, \mu)$ such that $f_n \overset{L^p}{\to} f$, where $p \in [1, +\infty]$.

1. Show that there exists a sub-sequence $(f_{n_k})_{k \geq 1}$ of $(f_n)_{n \geq 1}$, with:
\[
\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty
\]

2. Show that there exists $g \in L^p_C(\Omega, \mathcal{F}, \mu)$ such that $f_{n_k} \overset{\text{a.s.}}{\to} g$.

3. Show that $f_{n_k} \overset{L^p}{\to} g$ and $g = f$ $\mu$-a.s.

4. Conclude with the following:

**Theorem 45** Let $(f_n)_{n \geq 1}$ be in $L^p_C(\Omega, \mathcal{F}, \mu)$ and $f \in L^p_C(\Omega, \mathcal{F}, \mu)$ such that $f_n \overset{L^p}{\to} f$, where $p \in [1, +\infty]$. Then, we can extract a sub-sequence $(f_{n_k})_{k \geq 1}$ of $(f_n)_{n \geq 1}$ such that $f_{n_k} \overset{\text{a.s.}}{\to} f$.

**Exercise 15.** Prove the last theorem for $L^p_R(\Omega, \mathcal{F}, \mu)$.

**Exercise 16.** Let $(f_n)_{n \geq 1}$ be Cauchy in $L^p_C(\Omega, \mathcal{F}, \mu)$, $p \in [1, +\infty]$. 
1. Show that there exists a subsequence \((f_{n_k})_{k \geq 1}\) of \((f_n)_{n \geq 1}\) and \(f\) belonging to \(L^p_C(\Omega, \mathcal{F}, \mu)\), such that \(f_{n_k} \overset{L^p}{\rightharpoonup} f\).

2. Using the fact that \((f_n)_{n \geq 1}\) is Cauchy, show that \(f_n \overset{L^p}{\rightarrow} f\).

**Theorem 46** Let \(p \in [1, +\infty]\). Let \((f_n)_{n \geq 1}\) be a Cauchy sequence in \(L^p_C(\Omega, \mathcal{F}, \mu)\). Then, there exists \(f \in L^p_C(\Omega, \mathcal{F}, \mu)\) such that \(f_n \overset{L^p}{\rightharpoonup} f\).

**Exercise 17.** Prove the last theorem for \(L^p_R(\Omega, \mathcal{F}, \mu)\).
Solutions to Exercises

Exercise 1.

1. Since $p, q \in \mathbb{R}^+$, we have $p < +\infty$ and $q < +\infty$. From the inequality $1/p \leq 1/p + 1/q = 1$, we obtain $p \geq 1$. If $p = 1$, then $1/q = 0$, contradicting $q < +\infty$. So $p > 1$, and similarly $q > 1$. We have proved that $1 < p < +\infty$ and $1 < q < +\infty$.

2. Let $\alpha \in [0, +\infty]$ and $\phi = \phi^\alpha$. We want to prove that $\phi$ is continuous. For all $a \in \mathbb{R}^+$, it is clear that $\lim_{x \to a} \phi(x) = \phi(a)$. So $\phi$ is continuous at $x = a$. Furthermore, $\lim_{x \to +\infty} \phi(x) = \phi(+\infty)$. So $\phi$ is also continuous at $+\infty$. For many of us, this is sufficient proof of the fact that $\phi$ is a continuous map. However, for those who want to apply definition (27), the following can be said: let $V$ be open in $[0, +\infty]$. We want to show that $\phi^{-1}(V)$ is open in $[0, +\infty]$. Let $a \in \phi^{-1}(V)$. Then $\phi(a) \in V$. Since $\phi$ is continuous at $x = a$, there exists $U_a$ open in $[0, +\infty]$, containing $a$, such that $\phi(U_a) \subseteq V$. So $a \in U_a \subseteq \phi^{-1}(V)$. It follows that $\phi^{-1}(V)$ can be written as $\phi^{-1}(V) = \bigcup_{a \in \phi^{-1}(V)} U_a$, and $\phi^{-1}(V)$ is therefore open in $[0, +\infty]$. From definition (27), we conclude that $\phi : [0, +\infty] \to [0, +\infty]$ is a continuous map.

3. $f^p$ can be viewed as $f^p = \phi^p \circ f$, where $\phi^p$ is defined as in 2. We proved that $\phi^p$ is a continuous map. It is therefore measurable with respect to the Borel $\sigma$-algebra $B([0, +\infty])$ on $[0, +\infty]$. It follows that $f^p : (\Omega, \mathcal{F}) \to [0, +\infty]$ is a measurable map, which is also non-negative. Hence, the integral $\int f^p d\mu$ is a well-defined element of $[0, +\infty]$, and $A = (\int f^p d\mu)^{1/p}$ is also well-defined, being understood that $(+\infty)^{1/p} = +\infty$. Similarly, $B = (\int f^q d\mu)^{1/q}$ is a well-defined element of $[0, +\infty]$. Finally, the map $fg : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable, and $C = \int fg d\mu$ is a well-defined element of $[0, +\infty]$.

4. Suppose $A = 0$. Then $\int f^p d\mu = 0$, and since $f^p$ is non-negative, we see that $f^p = 0$ $\mu$-a.s., and consequently $f = 0$ $\mu$-a.s. So $fg = 0$ $\mu$-a.s., and finally $C = \int fg d\mu = 0$. So $C \leq AB$. Similarly, $B = 0$ implies $C = 0$, and therefore $C \leq AB$.

5. Suppose $A = +\infty$. Then, either $B = 0$ or $B > 0$. If $B = 0$, then $C \leq AB$ is true from 4. If $B > 0$, then $AB = +\infty$, and consequently $C \leq AB$. In any case, we see that $C \leq AB$. Similarly, $B = +\infty$ implies $C \leq AB$.

6. Suppose $A, B \in [0, +\infty]$. Let $F = f/A$ and $G = g/B$. We have:

$$\int F^p d\mu = \int (f/A)^p d\mu = \frac{1}{A^p} \int f^p d\mu = 1$$

and similarly, $\int G^p d\mu = 1$. 

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7. Let \( a, b \in [0, +\infty[ \). The map \( x \to -\ln(x) \) being convex on \( ]0, +\infty[ \), since 
\[ \frac{1}{p} + \frac{1}{q} = 1 \]
we have:
\[
-\ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \leq -\frac{1}{p} \ln(a^p) - \frac{1}{q} \ln(b^q) = -\ln(ab)
\]
and consequently \( \ln(ab) \leq \ln(a^p / p + b^q / q) \). The map \( x \to e^x \) being non-decreasing, we conclude that:
\[
ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \tag{1}
\]
It is easy to check that inequality (1) is in fact true for all \( a, b \in [0, +\infty[ \).

8. For all \( \omega \in \Omega \), \( F(\omega) \) and \( G(\omega) \) are elements of \( [0, +\infty[ \). From 7:
\[
F(\omega)G(\omega) \leq \frac{1}{p}F(\omega)^p + \frac{1}{q}G(\omega)^q
\]

9. Integrating on both side of 8., we obtain:
\[
\int FGd\mu \leq \frac{1}{p} \int F^p d\mu + \frac{1}{q} \int G^q d\mu = 1
\]
where we have used the fact that \( \int F^p d\mu = \int G^q d\mu = 1 \). Since \( \int FGd\mu = (\int f g d\mu) / AB = C / AB \), we conclude that \( C \leq AB \).

Exercise 1

Exercise 2.

1. \( f^p \), \( g^p \) and \( (f + g)^p \) are all non-negative and measurable. All three integrals 
\( \int f^p d\mu \), \( \int g^p d\mu \) and \( \int (f + g)^p d\mu \) are therefore well-defined. It follows that 
\( A, B \) and \( C \) are well-defined elements of \( [0, +\infty[ \).

2. Since \( p > 1 \), the map \( x \to x^p \) is convex on \( [0, +\infty[ \). In particular, for all 
\( a, b \in [0, +\infty[ \), we have \( (a + b)^p / 2 \leq (a^p + b^p) / 2 \). We conclude that:
\[
(a + b)^p \leq 2^{p-1}(a^p + b^p) \tag{2}
\]
In fact, it is easy to check that (2) holds for all \( a, b \in [0, +\infty[ \).

3. If \( A = +\infty \) or \( B = +\infty \), then \( A + B = +\infty \), and \( C \leq A + B \). If \( C = 0 \), 
then clearly \( C \leq A + B \).

4. Using 2., for all \( \omega \in \Omega \), we have:
\[
(f(\omega) + g(\omega))^p \leq 2^{p-1}(f(\omega)^p + g(\omega)^p)
\]
Integrating on both side of the inequality, we obtain:
\[
\int (f + g)^p d\mu \leq 2^{p-1} \left( \int f^p d\mu + \int g^p d\mu \right) \tag{3}
\]
If \( A < +\infty \) and \( B < +\infty \), then both integrals \( \int f^p d\mu \) and \( \int g^p d\mu \) are 
finite, and we see from (3) that \( \int (f + g)^p d\mu \) is itself finite. So \( C < +\infty \).
5. Take \( q = p/(p - 1) \). Since \( p \in ]1, +\infty[ \), \( q \) is a well-defined element of \( \mathbb{R}^+ \), and \( 1/p + 1/q = 1 \).

6. Let \( a, b \in [0, +\infty[ \). If \( a, b \in \mathbb{R}^+ \), then:
\[
(a + b)^p = (a + b)(a + b)^{p-1}
\]
If \( a = +\infty \) or \( b = +\infty \), then \( a + b = +\infty \) and both sides of (4) are equal to \(+\infty\). So (4) is true for all \( a, b \in [0, +\infty[ \).

7. Using holder inequality (41), since \( q(p - 1) = p \), we have:
\[
\int f.(f + g)^{p-1}d\mu \leq \left( \int f^p d\mu \right)^{\frac{1}{p}} \left( \int (f + g)^{q(p-1)} d\mu \right)^{\frac{1}{q}} = AC^\frac{p}{q}
\]
and:
\[
\int g.(f + g)^{p-1}d\mu \leq \left( \int g^p d\mu \right)^{\frac{1}{p}} \left( \int (f + g)^{q(p-1)} d\mu \right)^{\frac{1}{q}} = BC^\frac{p}{q}
\]

8. From 6., we have:
\[
\int (f + g)^p d\mu = \int f.(f + g)^{p-1}d\mu + \int g.(f + g)^{p-1}d\mu
\]
and using 7., we obtain:
\[
\int (f + g)^p d\mu \leq C^\frac{p}{q} (A + B)
\]

9. From 8., we have \( C^p \leq C^\frac{p}{q} (A+B) \). Having assumed in 5. that \( C \in ]0, +\infty[ \), we can divide both side of this inequality by \( C^\frac{p}{q} \), to obtain \( C^{p - \frac{p}{q}} \leq A + B \). Since \( p - p/q = 1 \), we conclude that \( C \leq A + B \).

10. If \( p = 1 \), then \( C = A + B \) is equivalent to:
\[
\int (f + g)d\mu = \int f d\mu + \int g d\mu
\]
which is true by linearity. In particular, \( C \leq A + B \). The purpose of this exercise is to prove minkowski inequality (43).

Exercise 2

Exercise 3.

1. Let \( f : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) be a map. Then, if \( f \) has values in \( \mathbb{R} \), i.e. \( f(\Omega) \subseteq \mathbb{R} \), then the measurability of \( f \) with respect to \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) is equivalent to its measurability with respect to \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). Hence:
\[
L^p_R(\Omega, \mathcal{F}, \mu) = \{ f \in L^p_C(\Omega, \mathcal{F}, \mu) \mid f(\Omega) \subseteq \mathbb{R} \}
\]
The equivalence of measurability with respect to \( \mathcal{B}(\mathbb{C}) \) and \( \mathcal{B}(\mathbb{R}) \) may be taken for granted by many. It is easily proved from the fact that \( \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{C})|_\mathbb{R} \), i.e. the Borel \( \sigma \)-algebra on \( \mathbb{R} \) is the trace on \( \mathbb{R} \), of the
Borel $\sigma$-algebra on $\mathbb{C}$. This fact can be seen from the trace theorem (10), and the fact that the usual topology on $\mathbb{R}$ is induced on $\mathbb{R}$, by the usual topology on $\mathbb{C}$.

2. Let $f, g \in L^p_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbb{R}$. The map $f + \alpha g$ is $\mathbb{R}$-valued and measurable. Moreover, we have $|f + \alpha g| \leq |f| + |\alpha||g|$. Since $p \geq 1$, (and in particular $p \geq 0$), the map $x \mapsto x^p$ is non-decreasing on $\mathbb{R}^+$, so $|f + \alpha g|^p \leq (|f| + |\alpha||g|)^p$. Hence, we see that $\int |f + \alpha g|^p d\mu \leq \int (|f| + |\alpha||g|)^p d\mu$.

However, using Minkowski inequality (43), we have:

$$\left( \int (|f| + |\alpha||g|)^p d\mu \right)^\frac{1}{p} \leq \left( \int |f|^p d\mu \right)^\frac{1}{p} + |\alpha| \left( \int |g|^p d\mu \right)^\frac{1}{p}$$

We conclude that $\int |f + \alpha g|^p d\mu \leq +\infty$. So $f + \alpha g \in L^p_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$, and we have proved that $L^p_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbb{R}$-linear combinations.

3. The fact that $L^p_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbb{C}$-linear combinations, is proved identically to 2., replacing $\mathbb{R}$ by $\mathbb{C}$.

4. Using $|f + g|^p \leq (|f| + |g|)^p$ and Minkowski inequality (43):

$$\left( \int (|f| + |g|)^p d\mu \right)^\frac{1}{p} \leq \left( \int |f|^p d\mu \right)^\frac{1}{p} + \left( \int |g|^p d\mu \right)^\frac{1}{p}$$

we see that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

5. Suppose $\|f\|_p = 0$. Then $\int |f|^p d\mu = 0$. Since $|f|^p$ is non-negative, $|f|^p = 0$ $\mu$-a.s., and consequently $f = 0$ $\mu$-a.s. Conversely, if $f = 0$ $\mu$-a.s., then $\|f\|_p = 0$, so $\int |f|^p d\mu = 0$ and finally $\|f\|_p = 0$.

6. Let $\alpha \in \mathbb{C}$. We have:

$$\|\alpha f\|_p = \left( \int |\alpha f|^p d\mu \right)^\frac{1}{p} = |\alpha| \left( \int |f|^p d\mu \right)^\frac{1}{p} = |\alpha| \cdot \|f\|_p$$

7. $\|f - g\|_p = 0$ only implies $f = g$ $\mu$-a.s., and not necessarily $f = g$. So $(f, g) \mapsto \|f - g\|_p$, may not be a metric on $L^p_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$.

Exercise 3

1. For all $f : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with values in $\mathbb{R}$, the measurability of $f$ with respect to $\mathcal{B}(\mathbb{C})$ is equivalent to its measurability with respect to $\mathcal{B}(\mathbb{R})$. Hence:

$$L^\infty_{\mathbb{R}}(\Omega, \mathcal{F}, \mu) = \{ f \in L^\infty_{\mathbb{C}}(\Omega, \mathcal{F}, \mu), f(\Omega) \subseteq \mathbb{R} \}$$

2. Since $\|f\|_{\infty} < +\infty$, for all $n \geq 1$, we have $\|f\|_{\infty} < \|f\|_{\infty} + 1/n$. $\|f\|_{\infty}$ being the greatest lower bound of all $\mu$-almost sure upper bounds of $|f|$, $\|f\|_{\infty} + 1/n$ cannot be such upper bound. There exists $M \in \mathbb{R}^+$, such that


|f| \leq M \ \mu\text{-a.s.}, \text{ and } M < \|f\|_\infty + 1/n. \text{ In particular, } |f| < \|f\|_\infty + 1/n \ \mu\text{-a.s.}

Let A_n be the set defined by A_n = \{|f| < \|f\|_\infty + 1/n \leq |f|\}. Then A_n \in \mathcal{F}

and \mu(A_n) = 0. Moreover, A_n \subseteq A_{n+1} \text{ and } \bigcup_{n=1}^{+\infty} A_n = \{|f| < \|f\|_\infty \}. \text{ It follows that } A_n \uparrow \{\|f\|_\infty < |f|\}, \text{ and from theorem (7), we see that:}

\[\mu(\{\|f\|_\infty < |f|\}) = \lim_{n \to +\infty} \mu(A_n) = 0\]

We conclude that |f| \leq \|f\|_\infty \ \mu\text{-a.s.}

3. Since |f + g| \leq |f| + |g|, using 2., we have:

\[|f + g| \leq \|f\|_\infty + \|g\|_\infty \ \mu\text{-a.s.}\]

Hence, \|f\|_\infty + \|g\|_\infty is a \mu\text{-almost sure upper bound of } |f + g|. \|f + g\|_\infty being a lower bound of all such upper bounds, we have \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.

4. Let \(f, g \in \mathbb{L}^2(\Omega, \mathcal{F}, \mu)\) and \(\alpha \in \mathbb{R}\). Then \(f + \alpha g\) is \(\mathbb{R}\)-valued and measurable. Furthermore, using 2., we have:

\[|f + \alpha g| \leq |f| + |\alpha| \cdot |g| \leq \|f\|_\infty + |\alpha| \cdot \|g\|_\infty \ \mu\text{-a.s.}\]

It follows that \(\|f + \alpha g\|_\infty \leq \|f\|_\infty + |\alpha| \cdot \|g\|_\infty < +\infty\). We conclude that \(f + \alpha g \in \mathbb{L}^2(\Omega, \mathcal{F}, \mu)\), and we have proved that \(\mathbb{L}^2(\Omega, \mathcal{F}, \mu)\) is closed under \(\mathbb{R}\)-linear combinations.

5. The fact that \(\mathbb{L}^\infty(\Omega, \mathcal{F}, \mu)\) is closed under \(\mathbb{C}\)-linear combinations can be proved identically, replacing \(\mathbb{R}\) by \(\mathbb{C}\).

6. Suppose \(\|f\|_\infty = 0\). Then \(|f| \leq 0 \ \mu\text{-a.s.}, \text{ and consequently } f = 0 \ \mu\text{-a.s.} \). Conversely, if \(f = 0 \ \mu\text{-a.s.}, \text{ then } |f| \leq 0 \ \mu\text{-a.s.}, \text{ and 0 is therefore a } \mu\text{-almost sure upper bound of } |f|\). So \(\|f\|_\infty \leq 0\). Since \(\|f\|_\infty\) is an infimum of a subset of \(\mathbb{R}^+\), it is either +\infty (when such subset is empty), or lies in \(\mathbb{R}^+\). So \(\|f\|_\infty \geq 0\) and finally \(\|f\|_\infty = 0\).

7. We have \(|\alpha f| \leq |\alpha| \cdot \|f\|_\infty \ \mu\text{-a.s.}, \text{ and hence } \|\alpha f\|_\infty \leq |\alpha| \cdot \|f\|_\infty\). if \(\alpha \neq 0\), we have:

\[\|f\|_\infty = \left|\frac{1}{\alpha} \cdot (\alpha f)\right|_\infty \leq \frac{1}{|\alpha|} \cdot \|\alpha f\|_\infty\]

It follows that \(\|\alpha f\|_\infty = |\alpha| \cdot \|f\|_\infty\), (also true if \(\alpha = 0\)).

8. \(\|f - g\|_\infty = 0\) implies \(f = g \ \mu\text{-a.s.}, \text{ but not } f = g\). It follows that \((f, g) \to \|f - g\|_\infty\) may not be a metric on \(\mathbb{L}^\infty(\Omega, \mathcal{F}, \mu)\).

Exercise 5.

1. Since \(N \neq \emptyset, 1_N \neq 0\), so \(f \neq g\). Since \(N \in \mathcal{F}\), the map \(f = 1_N\) is measurable, and being \(\mathbb{R}\)-valued, it is also \(\mathbb{C}\)-valued. Moreover, since \(\mu(N) = 0\), \(\|f\|_p = 0 < +\infty\) (whether \(p = +\infty\) or lies in \([1, +\infty]\)), and we see that \(f \in \mathbb{L}^p(\Omega, \mathcal{F}, \mu)\). Since \(g = 0\), it is \(\mathbb{C}\)-valued, measurable and \(\|g\|_p = 0 < +\infty\), so \(g \in \mathbb{L}^p(\Omega, \mathcal{F}, \mu)\).

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2. Let \( U \) be open in \( L^p_C(\Omega, \mathcal{F}, \mu) \), such that \( f \in U \). By definition (75), there exists \( \epsilon > 0 \), such that \( B(f, \epsilon) \subseteq U \). However, \( \|f - g\|_p = |f - g|_p = 0 \) (\( p = +\infty \) or \( p \in [1, +\infty) \)). So in particular \( \|f - g\|_p < \epsilon \). So \( g \in B(f, \epsilon) \) and finally \( g \in U \).

3. If \( L^p_C(\Omega, \mathcal{F}, \mu) \) was Hausdorff, since \( f \neq g \), there would exist \( U, V \) open sets in \( L^p_C(\Omega, \mathcal{F}, \mu) \) such that \( f \in U, g \in V \) and \( U \cap V = \emptyset \). However from 2., this is impossible, as \( g \) would always be an element of \( U \) as well as \( V \). We conclude similarly that \( L^p_R(\Omega, \mathcal{F}, \mu) \) is not Hausdorff.

**Exercise 5**

**Exercise 6.** Let \( L^p_R \) and \( L^p_C \) denote \( L^p_R(\Omega, \mathcal{F}, \mu) \) and \( L^p_C(\Omega, \mathcal{F}, \mu) \) respectively. Let \( \mathcal{T} \) be the usual topology on \( L^p_R \) and \( \mathcal{T}' \) be the usual topology on \( L^p_R \). We want to prove that \( \mathcal{T}' = \mathcal{T}_{|L^p_R} \), i.e. \( \mathcal{T}' \) is the topology on \( L^p_R \) induced by \( \mathcal{T} \). Given \( f \in L^p_R \) and \( \epsilon > 0 \), let \( B(f, \epsilon) \) denote the open ball in \( L^p_C \) and \( B'(f, \epsilon) \) denote the open ball the \( L^p_R \). Then \( B'(f, \epsilon) = B(f, \epsilon) \cap L^p_R \). It is a simple exercise to show that any open ball in \( L^p_R \) or \( L^p_C \), is in fact open with respect to their usual topology. Let \( U' \in \mathcal{T}' \). For all \( f \in U' \), there exists \( \epsilon_f > 0 \) such that \( f' \in B'(f, \epsilon_f) \subseteq U' \). It follows that:

\[
U' = \bigcup_{f \in U} B'(f, \epsilon_f) = (\bigcup_{f \in U} B(f, \epsilon_f)) \cap L^p_R
\]

and we see that \( U' \in \mathcal{T}_{|L^p_R} \). Conversely, let \( U' \in \mathcal{T}_{|L^p_R} \). There exists \( U \in \mathcal{T} \) such that \( U' = U \cap L^p_R \). Let \( f \in U' \). There exists \( \epsilon > 0 \) such that \( B(f, \epsilon) \subseteq U \). It follows that \( B'(f, \epsilon) = B(f, \epsilon) \cap L^p_R \subseteq U' \). So \( U' \) is open with respect to the usual topology in \( L^p_R \), i.e. \( U' \in \mathcal{T}' \). We have proved that \( \mathcal{T}_{|L^p_R} \subseteq \mathcal{T}' \), and finally \( \mathcal{T}' = \mathcal{T}_{|L^p_R} \).

**Exercise 6**

**Exercise 7.** Let \((E, \mathcal{T})\) be a topological space and \( E' \subseteq E \). Let \( \mathcal{T}' = \mathcal{T}_{|E'} \) be the induced topology on \( E' \). We assume that \((x_n)_{n \geq 1}\) is a sequence in \( E' \), and that \( x \in E' \). Suppose that \( x_n \xrightarrow{\mathcal{T}} x \). Let \( U' \in \mathcal{T}' \) be such that \( x \in U' \). There exists \( U \in \mathcal{T} \) such that \( U' = U \cap E' \). Since \( x \in U \) and \( x_n \xrightarrow{\mathcal{T}} x \), there exists \( n_0 \geq 1 \) such that \( x_n \in U \) for all \( n \geq n_0 \). But \( x_n \in E' \) for all \( n \geq 1 \). So \( x_n \in U \cap E' = U' \) for all \( n \geq n_0 \), and we see that \( x_n \xrightarrow{\mathcal{T}} x \). Conversely, suppose that \( x_n \xrightarrow{\mathcal{T}} x \). Let \( U \in \mathcal{T} \) be such that \( x \in U \). Then \( U \cap E' \in \mathcal{T}' \) and \( x \in U \cap E' \). There exists \( n_0 \geq 1 \), such that \( x_n \in U \cap E' \) for all \( n \geq n_0 \). In particular, \( x_n \in U \) for all \( n \geq n_0 \), and we see that \( x_n \xrightarrow{\mathcal{T}} x \). We have proved that \( x_n \xrightarrow{\mathcal{T}} x \) and \( x_n \xrightarrow{\mathcal{T}} x \) are equivalent.

**Exercise 7**

**Exercise 8.**

1. The notation \( f_n \to f \) has been used throughout these tutorials, to refer to a simple convergence, i.e. \( f_n(\omega) \to f(\omega) \) as \( n \to +\infty \), for all \( \omega \in \Omega \).

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2. Suppose $f_n \xrightarrow{L^p} f$. Let $\epsilon > 0$. The open ball $B(f, \epsilon)$ being open with respect to the usual topology in $L^p_C(\Omega, \mathcal{F}, \mu)$, there exists $n_0 \geq 1$, such that $f_n \in B(f, \epsilon)$ for all $n \geq n_0$, i.e.:

$$n \geq n_0 \Rightarrow \|f_n - f\|_p < \epsilon$$

So $\|f_n - f\|_p \to 0$. Conversely, suppose $\|f_n - f\|_p \to 0$. Let $U$ be open in $L^p_C(\Omega, \mathcal{F}, \mu)$, such that $f \in U$. From definition (75), there exists $\epsilon > 0$ such that $B(f, \epsilon) \subseteq U$. By assumption, there exists $n_0 \geq 0$, such that $\|f_n - f\|_p < \epsilon$ for all $n \geq n_0$. So $f_n \in B(f, \epsilon)$ for all $n \geq n_0$. Hence, we see that $f_n \in U$ for all $n \geq n_0$, and we have proved that $f_n \xrightarrow{L^p} f$. We conclude that $f_n \xrightarrow{L^p} f$ and $\|f_n - f\|_p \to 0$ are equivalent.

3. Suppose $f_n \xrightarrow{L^p} f$ and $f_n \xrightarrow{L^p} g$. From 2., we have $\|f_n - f\|_p \to 0$ and $\|f_n - g\|_p \to 0$. Using the triangle inequality (ex. (3) for $p \in [1, +\infty[$ and ex. (4) for $p = +\infty$):

$$\|f - g\|_p \leq \|f_n - f\|_p + \|f_n - g\|_p$$

for all $n \geq 1$. It follows that we have $\|f - g\|_p < \epsilon$ for all $\epsilon > 0$, and consequently $\|f - g\|_p = 0$. From ex. (3) and ex. (4) we conclude that $f = g$ $\mu$-a.s.

**Exercise 8**

**Exercise 9.** Take $f_n = 1_N = f$ for all $n \geq 1$. Take $g = 0$. Then $f_n, f$ and $g$ are all elements of $L^p_C(\Omega, \mathcal{F}, \mu)$, and $f \neq g$. Moreover, for all $n \geq 1$, we have $\|f_n - f\|_p = \|f_n - g\|_p = 0$. So $f_n \xrightarrow{L^p} f$ and $f_n \xrightarrow{L^p} g$. The purpose of this exercise is to show that a limit in $L^p$ may not be unique ($f \neq g$). However, it is unique, up to $\mu$-almost sure equality (See exercise (8)).

**Exercise 10.** Suppose $f_n \xrightarrow{L^p} f$. Let $\epsilon > 0$. There exists $n_0 \geq 1$, with:

$$n \geq n_0 \Rightarrow \|f_n - f\|_p \leq \epsilon/2$$

From the triangle inequality, for all $n, m \geq 1$:

$$\|f_n - f_m\|_p \leq \|f_n - f\|_p + \|f_m - f\|_p$$

It follows that we have:

$$n, m \geq n_0 \Rightarrow \|f_n - f_m\|_p \leq \epsilon$$

We conclude that $(f_n)_{n \geq 1}$ is a Cauchy sequence in $L^p_C(\Omega, \mathcal{F}, \mu)$.

**Exercise 11.**

1. Take $\epsilon = 1/2$. There exists $n_1 \geq 1$, such that:

$$n, m \geq n_1 \Rightarrow \|f_n - f_m\|_p \leq \frac{1}{2}$$
In particular, we have:

\[ n \geq n_1 \Rightarrow \|f_n - f_{n_1}\|_p \leq \frac{1}{2} \]

2. Let \( k \geq 1 \). We have \( n_1 < \ldots < n_k \), such that for all \( j = 1, \ldots, k \):

\[ n \geq n_j \Rightarrow \|f_n - f_{n_j}\|_p \leq \frac{1}{2^j} \]

Take \( \epsilon = 1/2^{k+1} \). There exists \( n_{k+1}' \geq 1 \), such that:

\[ n, m \geq n_{k+1}' \Rightarrow \|f_n - f_m\|_p \leq \frac{1}{2^{k+1}} \]

Take \( n_{k+1} = \max(n_k + 1, n_{k+1}') \). Then \( n_k < n_{k+1} \), and:

\[ n \geq n_{k+1} \Rightarrow \|f_n - f_{n_{k+1}}\|_p \leq \frac{1}{2^{k+1}} \]

3. By induction from 2., we can construct a strictly increasing sequence of integers \((n_k)_{k \geq 1}\), such that for all \( k \geq 1 \):

\[ n \geq n_k \Rightarrow \|f_n - f_{n_k}\|_p \leq \frac{1}{2^k} \]

In particular, \( \|f_{n_{k+1}} - f_{n_k}\|_p \leq 1/2^k \) for all \( k \geq 1 \). It follows that we have found a subsequence \((f_{n_k})_{k \geq 1}\) of \((f_n)_{n \geq 1}\), such that:

\[ \sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty \]

Exercise 12.

1. Each finite sum \( g_N = \sum_{n=1}^{N} |f_{n+1} - f_n| \) is well-defined and measurable. It follows that \( g = \sup_{N \geq 1} g_N \) is itself measurable. It is obviously non-negative.

2. Suppose \( p = +\infty \). From exercise (4), for all \( n \geq 1 \), we have:

\[ |f_{n+1} - f_n| \leq \|f_{n+1} - f_n\|_\infty, \mu\text{-a.s.} \]

The set \( N_n = \{ |f_{n+1} - f_n| > \|f_{n+1} - f_n\|_\infty \} \) which lies in \( \mathcal{F} \), is such that \( \mu(N_n) = 0 \). It follows that if \( N = \cup_{n \geq 1} N_n \), then \( \mu(N) = 0 \). However, for all \( \omega \in N^c \), we have:

\[ g(\omega) = \sum_{n=1}^{+\infty} |f_{n+1}(\omega) - f_n(\omega)| \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty \]

We conclude that \( g \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty \mu\text{-a.s.} \)
3. Let $p \in [1, +\infty]$ and $N \geq 1$. By the triangle inequality (ex. (3)):

$$\left\| \sum_{n=1}^{N} |f_{n+1} - f_n| \right\|_p \leq \sum_{n=1}^{N} \|f_{n+1} - f_n\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

4. Let $p \in [1, +\infty]$. Given $N \geq 1$, let $g_N = \sum_{n=1}^{N} |f_{n+1} - f_n|$. Then $g_N \to g$ as $N \to +\infty$. The map $x \to x^p$ being continuous on $[0, +\infty]$, we have $g_N^p \to g^p$, and in particular $g^p = \lim inf g_N^p$ as $N \to +\infty$. Using Fatou’s lemma (20), we see that:

$$\int g^p d\mu \leq \lim inf_{N \to +\infty} \int g_N^p d\mu \quad (5)$$

However, from 3., we have $\|g_N\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$, for all $N \geq 1$. Since $p \geq 0$, the map $x \to x^p$ is non-decreasing on $[0, +\infty]$, and therefore:

$$\int g_N^p d\mu \leq \left( \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p \right)^p \quad (6)$$

From inequalities (5) and (6), we conclude that:

$$\int g^p d\mu \leq \left( \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p \right)^p$$

and finally:

$$\left( \int g^p d\mu \right)^\frac{1}{p} \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

5. Let $p \in [1, +\infty]$. If $p = +\infty$, from 2. we have:

$$g \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p \text{ , } \mu\text{-a.s.} \quad (7)$$

By assumption, the series in (7) is finite. So $g < +\infty$ $\mu$-a.s.

If $p \in [1, +\infty[$, from 4. we have:

$$\left( \int g^p d\mu \right)^\frac{1}{p} \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

So $\int g^p d\mu < +\infty$. Since $(+\infty)\mu(\{g^p = +\infty\}) \leq \int g^p d\mu$, we see that $\mu(\{g^p = +\infty\}) = 0$ and finally $g < +\infty$ $\mu$-a.s.

6. Let $A = \{g < +\infty\}$. Let $\omega \in A$. Then $g(\omega) < +\infty$. The series $\sum_{n=1}^{+\infty} |f_{n+1}(\omega) - f_n(\omega)|$ is therefore finite. Let $\epsilon > 0$. There exists $n_0 \geq 1$, such that:

$$n \geq n_0 \Rightarrow \sum_{k=n}^{+\infty} |f_{k+1}(\omega) - f_k(\omega)| \leq \epsilon$$
Given \( m > n \geq n_0 \), we have:

\[
|f_m(\omega) - f_n(\omega)| \leq \sum_{k=n}^{m-1} |f_{k+1}(\omega) - f_k(\omega)| \leq \epsilon
\]

We conclude that the sequence \((f_n(\omega))_{n \geq 1}\) is Cauchy in \(C\). It therefore has a limit\(^1\), denoted \(z(\omega)\).

7. From 6, \(f_n(\omega) \to z(\omega) = f(\omega)\) for all \(\omega \in A\). Since by definition, \(f(\omega) = 0\) for all \(\omega \in A^c\), we see that \(f_n(\omega) \to f(\omega)\) for all \(\omega \in \Omega\). Hence, we have \(f_n1_A \to f\), and since \(f_n1_A\) is measurable for all \(n \geq 1\), we see from theorem (17) that \(f = \lim f_n1_A\) is itself measurable. Since \(\mu(A^c) = 0\) and \(f_n(\omega) \to f(\omega)\) on \(A\), we have \(f_n \to f\) \(\mu\)-a.s.

8. Suppose \(p = +\infty\). For all \(n \geq 1\), we have:

\[
|f_n - f_1| \leq \sum_{k=1}^{n-1} |f_{k+1} - f_k| \leq g
\]

So \(|f_n| \leq |f_1| + g\). Taking the limit as \(n \to +\infty\), we obtain \(|f| \leq |f_1| + g\) \(\mu\)-a.s. Let \(M = \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty\). Then by assumption, \(M < +\infty\) and from 2, we have \(g \leq M \mu\)-a.s. Moreover, since \(f_1 \in L_\infty^\infty(\Omega,F,\mu)\), using exercise (4), we have \(|f_1| \leq \|f_1\|_\infty \mu\)-a.s. with \(\|f_1\|_\infty < +\infty\). Hence, we see that \(|f| \leq \|f_1\|_\infty + M \mu\)-a.s., and consequently:

\[
\|f\|_\infty \leq \|f_1\|_\infty + \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty < +\infty
\]

\(f\) is therefore \(C\)-valued, measurable and with \(\|f\|_\infty < +\infty\). We have proved that \(f \in L_\infty^\infty(\Omega,F,\mu)\).

9. Let \(p \in [1, +\infty[\). The series \(\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p\) being finite, there exists \(n_0 \geq 1\), such that:

\[
n \geq n_0 \Rightarrow \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p \leq 1
\]

Let \(n \geq n_0\). By the triangle inequality:

\[
\|f_n - f_{n_0}\|_p \leq \sum_{k=n_0}^{n-1} \|f_{k+1} - f_k\|_p \leq 1
\]

Hence, we see that:

\[
n \geq n_0 \Rightarrow \int |f_n - f_{n_0}|^p d\mu \leq 1^p = 1
\]

\(^1\)The completeness of \(C\) is proved in the next Tutorial.
From 6, \( f_n(\omega) \to f(\omega) \) as \( n \to +\infty \), for all \( \omega \in A \), where \( \mu(A^c) = 0 \). In particular:
\[
1_A|f - f_{n_0}|^p = \liminf_{n \geq 1} 1_A|f_n - f_{n_0}|^p
\]
Using inequality (8) and Fatou lemma (20), we obtain:
\[
\int |f - f_{n_0}|^p d\mu \leq \liminf_{n \geq 1} \int |f_n - f_{n_0}|^p d\mu \leq 1
\]
In particular, \( \int |f - f_{n_0}|^p d\mu < +\infty \). Since \( f - f_{n_0} \) is \( \mathbf{C} \)-valued and measurable, we conclude that \( f - f_{n_0} \in L^p_C(\Omega, \mathcal{F}, \mu) \).

10. Let \( p \in [1, +\infty[ \). If \( p = +\infty \), then \( f \in L^\infty_C(\Omega, \mathcal{F}, \mu) \) was proved in 8. If \( p \in [1, +\infty[ \), we saw in 9, that \( f - f_{n_0} \in L^p_C(\Omega, \mathcal{F}, \mu) \) for some \( n_0 \geq 1 \). Since \( f_{n_0} \) is itself an element of \( L^p_C(\Omega, \mathcal{F}, \mu) \), we conclude from exercise (3) that \( f = (f - f_{n_0}) + f_{n_0} \) is also an element of \( L^p_C(\Omega, \mathcal{F}, \mu) \).

11. The purpose of this exercise is to prove that given a sequence \((f_n)_{n \geq 1}\) in \( L^p_C(\Omega, \mathcal{F}, \mu) \) such that \( \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty \), there exists \( f \in L^p_C(\Omega, \mathcal{F}, \mu) \), such that \( f_n \to f \) \( \mu \)-a.s. We now assume that all \( f_n \)'s are in fact \( \mathbf{R} \)-valued, i.e. \( f_n \in \bigcap_{n=1}^{+\infty} L^p(\Omega, \mathcal{F}, \mu) \). There exists \( f^* \in L^p_C(\Omega, \mathcal{F}, \mu) \) such that \( f_n \to f^* \) \( \mu \)-a.s. However, \( f^*(\omega) \) may not be \( \mathbf{R} \)-valued for all \( \omega \in \Omega \). Yet, if \( N \in \mathcal{F} \) is such that \( \mu(N) = 0 \) and \( f_n(\omega) \to f^*(\omega) \) for all \( \omega \in N^c \), then \( f^* \) is \( \mathbf{R} \)-valued on \( N^c \) (as a limit of an \( \mathbf{R} \)-valued sequence). If we define \( f = f^*1_{N^c} \), then \( f \) is \( \mathbf{R} \)-valued and measurable, with \( \|f\|_p = \|f^*\|_p < +\infty \). So \( f \in L^p_R(\Omega, \mathcal{F}, \mu) \) and furthermore since \( f = f^* \) \( \mu \)-a.s., \( f_n \to f \) \( \mu \)-a.s.

**Exercise 12**

1. Yes, there does exist \( f \in L^\infty_C(\Omega, \mathcal{F}, \mu) \) such that \( f_n \to f \) \( \mu \)-a.s. This was precisely the object of the previous exercise.

2. Suppose \( p = +\infty \), and let \( n < m \). From exercise (4), we have \( |f_{m+1} - f_n| \leq \|f_{m+1} - f_n\|_\infty \) \( \mu \)-a.s. Furthermore, from the triangle inequality, \( \|f_{m+1} - f_n\|_\infty \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty \). It follows that:
\[
|f_{m+1} - f_n| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty , \quad \mu \text{-a.s.}
\]  

3. Suppose \( p = +\infty \) and let \( n \geq 1 \). For all \( m > n \), let \( N_m \in \mathcal{F} \) be such that \( \mu(N_m) = 0 \), and inequality (9) holds for all \( \omega \in N_{m} \). Furthermore, since \( f_{m+1} \to f \) \( \mu \)-a.s., let \( M \in \mathcal{F} \) be such that \( \mu(M) = 0 \), and \( f_{m+1}(\omega) \to f(\omega) \) for all \( \omega \in M^c \). Then, if \( N = M \cup (\cup_{m>n} N_m) \), we have \( N \in \mathcal{F} \), \( \mu(N) = 0 \) and for all \( \omega \in N^c \), \( f_{m+1}(\omega) \to f(\omega) \), together with, for all \( m > n \):
\[
|f_{m+1}(\omega) - f_n(\omega)| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty
\]

\[\text{Note that } n \geq n_0 \Rightarrow u_n \leq 1 \text{ is enough to ensure } \liminf_{n \geq 1} u_n \leq 1.\]
Taking the limit as \( m \to +\infty \), we obtain:

\[
|f(\omega) - f_n(\omega)| \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_{\infty}
\]

This being true for all \( \omega \in N^c \), we have proved that:

\[
|f - f_n| \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_{\infty}, \text{ } \mu-\text{a.s.}
\]

From definition (74), we conclude that:

\[
\|f - f_n\|_{\infty} \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_{\infty}
\]

4. Let \( p \in [1, +\infty[ \) and \( n < m \). From exercise (3), we have:

\[
\left( \int |f_{m+1} - f_n|^p d\mu \right)^{\frac{1}{p}} = \|f_{m+1} - f_n\|_p \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_p
\]

5. Let \( p \in [1, +\infty[ \) and \( n \geq 1 \). Let \( N \in \mathcal{F} \) be such that \( \mu(N) = 0 \), and \( f_{m+1}(\omega) \to f(\omega) \) for all \( \omega \in N^c \). Then, we have:

\[
|f - f_n|^p 1_{N^c} = \liminf_{m>n} |f_{m+1} - f_n|^p 1_{N^c}
\]

Using Fatou lemma (20), we obtain:

\[
\int |f - f_n|^p d\mu \leq \liminf_{m>n} \int |f_{m+1} - f_n|^p d\mu
\]

Hence, from 4. we see that:

\[
\int |f - f_n|^p d\mu \leq \left( \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p \right)^{\frac{1}{p}}
\]

and consequently:

\[
\|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p
\]

6. Let \( p \in [1, +\infty] \), whether \( p = +\infty \) or \( p \in [1, +\infty[ \), from 3. and 5., for all \( n \geq 1 \), we have \( \|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p \). Since by assumption, the series \( \sum_{k=1}^{+\infty} \|f_{k+1} - f_k\|_p \) is finite, we conclude that \( \|f - f_n\|_p \to 0 \), as \( n \to +\infty \). It follows that not only \( f_n \to f \) \( \mu \)-a.s., but also \( f_n \overset{L^p}{\to} f \).

7. Suppose \( g \in L^p_C(\Omega, \mathcal{F}, \mu) \) is such that \( f_n \overset{L^p}{\to} g \). Then \( f_n \overset{L^p}{\to} f \) together with \( f_n \overset{L^p}{\to} g \). From ex. (8), \( f = g \) \( \mu \)-a.s. Furthermore, since \( f_n \to f \) \( \mu \)-a.s., we see that \( f_n \to g \) \( \mu \)-a.s. The purpose of this exercise (and the previous) is to prove theorem (44).
Exercise 13

Exercise 14.

1. Since \( f_n \xrightarrow{L^p} f \), from exercise (10), \( (f_n)_{n \geq 1} \) is a Cauchy sequence in \( L^p_c(\Omega, \mathcal{F}, \mu) \). Using exercise (11), there exists a sub-sequence \( (f_{n_k})_{k \geq 1} \) of \( (f_n)_{n \geq 1} \), such that \( \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty \).

2. Applying theorem (44) to the sequence \( (f_{n_k})_{k \geq 1} \), there exists \( g \in L^p_c(\Omega, \mathcal{F}, \mu) \), such that \( f_{n_k} \xrightarrow{L^p} g \mu\text{-a.s.} \).

3. Also from theorem (44), the convergence \( f_{n_k} \xrightarrow{L^p} g \mu\text{-a.s.} \) and \( f_{n_k} \xrightarrow{L^p} g \) are equivalent. Hence, we also have \( f_{n_k} \xrightarrow{L^p} f \). However, since by assumption \( f_n \xrightarrow{L^p} f \), we see that \( f_{n_k} \xrightarrow{L^p} f \), and consequently from exercise (8), \( f = g \mu\text{-a.s.} \).

4. From 2., \( f_{n_k} \xrightarrow{L^p} g \mu\text{-a.s.} \), and from 3., \( f = g \mu\text{-a.s.} \). It follows that \( f_{n_k} \xrightarrow{L^p} f \mu\text{-a.s.} \). Given a sequence \( (f_n)_{n \geq 1} \) and \( f \) in \( L^p_c(\Omega, \mathcal{F}, \mu) \), such that \( f_n \xrightarrow{L^p} f \), we have been able to extract a sub-sequence \( (f_{n_k})_{k \geq 1} \) such that \( f_{n_k} \xrightarrow{L^p} f \mu\text{-a.s.} \). This proves theorem (45).

Exercise 14

Exercise 15. Suppose \( (f_n)_{n \geq 1} \) is a sequence in \( L^p_R(\Omega, \mathcal{F}, \mu) \), and \( f \in L^p_R(\Omega, \mathcal{F}, \mu) \) such that \( f_n \xrightarrow{L^p} f \). Then in particular, all \( f_n \)'s and \( f \) are elements of \( L^p_c(\Omega, \mathcal{F}, \mu) \) with \( \|f - f_n\|_p \to 0 \) as \( n \to +\infty \). From theorem (45), we can extract a sub-sequence \( (f_{n_k})_{k \geq 1} \) of \( (f_n)_{n \geq 1} \), such that \( f_{n_k} \xrightarrow{L^p} f \mu\text{-a.s.} \). This proves theorem (45), where \( L^p_c(\Omega, \mathcal{F}, \mu) \) is replaced by \( L^p_R(\Omega, \mathcal{F}, \mu) \). Anyone who feels there was very little to prove in this exercise, could make a very good point.

Exercise 15

Exercise 16.

1. Since \( (f_n)_{n \geq 1} \) is Cauchy in \( L^p_c(\Omega, \mathcal{F}, \mu) \), from exercise (11), we can extract a sub-sequence \( (f_{n_k})_{k \geq 1} \) of \( (f_n)_{n \geq 1} \), such that:
   \[
   \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty
   \]
   From theorem (44), there exists \( f \in L^p_c(\Omega, \mathcal{F}, \mu) \), such that \( f_{n_k} \xrightarrow{L^p} f \mu\text{-a.s.} \), as well as \( f_n \xrightarrow{L^p} f \).

2. Let \( \epsilon > 0 \). \( (f_n)_{n \geq 1} \) being Cauchy, there exists \( n_0 \geq 1 \), such that:
   \[
   n, m \geq n_0 \implies \|f_m - f_n\|_p \leq \frac{\epsilon}{2}
   \]
   Furthermore, since \( f_{n_k} \xrightarrow{L^p} f \), there exists \( k_0 \geq 1 \), such that:
   \[
   k \geq k_0 \implies \|f - f_{n_k}\|_p \leq \frac{\epsilon}{2}
   \]
However, $n_k \uparrow +\infty$ as $k \to +\infty$. There exists $k_0' \geq 1$, such that $k \geq k_0' \Rightarrow n_k \geq n_0$. Choose an arbitrary $k \geq \max(k_0, k_0')$. Then $\|f - f_{n_k}\|_p \leq \epsilon/2$ together with $n_k \geq n_0$. Hence, for all $n \geq n_0$, we have:

$$\|f - f_n\|_p \leq \|f - f_{n_k}\|_p + \|f_{n_k} - f_n\|_p \leq \epsilon$$

We have found $n_0 \geq 1$ such that:

$$n \geq n_0 \Rightarrow \|f - f_n\|_p \leq \epsilon$$

This shows that $f_n \overset{L^p}{\to} f$. The purpose of this exercise, is to prove theorem (46). It is customary to say in light of this theorem, that $L^p_C(\Omega, \mathcal{F}, \mu)$ is complete, even though as defined in these tutorials, $L^p_C(\Omega, \mathcal{F}, \mu)$ is not strictly speaking a metric space.

Exercise 16

**Exercise 17.** Let $(f_n)_{n \geq 1}$ be a Cauchy sequence in $L^p_R(\Omega, \mathcal{F}, \mu)$. Then in particular, it is a Cauchy sequence in $L^p_C(\Omega, \mathcal{F}, \mu)$. From theorem (46), there exists $f^* \in L^p_C(\Omega, \mathcal{F}, \mu)$ such that $f_n \overset{L^p}{\to} f^*$. Furthermore, from theorem (45), there exists a sub-sequence $(f_{n_k})_{k \geq 1}$ of $(f_n)_{n \geq 1}$, such that $f_{n_k} \overset{\mu}{\to} f^*$ $\mu$-a.s. It follows that $f^*$ is in fact $R$-valued $\mu$-almost surely. There exists $N \in \mathcal{F}$, $\mu(N) = 0$, such that $f^*(\omega) \in R$ for all $\omega \in N^c$. Take $f = f^*1_{N^c}$. Then $f$ is $R$-valued, measurable and $\|f\|_p = \|f^*\|_p < +\infty$. So $f \in L^p_R(\Omega, \mathcal{F}, \mu)$. Furthermore, $\|f - f_n\|_p = \|f^* - f_n\|_p \to 0$, which shows that $f_n \overset{L^p}{\to} f$. This proves theorem (46), where $L^p_C(\Omega, \mathcal{F}, \mu)$ is replaced by $L^p_R(\Omega, \mathcal{F}, \mu)$.

Exercise 17