13. Regular Measure

In the following, \( K \) denotes \( \mathbb{R} \) or \( \mathbb{C} \).

**Definition 99** Let \( (\Omega, \mathcal{F}) \) be a measurable space. We say that a map \( s : \Omega \to \mathbb{C} \) is a complex simple function on \( (\Omega, \mathcal{F}) \), if and only if it is of the form:

\[
s = \sum_{i=1}^{n} \alpha_i 1_{A_i}
\]

where \( n \geq 1 \), \( \alpha_i \in \mathbb{C} \) and \( A_i \in \mathcal{F} \) for all \( i \in \mathbb{N}_n \). The set of all complex simple functions on \( (\Omega, \mathcal{F}) \) is denoted \( S_C(\Omega, \mathcal{F}) \). The set of all \( \mathbb{R} \)-valued complex simple functions in \( (\Omega, \mathcal{F}) \) is denoted \( S_R(\Omega, \mathcal{F}) \).

Recall that a simple function on \( (\Omega, \mathcal{F}) \), as defined in (40), is just a non-negative element of \( S_R(\Omega, \mathcal{F}) \).

**Exercise 1.** Let \( (\Omega, \mathcal{F}, \mu) \) be a measure space and \( p \in [1, +\infty[ \). Let \( f \) be a non-negative element of \( L^p_R(\Omega, \mathcal{F}, \mu) \).

1. Suppose \( s : \Omega \to \mathbb{C} \) is of the form

\[
s = \sum_{i=1}^{n} \alpha_i 1_{A_i}
\]

where \( n \geq 1 \), \( \alpha_i \in \mathbb{C} \), \( A_i \in \mathcal{F} \) and \( \mu(A_i) < +\infty \) for all \( i \in \mathbb{N}_n \). Show that \( s \in L^p_C(\Omega, \mathcal{F}, \mu) \cap S_C(\Omega, \mathcal{F}) \).

2. Show that any \( s \in S_C(\Omega, \mathcal{F}) \) can be written as:

\[
s = \sum_{i=1}^{n} \alpha_i 1_{A_i}
\]

where \( n \geq 1 \), \( \alpha_i \in \mathbb{C} \setminus \{0\} \), \( A_i \in \mathcal{F} \) and \( A_i \cap A_j = \emptyset \) for \( i \neq j \).

3. Show that any \( s \in L^p_C(\Omega, \mathcal{F}, \mu) \cap S_C(\Omega, \mathcal{F}) \) is of the form:

\[
s = \sum_{i=1}^{n} \alpha_i 1_{A_i}
\]

where \( n \geq 1 \), \( \alpha_i \in \mathbb{C} \), \( A_i \in \mathcal{F} \) and \( \mu(A_i) < +\infty \), for all \( i \in \mathbb{N}_n \).

4. Show that \( L^p_C(\Omega, \mathcal{F}, \mu) \cap S_C(\Omega, \mathcal{F}) = S_C(\Omega, \mathcal{F}) \).

**Exercise 2.** Let \( (\Omega, \mathcal{F}, \mu) \) be a measure space and \( p \in [1, +\infty[ \). Let \( f \) be a non-negative element of \( L^p_R(\Omega, \mathcal{F}, \mu) \).

1. Show the existence of a sequence \( (s_n)_{n \geq 1} \) of non-negative functions in \( L^p_R(\Omega, \mathcal{F}, \mu) \cap S_R(\Omega, \mathcal{F}) \) such that \( s_n \uparrow f \).

2. Show that:

\[
\lim_{n \to +\infty} \int |s_n - f|^p d\mu = 0
\]
3. Show that there exists \( s \in L^p_R(\Omega, \mathcal{F}, \mu) \cap S_R(\Omega, \mathcal{F}) \) such that \( \|f - s\|_p \leq \epsilon \), for all \( \epsilon > 0 \).

4. Show that \( L^p_K(\Omega, \mathcal{F}, \mu) \cap S_K(\Omega, \mathcal{F}) \) is dense in \( L^p_K(\Omega, \mathcal{F}, \mu) \).

EXERCISE 3. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space. Let \( f \) be a non-negative element of \( L^1_R(\Omega, \mathcal{F}, \mu) \). For all \( n \geq 1 \), we define:

\[
s_n = \sum_{k=0}^{n^2-1} \frac{k}{2^n} 1_{\{k/2^n \leq f < (k+1)/2^n\}} + n1_{\{n \leq f\}}
\]

1. Show that for all \( n \geq 1 \), \( s_n \) is a simple function.

2. Show there exists \( n_0 \geq 1 \) and \( N \in \mathcal{F} \) with \( \mu(N) = 0 \), such that:

\[\forall \omega \in N^c, \quad 0 \leq f(\omega) < n_0\]

3. Show that for all \( n \geq n_0 \) and \( \omega \in N^c \), we have:

\[0 \leq f(\omega) - s_n(\omega) < \frac{1}{2^n}\]

4. Conclude that:

\[
\lim_{n \to +\infty} \|f - s_n\|_\infty = 0
\]

5. Show the following:

**Theorem 67** Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and \( p \in [1, +\infty] \). Then, \( L^p_K(\Omega, \mathcal{F}, \mu) \cap S_K(\Omega, \mathcal{F}) \) is dense in \( L^p_K(\Omega, \mathcal{F}, \mu) \).

EXERCISE 4. Let \((\Omega, \mathcal{T})\) be a metrizable topological space, and \( \mu \) be a finite measure on \((\Omega, B(\Omega))\). We define \( \Sigma \) as the set of all \( B \in B(\Omega) \) such that for all \( \epsilon > 0 \), there exist \( F \) closed and \( G \) open in \( \Omega \), with:

\[F \subseteq B \subseteq G, \quad \mu(G \setminus F) \leq \epsilon\]

Given a metric \( d \) on \((\Omega, \mathcal{T})\) inducing the topology \( \mathcal{T} \), we define:

\[d(x, A) \triangleq \inf \{d(x, y) : y \in A\}\]

for all \( A \subseteq \Omega \) and \( x \in \Omega \).

1. Show that \( x \to d(x, A) \) from \( \Omega \) to \( \mathbb{R} \) is continuous for all \( A \subseteq \Omega \).

2. Show that if \( F \) is closed in \( \Omega \), \( x \in F \) is equivalent to \( d(x, F) = 0 \).

EXERCISE 5. Further to exercise (4), we assume that \( F \) is a closed subset of \( \Omega \). For all \( n \geq 1 \), we define:

\[G_n \triangleq \{x \in \Omega : d(x, F) < \frac{1}{n}\}\]
1. Show that $G_n$ is open for all $n \geq 1$.

2. Show that $G_n \uparrow F$.

3. Show that $F \in \Sigma$.

4. Was it important to assume that $\mu$ is finite?

5. Show that $\Omega \in \Sigma$.

6. Show that if $B \in \Sigma$, then $B^c \in \Sigma$.

**Exercise 6.** Further to exercise (5), let $(B_n)_{n \geq 1}$ be a sequence in $\Sigma$. Define $B = \bigcup_{n=1}^{+\infty} B_n$ and let $\epsilon > 0$.

1. Show that for all $n$, there is $F_n$ closed and $G_n$ open in $\Omega$, with:
   
   $$F_n \subseteq B_n \subseteq G_n \ , \ \mu(G_n \setminus F_n) \leq \frac{\epsilon}{2^n}$$

2. Show the existence of some $N \geq 1$ such that:
   
   $$\mu\left(\left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus \left(\bigcup_{n=1}^{N} F_n\right)\right) \leq \epsilon$$

3. Define $G = \bigcup_{n=1}^{+\infty} G_n$ and $F = \bigcup_{n=1}^{N} F_n$. Show that $F$ is closed, $G$ is open and $F \subseteq B \subseteq G$.

4. Show that:
   
   $$G \setminus F \subseteq G \setminus \left(\bigcup_{n=1}^{+\infty} F_n\right) \cup \left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus F$$

5. Show that:
   
   $$G \setminus \left(\bigcup_{n=1}^{+\infty} F_n\right) \subseteq \bigcup_{n=1}^{+\infty} G_n \setminus F_n$$

6. Show that $\mu(G \setminus F) \leq 2\epsilon$.

7. Show that $\Sigma$ is a $\sigma$-algebra on $\Omega$, and conclude that $\Sigma = \mathcal{B}(\Omega)$.

**Theorem 68** Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and $\mu$ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $B \in \mathcal{B}(\Omega)$ and $\epsilon > 0$, there exist $F$ closed and $G$ open in $\Omega$ such that:

$$F \subseteq B \subseteq G \ , \ \mu(G \setminus F) \leq \epsilon$$

**Definition 100** Let $(\Omega, \mathcal{T})$ be a topological space. We denote $C^b_K(\Omega)$ the $K$-vector space of all continuous, bounded maps $\phi : \Omega \rightarrow K$, where $K = \mathbb{R}$ or $K = \mathbb{C}$. 

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Exercise 7. Let $(\Omega, T)$ be a metrizable topological space with some metric $d$. Let $\mu$ be a finite measure on $(\Omega, B(\Omega))$ and $F$ be a closed subset of $\Omega$. For all $n \geq 1$, we define $\phi_n : \Omega \to \mathbb{R}$ by:

$$\forall x \in \Omega, \quad \phi_n(x) \triangleq 1 - 1 \wedge (nd(x, F))$$

1. Show that for all $p \in [1, +\infty]$, we have $C_b^k(\Omega) \subseteq L^p_k(\Omega, B(\Omega), \mu)$.
2. Show that for all $n \geq 1$, $\phi_n \in C_b^k(\Omega)$.
3. Show that $\phi_n \to 1_F$.
4. Show that for all $p \in [1, +\infty]$, we have:

$$\lim_{n \to +\infty} \int |\phi_n - 1_F|^p d\mu = 0$$

5. Show that for all $p \in [1, +\infty]$ and $\epsilon > 0$, there exists $\phi \in C_b^k(\Omega)$ such that $\|\phi - 1_F\|_p \leq \epsilon$.
6. Let $\nu \in M^1(\Omega, B(\Omega))$. Show that $C_b^k(\Omega) \subseteq L^1(\Omega, B(\Omega), \nu)$ and:

$$\nu(F) = \lim_{n \to +\infty} \int \phi_n d\nu$$

7. Prove the following:

**Theorem 69** Let $(\Omega, T)$ be a metrizable topological space and $\mu, \nu$ be two complex measures on $(\Omega, B(\Omega))$ such that:

$$\forall \phi \in C_b^k(\Omega), \quad \int \phi d\mu = \int \phi d\nu$$

Then $\mu = \nu$.

Exercise 8. Let $(\Omega, T)$ be a metrizable topological space and $\mu$ be a finite measure on $(\Omega, B(\Omega))$. Let $s \in S_C(\Omega, B(\Omega))$ be a complex simple function:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbb{C}$, $A_i \in B(\Omega)$ for all $i \in \mathbb{N}_n$. Let $p \in [1, +\infty[$.

1. Show that given $\epsilon > 0$, for all $i \in \mathbb{N}_n$ there is a closed subset $F_i$ of $\Omega$ such that $F_i \subseteq A_i$ and $\mu(A_i \setminus F_i) \leq \epsilon$. Let:

$$s' \triangleq \sum_{i=1}^{n} \alpha_i 1_{F_i}$$

2. Show that:

$$\|s - s'\|_p \leq \left( \sum_{i=1}^{n} |\alpha_i| \right)^{\frac{1}{p}} \epsilon^{\frac{1}{p}}$$
3. Conclude that given $\epsilon > 0$, there exists $\phi \in C^b_0(\Omega)$ such that:

$$||\phi - s||_p \leq \epsilon$$

4. Prove the following:

**Theorem 70** Let $(\Omega, T)$ be a metrizable topological space and $\mu$ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $p \in [1, +\infty]$, $C^b_0(\Omega)$ is dense in $L^p_K(\Omega, \mathcal{B}(\Omega), \mu)$.

**Definition 101** A topological space $(\Omega, T)$ is said to be $\sigma$-compact if and only if, there exists a sequence $(K_n)_{n \geq 1}$ of compact subsets of $\Omega$ such that $K_n \uparrow \Omega$.

**Exercise 9.** Let $(\Omega, T)$ be a metrizable and $\sigma$-compact topological space, with metric $d$. Let $\Omega'$ be open in $\Omega$. For all $n \geq 1$, we define:

$$F_n = \{ x \in \Omega : d(x, (\Omega')^c) \geq 1/n \}$$

Let $(K_n)_{n \geq 1}$ be a sequence of compact subsets of $\Omega$ such that $K_n \uparrow \Omega$.

1. Show that for all $n \geq 1$, $F_n$ is closed in $\Omega$.
2. Show that $F_n \uparrow \Omega'$.
3. Show that $F_n \cap K_n \uparrow \Omega'$.
4. Show that $F_n \cap K_n$ is closed in $K_n$ for all $n \geq 1$.
5. Show that $F_n \cap K_n$ is a compact subset of $\Omega'$ for all $n \geq 1$.
6. Prove the following:

**Theorem 71** Let $(\Omega, T)$ be a metrizable and $\sigma$-compact topological space. Then, for all $\Omega'$ open subsets of $\Omega$, the induced topological space $(\Omega', T_{|\Omega'})$ is itself metrizable and $\sigma$-compact.

**Definition 102** Let $(\Omega, T)$ be a topological space and $\mu$ be a measure on $(\Omega, \mathcal{B}(\Omega))$. We say that $\mu$ is locally finite, if and only if, every $x \in \Omega$ has an open neighborhood of finite $\mu$-measure, i.e.

$$\forall x \in \Omega, \exists U \in T, x \in U, \mu(U) < +\infty$$

**Definition 103** If $\mu$ is a measure on a Hausdorff topological space $\Omega$:

We say that $\mu$ is inner-regular, if and only if, for all $B \in \mathcal{B}(\Omega)$:

$$\mu(B) = \sup \{ \mu(K) : K \subseteq B, K \text{ compact} \}$$

We say that $\mu$ is outer-regular, if and only if, for all $B \in \mathcal{B}(\Omega)$:

$$\mu(B) = \inf \{ \mu(G) : B \subseteq G, G \text{ open} \}$$

We say that $\mu$ is regular if it is both inner and outer-regular.
Exercise 10. Let \((\Omega, T)\) be a Hausdorff topological space, \(\mu\) a locally finite measure on \((\Omega, \mathcal{B}(\Omega))\), and \(K\) a compact subset of \(\Omega\).

1. Show the existence of open sets \(V_1, \ldots, V_n\) with \(\mu(V_i) < +\infty\) for all \(i \in \mathbb{N}_n\) and \(K \subseteq V_1 \cup \ldots \cup V_n\), where \(n \geq 1\).

2. Conclude that \(\mu(K) < +\infty\).

Exercise 11. Let \((\Omega, T)\) be a metrizable and \(\sigma\)-compact topological space. Let \(\mu\) be a finite measure on \((\Omega, \mathcal{B}(\Omega))\). Let \((K_n)_{n \geq 1}\) be a sequence of compact subsets of \(\Omega\) such that \(K_n \uparrow \Omega\). Let \(B \in \mathcal{B}(\Omega)\). We define \(\alpha = \sup\{\mu(K) : K \subseteq B, K \text{ compact}\}\).

1. Show that given \(\epsilon > 0\), there exists \(F\) closed in \(\Omega\) such that \(F \subseteq B\) and \(\mu(B \setminus F) \leq \epsilon\).

2. Show that \(F \setminus (K_n \cap F) \downarrow \emptyset\).

3. Show that \(K_n \cap F\) is closed in \(K_n\).

4. Show that \(K_n \cap F\) is compact.

5. Conclude that given \(\epsilon > 0\), there exists \(K\) compact subset of \(\Omega\) such that \(K \subseteq F\) and \(\mu(F \setminus K) \leq \epsilon\).

6. Show that \(\mu(B) \leq \mu(K) + 2\epsilon\).

7. Show that \(\mu(B) \leq \alpha\) and conclude that \(\mu\) is inner-regular.

Exercise 12. Let \((\Omega, T)\) be a metrizable and \(\sigma\)-compact topological space. Let \(\mu\) be a locally finite measure on \((\Omega, \mathcal{B}(\Omega))\). Let \((K_n)_{n \geq 1}\) be a sequence of compact subsets of \(\Omega\) such that \(K_n \uparrow \Omega\). Let \(B \in \mathcal{B}(\Omega)\), and \(\alpha \in \mathbb{R}\) be such that \(\alpha < \mu(B)\).

1. Show that \(\mu(K_n \cap B) \uparrow \mu(B)\).

2. Show the existence of \(K \subseteq \Omega\) compact, with \(\alpha < \mu(K \cap B)\).

3. Let \(\mu^K = \mu(K \cap \cdot)\). Show that \(\mu^K\) is a finite measure, and conclude that \(\mu^K(B) = \sup\{\mu^K(K^*) : K^* \subseteq B, K^* \text{ compact}\}\).

4. Show the existence of a compact subset \(K^*\) of \(\Omega\), such that \(K^* \subseteq B\) and \(\alpha < \mu(K \cap K^*)\).

5. Show that \(K^*\) is closed in \(\Omega\).

6. Show that \(K \cap K^*\) is closed in \(K\).

7. Show that \(K \cap K^*\) is compact.

8. Show that \(\alpha < \sup\{\mu(K') : K' \subseteq B, K' \text{ compact}\}\).
9. Show that $\mu(B) \leq \sup \{ \mu(K') : K' \subseteq B \text{ } , \text{ } K' \text{ } \text{compact} \}$. 

10. Conclude that $\mu$ is inner-regular.

**EXERCISE 13.** Let $(\Omega, T)$ be a metrizable topological space.

1. Show that $(\Omega, T)$ is separable if and only if it has a countable base.

2. Show that if $(\Omega, T)$ is compact, for all $n \geq 1$, $\Omega$ can be covered by a finite number of open balls with radius $1/n$.

3. Show that if $(\Omega, T)$ is compact, then it is separable.

**EXERCISE 14.** Let $(\Omega, T)$ be a metrizable and $\sigma$-compact topological space with metric $d$. Let $(K_n)_{n \geq 1}$ be a sequence of compact subsets of $\Omega$ such that $K_n \uparrow \Omega$.

1. For all $n \geq 1$, give a metric on $K_n$ inducing the topology $T_{|K_n}$.

2. Show that $(K_n, T_{|K_n})$ is separable.

3. Let $(x_n^p)_{p \geq 1}$ be an at most countable sequence of $(K_n, T_{|K_n})$, which is dense. Show that $(x_n^p)_{n \geq 1}$ is an at most countable dense family of $(\Omega, T)$, and conclude with the following:

**Theorem 72** Let $(\Omega, T)$ be a metrizable and $\sigma$-compact topological space. Then, $(\Omega, T)$ is separable and has a countable base.

**EXERCISE 15.** Let $(\Omega, T)$ be a metrizable and $\sigma$-compact topological space. Let $\mu$ be a locally finite measure on $(\Omega, B(\Omega))$. Let $\mathcal{H}$ be a countable base of $(\Omega, T)$.

We define $\mathcal{H}' = \{ V \in \mathcal{H} : \mu(V) < +\infty \}$.

1. Show that for all $U$ open in $\Omega$ and $x \in U$, there is $U_x$ open in $\Omega$ such that $x \in U_x \subseteq U$ and $\mu(U_x) < +\infty$.

2. Show the existence of $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U_x$.

3. Conclude that $\mathcal{H}'$ is a countable base of $(\Omega, T)$.

4. Show the existence of a sequence $(V_n)_{n \geq 1}$ of open sets in $\Omega$ with:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \text{ , } \mu(V_n) < +\infty \text{ , } \forall n \geq 1$$

**EXERCISE 16.** Let $(\Omega, T)$ be a metrizable and $\sigma$-compact topological space. Let $\mu$ be a locally finite measure on $(\Omega, B(\Omega))$. Let $(V_n)_{n \geq 1}$ a sequence of open subsets of $\Omega$ such that:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \text{ , } \mu(V_n) < +\infty \text{ , } \forall n \geq 1$$

Let $B \in B(\Omega)$ and $\alpha = \inf \{ \mu(G) : B \subseteq G \text{ , } G \text{ open} \}$. 

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1. Given \( \epsilon > 0 \), show that there exists \( G_n \) open in \( \Omega \) such that \( B \subseteq G_n \) and 
\[ \mu^V_n (G_n \setminus B) \leq \epsilon / 2^n, \] 
where \( \mu^V_n = \mu (V_n \cap \cdot) \).

2. Let \( G = \bigcup_{n=1}^{+\infty} (V_n \cap G_n) \). Show that \( G \) is open in \( \Omega \), and \( B \subseteq G \).

3. Show that \( G \setminus B = \bigcup_{n=1}^{+\infty} V_n \cap (G_n \setminus B) \).

4. Show that \( \mu (G) \leq \mu (B) + \epsilon \).

5. Show that \( \alpha \leq \mu (B) \).

6. Conclude that is \( \mu \) outer-regular.

7. Show the following:

**Theorem 73** Let \( \mu \) be a locally finite measure on a metrizable and \( \sigma \)-compact topological space \( (\Omega, T) \). Then, \( \mu \) is regular, i.e.:

\[
\mu (B) = \sup \{ \mu (K) : K \subseteq B, K \text{ compact} \} = \inf \{ \mu (G) : B \subseteq G, G \text{ open} \}
\]

for all \( B \in \mathcal{B} (\Omega) \).

**Exercise 17.** Show the following:

**Theorem 74** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \), where \( n \geq 1 \). Any locally finite measure on \( (\Omega, \mathcal{B}(\Omega)) \) is regular.

**Definition 104** We call strongly \( \sigma \)-compact topological space, a topological space \( (\Omega, T) \), for which there exists a sequence \( (V_n)_{n \geq 1} \) of open sets with compact closure, such that \( V_n \uparrow \Omega \).

**Definition 105** We call locally compact topological space, a topological space \( (\Omega, T) \), for which every \( x \in \Omega \) has an open neighborhood with compact closure, i.e. such that:

\[
\forall x \in \Omega \ , \ \exists U \in T : \ x \in U \ , \ \bar{U} \text{ is compact}
\]

**Exercise 18.** Let \( (\Omega, T) \) be a \( \sigma \)-compact and locally compact topological space. Let \( (K_n)_{n \geq 1} \) be a sequence of compact subsets of \( \Omega \) such that \( K_n \uparrow \Omega \).

1. Show that for all \( n \geq 1 \), there are open sets \( V_1^n, \ldots, V_{p_n}^n \), \( p_n \geq 1 \), such that \( K_n \subseteq V_1^n \cup \ldots \cup V_{p_n}^n \) and \( V_1^n, \ldots, V_{p_n}^n \) are compact subsets of \( \Omega \).

2. Define \( W_n = V_1^n \cup \ldots \cup V_{p_n}^n \) and \( \bar{V}_n = \bigcup_{k=1}^{p_n} W_k \), for \( n \geq 1 \). Show that \( (V_n)_{n \geq 1} \) is a sequence of open sets in \( \Omega \) with \( V_n \uparrow \Omega \).

3. Show that \( \bar{W}_n = \bar{V}_1^n \cup \ldots \cup \bar{V}_{p_n}^n \) for all \( n \geq 1 \).
4. Show that $W_n$ is compact for all $n \geq 1$.
5. Show that $V_n$ is compact for all $n \geq 1$
6. Conclude with the following:

**Theorem 75** A topological space $(\Omega, T)$ is strongly $\sigma$-compact, if and only if it is $\sigma$-compact and locally compact.

**Exercise 19.** Let $(\Omega, T)$ be a topological space and $\Omega'$ be a subset of $\Omega$. Let $A \subseteq \Omega'$. We denote $\overline{A'}$ the closure of $A$ in the induced topological space $(\Omega', T_{\Omega'})$, and $\overline{A}$ its closure in $\Omega$.

1. Show that $A \subseteq \Omega' \cap \overline{A}$.
2. Show that $\Omega' \cap \overline{A}$ is closed in $\Omega'$.
3. Show that $\overline{A'} \subseteq \Omega' \cap \overline{A}$.
4. Let $x \in \Omega' \cap A$. Show that if $x \in U' \in T_{\Omega'}$, then $A \cap U' \neq \emptyset$.
5. Show that $\overline{A'} = \Omega' \cap \overline{A}$.

**Exercise 20.** Let $(\Omega, d)$ be a metric space.

1. Show that for all $x \in \Omega$ and $\epsilon > 0$, we have:
   \[ \overline{B(x, \epsilon)} \subseteq \{ y \in \Omega : \: d(x, y) \leq \epsilon \} \]
2. Take $\Omega = [0, 1/2] \cup \{1\}$. Show that $\overline{B(0, 1)} = [0, 1/2]$.
3. Show that $[0, 1/2]$ is closed in $\Omega$.
4. Show that $\overline{B(0, 1)} = [0, 1/2]$.
5. Conclude that $\overline{B(0, 1)} \neq \{ y \in \Omega : \: |y| \leq 1 \} = \Omega$.

**Exercise 21.** Let $(\Omega, d)$ be a locally compact metric space. Let $\Omega'$ be an open subset of $\Omega$. Let $x \in \Omega'$.

1. Show there exists $U$ open with compact closure, such that $x \in U$.
2. Show the existence of $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U \cap \Omega'$.
3. Show that $\overline{B(x, \epsilon/2)} \subseteq \hat{U}$.
4. Show that $\overline{B(x, \epsilon/2)}$ is closed in $\hat{U}$.
5. Show that $\overline{B(x, \epsilon/2)}$ is a compact subset of $\Omega$.
6. Show that $\overline{B(x, \epsilon/2)} \subseteq \Omega'$.
7. Let $U' = B(x, \epsilon/2) \cap \Omega' = B(x, \epsilon/2)$. Show $x \in U' \in T_{\Omega'}$, and:

$$U'^{\Omega'} = B(x, \epsilon/2)$$

8. Show that the induced topological space $\Omega'$ is locally compact.

9. Prove the following:

**Theorem 76** Let $(\Omega, T)$ be a metrizable and strongly $\sigma$-compact topological space. Then, for all $\Omega'$ open subsets of $\Omega$, the induced topological space $(\Omega', T_{\Omega'})$ is itself metrizable and strongly $\sigma$-compact.

**Definition 106** Let $(\Omega, T)$ be a topological space, and $\phi : \Omega \to \mathbb{C}$ be a map. We call **support** of $\phi$, the closure of the set $\{\phi \neq 0\}$, i.e.:

$$\text{supp}(\phi) \triangleq \{\omega \in \Omega : \phi(\omega) \neq 0\}$$

**Definition 107** Let $(\Omega, T)$ be a topological space. We denote $C^c_K(\Omega)$ the $K$-vector space of all **continuous** map with **compact support** $\phi : \Omega \to K$, where $K = \mathbb{R}$ or $K = \mathbb{C}$.

**Exercise 22.** Let $(\Omega, T)$ be a topological space.

1. Show that $0 \in C^c_K(\Omega)$.

2. Show that $C^c_K(\Omega)$ is indeed a $K$-vector space.

3. Show that $C^c_K(\Omega) \subseteq C^c_K(\Omega)$.

**Exercise 23.** Let $(\Omega, d)$ be a locally compact metric space. Let $K$ be a compact subset of $\Omega$, and $G$ be open in $\Omega$, with $K \subseteq G$.

1. Show the existence of open sets $V_1, \ldots, V_n$ such that:

$$K \subseteq V_1 \cup \ldots \cup V_n$$

and $\hat{V}_1, \ldots, \hat{V}_n$ are compact subsets of $\Omega$.

2. Show that $V = (V_1 \cup \ldots \cup V_n) \cap G$ is open in $\Omega$, and $K \subseteq V \subseteq G$.

3. Show that $\hat{V} \subseteq \hat{V}_1 \cup \ldots \cup \hat{V}_n$.

4. Show that $\hat{V}$ is compact.

5. We assume $K \neq \emptyset$ and $V \neq \Omega$, and we define $\phi : \Omega \to \mathbb{R}$ by:

$$\forall x \in \Omega \; , \; \phi(x) \triangleq \frac{d(x, V^c)}{d(x, V^c) + d(x, K)}$$

Show that $\phi$ is well-defined and continuous.
6. Show that \( \{ \phi \neq 0 \} = V \).

7. Show that \( \phi \in C^e_K(\Omega) \).

8. Show that \( 1_K \leq \phi \leq 1_G \).

9. Show that if \( K = \emptyset \), there is \( \phi \in C^e_K(\Omega) \) with \( 1_K \leq \phi \leq 1_G \).

10. Show that if \( V = \Omega \) then \( \Omega \) is compact.

11. Show that if \( V = \Omega \), there \( \phi \in C^e_K(\Omega) \) with \( 1_K \leq \phi \leq 1_G \).

**Theorem 77** Let \((\Omega, T)\) be a metrizable and locally compact topological space. Let \( K \) be a compact subset of \( \Omega \), and \( G \) be an open subset of \( \Omega \) such that \( K \subseteq G \). Then, there exists \( \phi \in C^e_K(\Omega) \), real-valued continuous map with compact support, such that:

\[
1_K \leq \phi \leq 1_G
\]

**Exercise 24.** Let \((\Omega, T)\) be a metrizable and strongly \( \sigma \)-compact topological space. Let \( \mu \) be a locally finite measure on \((\Omega, B(\Omega))\). Let \( B \in B(\Omega) \) be such that \( \mu(B) < +\infty \). Let \( p \in [1, +\infty[ \).

1. Show that \( C^e_K(\Omega) \subseteq L^p_K(\Omega, B(\Omega), \mu) \).

2. Let \( \epsilon > 0 \). Show the existence of \( K \) compact and \( G \) open, with:

\[
K \subseteq B \subseteq G \ , \ \mu(G \setminus K) \leq \epsilon
\]

3. Where did you use the fact that \( \mu(B) < +\infty \)?

4. Show the existence of \( \phi \in C^e_K(\Omega) \) with \( 1_K \leq \phi \leq 1_G \).

5. Show that:

\[
\int |\phi - 1_B|^p d\mu \leq \mu(G \setminus K)
\]

6. Conclude that for all \( \epsilon > 0 \), there exists \( \phi \in C^e_K(\Omega) \) such that:

\[
\|\phi - 1_B\|_p \leq \epsilon
\]

7. Let \( s \in S_G(\Omega, B(\Omega)) \cap L^p_B(\Omega, B(\Omega), \mu) \). Show that for all \( \epsilon > 0 \), there exists \( \phi \in C^e_G(\Omega) \) such that \( \|\phi - s\|_p \leq \epsilon \).

8. Prove the following:

**Theorem 78** Let \((\Omega, T)\) be a metrizable and strongly \( \sigma \)-compact topological space\(^1\). Let \( \mu \) be a locally finite measure on \((\Omega, B(\Omega))\). Then, for all \( p \in [1, +\infty[ \), the space \( C^e_K(\Omega) \) of \( K \)-valued, continuous maps with compact support, is dense in \( L^p_K(\Omega, B(\Omega), \mu) \).

\(^1\)i.e. a metrizable topological space for which there exists a sequence \((V_n)_{n \geq 1}\) of open sets with compact closure, such that \( V_n \uparrow \Omega \).

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Exercise 25. Prove the following:

**Theorem 79** Let $\Omega$ be an open subset of $\mathbb{R}^n$, where $n \geq 1$. Then, for any locally finite measure $\mu$ on $(\Omega, \mathcal{B}(\Omega))$ and $p \in [1, +\infty[$, $C^\infty_K(\Omega)$ is dense in $L^p_K(\Omega, \mathcal{B}(\Omega), \mu)$. 
Solutions to Exercises

Exercise 1.

1. From definition (99), \( s \) is clearly an element of \( S_C(\Omega, \mathcal{F}) \). Furthermore, for all \( i \in \mathbb{N}_n, 1_{A_i} \) is measurable, and:

\[
\int |1_{A_i}|^p d\mu = \int 1_{A_i} d\mu = \mu(A_i) < +\infty
\]

So \( 1_{A_i} \in L_C^p(\Omega, \mathcal{F}, \mu) \). \( s \) being a linear combination of the \( 1_{A_i} \)'s is also an element of \( L_C^p(\Omega, \mathcal{F}, \mu) \). We have proved that \( s \) is an element of \( L_C^p(\Omega, \mathcal{F}, \mu) \cap S_C(\Omega, \mathcal{F}) \).

2. Let \( s \in S_C(\Omega, \mathcal{F}) \). From definition (99), \( s \) is of the form:

\[
s = \sum_{j=1}^m \beta_j 1_{B_j}
\]

where \( m \geq 1, \beta_j \in \mathbb{C}, \) and \( B_j \in \mathcal{F} \) for all \( j \in \mathbb{N}_m \). If \( s = 0 \), it can be written as \( s = 1 \times 1_{\emptyset} \) and there is nothing further to prove. We assume that \( s \neq 0 \). The map \( \theta : \{0, 1\}^m \rightarrow \mathbb{C} \) given by \( \theta(\epsilon_1, \ldots, \epsilon_m) = \sum_{j=1}^m \beta_j \epsilon_j \) being defined on a finite set, has a finite range. Since \( s(\Omega) \) is a subset of \( \theta(\{0, 1\}^m) \), \( s(\Omega) \) is also a finite set. Having assumed that \( s \neq 0 \), the set \( s(\Omega) \setminus \{0\} \) is therefore non-empty and finite. Let \( n \geq 1 \) be its cardinal, and \( \alpha : \mathbb{N}_n \rightarrow s(\Omega) \setminus \{0\} \) be an arbitrary bijection. For all \( \omega \in \Omega \), we have:

\[
s(\omega) = \sum_{i=1}^n \alpha(i) 1_{\{s = \alpha(i)\}}
\]

Since \( B_j \in \mathcal{F} \) for all \( j \)'s, \( s \) is a measurable map. If we define \( A_i = \{s = \alpha(i)\} \) for \( i \in \mathbb{N}_n \), we have \( A_i \in \mathcal{F} \). Furthermore, it is clear that \( A_i \cap A_j = \emptyset \) for \( i \neq j \). We conclude from (2) that \( s \) can be written as:

\[
s = \sum_{i=1}^n \alpha(i) 1_{A_i}
\]

where \( n \geq 1, \alpha(i) \in \mathbb{C} \setminus \{0\}, A_i \in \mathcal{F}, \) and \( A_i \cap A_j = \emptyset \) for \( i \neq j \).

3. Let \( s \in L_C^p(\Omega, \mathcal{F}, \mu) \cap S_C(\Omega, \mathcal{F}) \). From 2. \( s \) can be expressed as:

\[
s = \sum_{i=1}^n \alpha_i 1_{A_i}
\]

where \( n \geq 1, \alpha_i \neq 0, A_i \in \mathcal{F} \) and \( A_i \cap A_j = \emptyset \) for \( i \neq j \). Let \( A = A_1 \cup \ldots \cup A_n \). Then \( s(\omega) = 0 \) for all \( \omega \in A^c \) and furthermore \( 1_A = 1_{A_1} + \ldots + 1_{A_n} \). Hence:

\[
\int |s|^p d\mu = \sum_{i=1}^n \int |s|^p 1_{A_i} d\mu = \sum_{i=1}^n |\alpha_i|^p \mu(A_i) < +\infty
\]
Exercise 2.

1. If \( f \) being non-negative and measurable, from theorem (18) there exists a sequence \((s_n)_{n \geq 1}\) of simple functions on \((\Omega, \mathcal{F})\) such that \( s_n \uparrow f \). In particular, each \( s_n \) is a non-negative element of \( S_\mathbb{R}(\Omega, \mathcal{F}) \). Furthermore, \( s_n \leq f \) for all \( n \geq 1 \) and having assumed that \( f \in L^p_\mathbb{R}(\Omega, \mathcal{F}, \mu) \), we have:

\[
\int s_n^p d\mu \leq \int f^p d\mu < +\infty
\]

We conclude that \((s_n)_{n \geq 1}\) is a sequence of non-negative elements of \( L^p_\mathbb{R}(\Omega, \mathcal{F}, \mu) \cap S_\mathbb{R}(\Omega, \mathcal{F}) \) such that \( s_n \uparrow f \).

2. Since \( s_n \to f \), we have \(|s_n - f|^p \to 0\) as \( n \to +\infty \). Furthermore:

\[
|s_n - f|^p \leq (s_n + f)^p \leq 2f^p \in L^1_\mathbb{R}(\Omega, \mathcal{F}, \mu)
\]

From the dominated convergence theorem (23), we obtain:

\[
\lim_{n \to +\infty} \int |s_n - f|^p d\mu = 0
\]

3. Given \( \epsilon > 0 \), from 2. there exists \( N \geq 1 \) such that:

\[
n \geq N \Rightarrow \int |s_n - f|^p d\mu \leq \epsilon
\]

In particular, taking \( s = s_N \), we have found \( s \) belonging to the set \( L^p_\mathbb{R}(\Omega, \mathcal{F}, \mu) \cap S_\mathbb{R}(\Omega, \mathcal{F}) \) such that \( \|f - s\|_p \leq \epsilon \).

4. Let \( A_K = L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \cap S_\mathbb{K}(\Omega, \mathcal{F}) \). We claim that \( A_K \) is dense in \( L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \), i.e. that \( A_K = L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \) where \( A_K \) is the closure of \( A_K \) in \( L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \). Recall from definition (75) that for any open set \( U \) in \( L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \) and \( f \in U \), there exists \( \epsilon > 0 \) such that \( B(f, \epsilon) \subseteq U \). Hence, all we need to prove is that given \( f \in L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \) and \( \epsilon > 0 \), there exists \( s \in A_K \) such that \( \|f - s\|_p \leq \epsilon \). Indeed, if such property is proved, then for any \( f \in L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \) and \( U \) open containing \( f \), we have \( A_K \cap U \neq \emptyset \) and consequently \( f \in A_K \). So \( L^p_\mathbb{K}(\Omega, \mathcal{F}, \mu) \subseteq A_K \). Now, if \( f \in L^p_\mathbb{R}(\Omega, \mathcal{F}, \mu) \) and \( \epsilon > 0 \), the existence of \( s \in A_R \) such that \( \|f - s\|_p \leq \epsilon \) has already been proved when \( f \) is non-negative. Suppose \( f \in L^p_\mathbb{R}(\Omega, \mathcal{F}, \mu) \). Then \( f = f^+ - f^- \) where each \( f^+, f^- \) is a non-negative element of \( L^p_\mathbb{R}(\Omega, \mathcal{F}, \mu) \). There exists \( s^+, s^- \in A_R \) such that \( \|f^+ - s^+\|_p \leq \epsilon/2 \) and \( \|f^- - s^-\|_p \leq \epsilon/2 \). Taking \( s = s^+ - s^- \), we have found \( s \in A_R \) such that:

\[
\|f - s\|_p \leq \|f^+ - s^+\|_p + \|f^- - s^-\|_p \leq \epsilon
\]

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and the property is proved for \( f \in L^p_R(\Omega, \mathcal{F}, \mu) \). If \( f \) is an element of \( L^p_C(\Omega, \mathcal{F}, \mu) \), then \( f = f_1 + if_2 \) where each \( f_1, f_2 \) lies in \( L^p_R(\Omega, \mathcal{F}, \mu) \). There exists \( s_1, s_2 \in A_R \) such that \( \|f_1 - s_1\|_p \leq \epsilon/2 \) and \( \|f_2 - s_2\|_p \leq \epsilon/2 \). Taking \( s = s_1 + is_2 \), we have found \( s \in A_C \) such that:
\[
\|f - s\|_p \leq \|f_1 - s_1\|_p + \|f_2 - s_2\|_p \leq \epsilon
\]
and the property is proved for \( f \in L^p_C(\Omega, \mathcal{F}, \mu) \).

Exercise 3.

1. Given \( n \geq 1 \), \( s_n \) is of the form:
\[
s_n = \sum_{i=1}^{p} \alpha_i 1_{A_i}
\]
where \( p \geq 1 \), \( \alpha_i \in \mathbb{R}^+ \) and \( A_i \in \mathcal{F} \) for all \( i \in \mathbb{N}_p \). From definition (40), it is therefore a simple function on \((\Omega, \mathcal{F})\) (or indeed a complex simple function on \((\Omega, \mathcal{F})\) with values in \( \mathbb{R}^+ \)).

2. Since \( f \) is an element of \( L^\infty_R(\Omega, \mathcal{F}, \mu) \), we have:
\[
\|f\|_\infty \triangleq \inf \{ M \in \mathbb{R}^+ : |f| \leq M \, \mu\text{-a.s.}\} < +\infty
\]
It is therefore possible to find an integer \( n_0 \geq 1 \) such that \( \|f\|_\infty < n_0 \). Since \( \|f\|_\infty \) is the greatest lower bound all \( M \)'s such that \( |f| \leq M \, \mu\text{-a.s.} \), \( n_0 \) cannot be such lower bound. Hence, there exists \( M_0 \in \mathbb{R}^+ \) such that \( |f| \leq M_0 \, \mu\text{-a.s.} \) and \( M_0 < n_0 \). Thus, there exists \( N \in \mathcal{F} \) with \( \mu(N) = 0 \), and:
\[
\forall \omega \in N^c, \, |f(\omega)| \leq M_0 < n_0
\]
In particular, since \( f \) is a non-negative element of \( L^\infty_R(\Omega, \mathcal{F}, \mu) \):
\[
\forall \omega \in N^c, \, 0 \leq f(\omega) < n_0
\]

3. Let \( n \geq n_0 \) and \( \omega \in N^c \). From 2, we have \( 0 \leq f(\omega) < n_0 \) and consequently \( s_n(\omega) = k/2^n \), where \( k \) is the unique integer of \( \{0, \ldots, n2^n - 1\} \) such that \( f(\omega) \in [k/2^n, (k+1)/2^n] \). So:
\[
0 \leq f(\omega) - s_n(\omega) < \frac{1}{2^n} \tag{4}
\]

4. From 3, we have \( N \in \mathcal{F} \) with \( \mu(N) = 0 \) such that for all \( \omega \in N^c \), inequality (4) holds for all \( n \geq n_0 \). So \( |f - s_n| < 1/2^n \, \mu\text{-a.s.} \) for all \( n \geq n_0 \). Since \( \|f - s_n\|_\infty \) is a lower bound of all \( M \)'s such that \( |f - s_n| \leq M \, \mu\text{-a.s.} \), we conclude that \( \|f - s_n\|_\infty \leq 1/2^n \) for all \( n \geq n_0 \), and in particular:
\[
\lim_{n \to +\infty} \|f - s_n\|_\infty = 0 \tag{5}
\]
5. Let \( p \in [1, +\infty) \) be given and \( A_K = L^p_K(\Omega, \mathcal{F}, \mu) \cap S_K(\Omega, \mathcal{F}) \). If \( p \in [1, +\infty[ \), we have already proved in exercise (2) that \( A_K \) is dense in \( L^p_K(\Omega, \mathcal{F}, \mu) \). We assume that \( p = +\infty \) and we claim likewise that \( A_K \) is dense in \( L^\infty_K(\Omega, \mathcal{F}, \mu) \) (note that \( A_K \) and \( S_K(\Omega, \mathcal{F}) \) coincide when \( p = +\infty \)).

Given \( f \in L^\infty_K(\Omega, \mathcal{F}, \mu) \) and \( \epsilon > 0 \), we need to show the existence of \( s \in A_K \) such that \( \| f - s \|_\infty \leq \epsilon \). When \( K = \mathbb{R} \) and \( f \) is a non-negative element of \( L^\infty(\Omega, \mathcal{F}, \mu) \), then such existence is guaranteed by (5), (keeping in mind that simple functions on \((\Omega, \mathcal{F})\) are elements of \( S_{\mathbb{R}}(\Omega, \mathcal{F}) = A_{\mathbb{R}} \)).

If \( f \in L^\infty(\Omega, \mathcal{F}, \mu) \), then \( f = f^+ - f^- \) where each \( f^+ \), \( f^- \) is a non-negative element of \( L^\infty(\Omega, \mathcal{F}, \mu) \). There exists \( s^+ \), \( s^- \) in \( A_{\mathbb{R}} \) such that \( \| f^+ - s^+ \|_\infty \leq \epsilon/2 \) and \( \| f^- - s^- \|_\infty \leq \epsilon/2 \). Taking \( s = s^+ - s^- \) we obtain \( s \in A_{\mathbb{R}} \) and \( \| f - s \|_\infty \leq \epsilon \). This completes the proof of theorem (67) when \( K = \mathbb{R} \). If \( f \in L^\infty(\Omega, \mathcal{F}, \mu) \), then \( f = f_1 + if_2 \) where each \( f_1, f_2 \) is an element of \( L^\infty(\Omega, \mathcal{F}, \mu) \). Approximating \( f_1 \) and \( f_2 \) by elements \( s_1, s_2 \) of \( A_{\mathbb{R}} \), we obtain likewise an element \( s = s_1 + is_2 \) of \( A_{\mathbb{C}} \) with \( \| f - s \|_\infty \leq \epsilon \). This proves theorem (67).

Exercise 4.

1. Let \( A \subseteq \Omega \). If \( A = \emptyset \), then \( d(x, A) = +\infty \) for all \( x \in \Omega \). In particular, the map \( x \to d(x, A) \) is a continuous map. If \( A \neq \emptyset \) and \( y \in A \), then \( d(x, A) \leq d(x, y) \). In particular \( d(x, A) < +\infty \) for all \( x \in \Omega \). Furthermore, for all \( x, x' \in \Omega \) and \( y \in A \):

\[
d(x, A) \leq d(x, y) \leq d(x, x') + d(x', y)
\]

Consequently, \( d(x, A) - d(x, x') \) is a lower bound of all \( d(x', y) \), as \( y \) ranges through \( A \). \( d(x', A) \) being the greatest of such lower bounds, we have:

\[
d(x, A) \leq d(x, x') + d(x', A)
\]

Interchanging the roles of \( x \) and \( x' \) we obtain:

\[
d(x', A) \leq d(x, x') + d(x, A)
\]

from which we see that:

\[
\forall x, x' \in \Omega, \quad |d(x, A) - d(x', A)| \leq d(x, x') \tag{6}
\]

We conclude from (6) that \( x \to d(x, A) \) is continuous.

2. Let \( F \) be a closed subset of \( \Omega \). If \( x \in F \), \( d(x, F) \leq d(x, x) = 0 \) and consequently \( d(x, F) = 0 \). Conversely, suppose \( d(x, F) = 0 \). We shall show that \( x \notin F \) is impossible. Indeed, if \( x \in F^c \), since \( F^c \) is open, there exists \( \epsilon > 0 \) such that \( B(x, \epsilon) \subseteq F^c \). However, \( d(x, F) = 0 \) implies in particular that \( d(x, F) < \epsilon \). Since \( d(x, F) \) is the greatest of all lower bounds of \( d(x, y) \), as \( y \) range through \( F \), \( \epsilon \) cannot be such a lower bound. Hence, there exists \( y \in F \) such that \( d(x, y) < \epsilon \). So \( y \in B(x, \epsilon) \cap F \neq \emptyset \) which is a contradiction. We have proved that \( x \in F \) is equivalent to
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$d(x, F) = 0$, whenever $F$ is a closed subset of $\Omega$. This exercise is in fact a repetition of exercise (22) of Tutorial 4.

Exercise 4

Exercise 5.

1. $G_n = \{ x \in \Omega : d(x, F) < 1/n \}$ can be written as $\Phi_F^{-1}([-\infty, 1/n])$ where $\Phi_F$ is the map defined on $\Omega$ by $\Phi_F(x) = d(x, F)$. Having proved in exercise (4) that $\Phi_F$ is continuous, and since $[-\infty, 1/n]$ is open in $\mathbb{R}$, we conclude that $G_n$ is an open subset of $\Omega$.

2. It is clear that $G_{n+1} \subseteq G_n$ and $F \subseteq \cap_{n \geq 1} G_n$. Suppose that $x \in \cap_{n \geq 1} G_n$. Then $d(x, F) < 1/n$ for all $n \geq 1$ and consequently $d(x, F) = 0$. From exercise (4), $F$ being a closed subset of $\Omega$, it follows that $x \in F$. This shows that $\cap_{n \geq 1} G_n \subseteq F$ and finally $\cap_{n \geq 1} G_n = F$. So $G_n \downarrow F$.

3. Since $\mu$ is a finite measure on $(\Omega, \mathcal{B}(\Omega))$, from theorem (8) and $G_n \downarrow F$ we obtain $\mu(G_n) \rightarrow \mu(F)$ as $n \rightarrow +\infty$. Furthermore, since $F \subseteq G_n$ for all $n \geq 1$, we have:

$$\mu(G_n \setminus F) = \mu(G_n \setminus F) + \mu(F) - \mu(F) = \mu(G_n) - \mu(F)$$

It follows that $\mu(G_n \setminus F) \rightarrow 0$ as $n \rightarrow +\infty$. Given $\epsilon > 0$, there exists $N \geq 1$, such that:

$$n \geq N \Rightarrow \mu(G_n \setminus F) \leq \epsilon$$

In particular, taking $F' = F$ and $G' = G_N$, $F'$ and $G'$ are respectively closed and open subsets of $\Omega$, with $F' \subseteq F \subseteq G'$ and $\mu(G' \setminus F') \leq \epsilon$. This shows that $F \in \Sigma$. We have proved that any closed subset $F$ of $\Omega$ is an element of $\Sigma$.

4. The application of theorem (8) requires some finiteness property.

5. $\Omega$ is a closed subset of $\Omega$. So $\Omega \in \Sigma$.

6. Let $B \in \Sigma$. For all $\epsilon > 0$, there exist $F$ and $G$ respectively closed and open subsets of $\Omega$, such that $F \subseteq B \subseteq G$ and $\mu(G \setminus F) \leq \epsilon$. Since $F^c \cap G = F^c \cap G = G \setminus F$, it follows that $G^c \subseteq B^c \subseteq F^c$ and $\mu(G^c \setminus F^c) \leq \epsilon$. This shows that $B^c \in \Sigma$, since $G^c$ and $F^c$ are respectively closed and open subsets of $\Omega$. We have proved that $\Sigma$ is closed under complementation.

Exercise 6.

1. Let $n \geq 1$. By assumption $B_n$ is an element of $\Sigma$. For all $\epsilon' > 0$, and in particular for $\epsilon' = \epsilon/2^n$, there exist $F_n$ and $G_n$ respectively closed and open subsets of $\Omega$, with $F_n \subseteq B_n \subseteq G_n$ and $\mu(G_n \setminus F_n) \leq \epsilon'$.
2. Let \( H_n = \bigcup_{k=1}^{n} F_k \) and \( H = \bigcup_{k \geq 1} F_k \). Then \( H_n \uparrow H \), and consequently from theorem (7), \( \mu(H_n) \to \mu(H) \) as \( n \to +\infty \). \( \mu \) being a finite measure, we obtain:

\[
\lim_{n \to +\infty} \mu(H \setminus H_n) = \lim_{n \to +\infty} \mu(H) - \mu(H_n) = 0
\]

In particular, there exists \( N \geq 1 \) such that \( \mu(H \setminus H_N) \leq \epsilon \), or equivalently:

\[
\mu \left( \left( \bigcup_{n=1}^{+\infty} F_n \right) \setminus \left( \bigcup_{n=1}^{N} F_n \right) \right) \leq \epsilon 
\] (7)

3. Let \( G = \bigcup_{n \geq 1} G_n \) and \( F = \bigcup_{n=1}^{N} F_n \). \( G \) being a union of open subsets of \( \Omega \), is itself an open subset of \( \Omega \). \( F \) being a finite union of closed subsets of \( \Omega \), is itself a closed subset of \( \Omega \). Since \( F_n \subseteq B_n \subseteq G_n \) for all \( n \geq 1 \) and \( B = \bigcup_{n \geq 1} B_n \), it is clear that \( F \subseteq B \subseteq G \).

4. Let \( H = \bigcup_{n \geq 1} F_n \). The sets \( G \setminus H \) and \( H \setminus F \) are clearly disjoint. Furthermore if \( x \in G \setminus F = G \cap F^c \), then either \( x \in H \) or \( x \notin H \). If \( x \in H \) then \( x \in H \setminus F \). If \( x \notin H \) then \( x \in G \setminus H \). In any case, \( x \in G \setminus H \cup H \setminus F \). This shows that \( G \setminus F \subseteq G \setminus H \cup H \setminus F \).

5. Let \( H = \bigcup_{n \geq 1} F_n \) and \( x \in G \setminus H \). Since \( x \in G \), there exists \( n \geq 1 \) such that \( x \in G_n \). But \( x \in H^c \cap \bigcap_{k \geq 1} F_k^c \). So in particular \( x \in F_k^c \) and consequently \( x \in G_n \setminus F_n \). This shows that \( G \setminus H \subseteq \bigcup_{n \geq 1} G_n \setminus F_n \).

6. Applying 4. and 5. with \( H = \bigcup_{n \geq 1} F_n \), we have:

\[
G \setminus F \subseteq \left( \bigcup_{n \geq 1} G_n \setminus F_n \right) \cup H \setminus F
\]

It follows that:

\[
\mu(G \setminus F) \leq \sum_{n=1}^{+\infty} \mu(G_n \setminus F_n) + \mu(H \setminus F)
\]

Having chosen \( F_n \) and \( G_n \) such that \( \mu(G_n \setminus F_n) \leq \epsilon/2^n \) and having defined \( F \) from 2. such that \( \mu(H \setminus F) \leq \epsilon \), we conclude that \( \mu(G \setminus F) \leq 2\epsilon \).

7. Given a sequence \( (B_n)_{n \geq 1} \) in \( \Sigma \) and \( B = \bigcup_{n \geq 1} B_n \), given an arbitrary \( \epsilon > 0 \), we have shown the existence of \( F \) and \( G \) respectively closed and open subsets of \( \Omega \), such that \( F \subseteq B \subseteq G \) (see 3.) and \( \mu(G \setminus F) \leq \epsilon \) (see 6.). It follows that \( B \in \Sigma \). This shows that \( \Sigma \) is closed under countable union.

Since \( \Omega \in \Sigma \) and \( \Sigma \) is closed under complementation (see exercise (5)), \( \Sigma \) is therefore a \( \sigma \)-algebra on \( \Omega \). Furthermore, still from exercise (5), \( \Sigma \) contains every closed subset of \( \Omega \). Being closed under complementation, it also contains every open subset of \( \Omega \). In other words, the topology \( T \) is a subset of \( \Sigma \), i.e. \( T \subseteq \Sigma \). The \( \sigma \)-algebra \( \sigma(T) \) being the smallest \( \sigma \)-algebra on \( \Omega \) containing \( T \) (containing in the inclusion sense), the fact that \( \Sigma \) is a \( \sigma \)-algebra on \( \Omega \) implies that \( B(\Omega) = \sigma(T) \subseteq \Sigma \). \( \Sigma \) being a subset of the Borel \( \sigma \)-algebra \( B(\Omega) \), we conclude that \( \Sigma = B(\Omega) \). Hence, for all \( B \in B(\Omega) \) and \( \epsilon > 0 \), there exist \( F \) and \( G \) respectively closed and open subsets of \( \Omega \), such that \( F \subseteq B \subseteq G \) and \( \mu(G \setminus F) \leq \epsilon \). This proves theorem (68).
Exercise 7.

1. Let $p \in [1, +\infty]$ and $f \in C^b_k(\Omega)$. Since $f$ is continuous, $f$ is Borel measurable. Furthermore, since $f$ is bounded, there exists $M \in \mathbb{R}^+$ such that $|f| \leq M$. This implies that $\|f\|_\infty \leq M$ and in particular $\|f\|_\infty < +\infty$. So $f \in L^p_k(\Omega, \mathcal{B}(\Omega), \mu)$. Moreover, if $p \in [1, +\infty[$, $\mu$ being a finite measure on $(\Omega, \mathcal{B}(\Omega))$:

$$\int |f|^p d\mu \leq M^p \mu(\Omega) < +\infty$$

so $f \in L^p_k(\Omega, \mathcal{B}(\Omega), \mu)$, and finally $C^b_k(\Omega) \subseteq L^p_k(\Omega, \mathcal{B}(\Omega), \mu)$.

2. Let $n \geq 1$ and $\phi_n$ be defined by $\phi_n(x) = 1 - 1 \wedge (nd(x, F))$. From exercise (4), the map $x \to d(x, F)$ is continuous. So $\phi_n$ is also continuous, and furthermore it is clear that $|\phi_n(x)| \leq 1$ for all $x \in \Omega$. So $\phi_n \in C^b_k(\Omega)$.

3. Let $x \in \Omega$. If $x \in F$, then $d(x, F) = 0$ and $\phi_n(x) = 1$ for all $n \geq 1$. In particular, $\phi_n(x) \to 1_F(x)$ as $n \to +\infty$. If $x \not\in F$, then from exercise (4), $F$ being a closed subset of $\Omega$, we have $d(x, F) > 0$. It follows that:

$$\lim_{n \to +\infty} \phi_n(x) = 1 - \lim_{n \to +\infty} 1 \wedge (nd(x, F)) = 0$$

In particular, $\phi_n(x) \to 1_F(x)$ as $n \to +\infty$. So $\phi_n \to 1_F$.

4. Let $p \in [1, +\infty[$. From 3, we have $\phi_n \to 1_F$ and consequently $|\phi_n - 1_F|^p \to 0$ as $n \to +\infty$. Furthermore, for all $n \geq 1$:

$$|\phi_n - 1_F|^p \leq (|\phi_n| + |1_F|)^p \leq 2^p$$

$\mu$ being a finite measure on $(\Omega, \mathcal{B}(\Omega))$, from the dominated convergence theorem (23) we conclude that:

$$\lim_{n \to +\infty} \int |\phi_n - 1_F|^p d\mu = 0$$

5. Let $p \in [1, +\infty[$ and $\epsilon > 0$. From 4, there is $N \geq 1$ such that:

$$n \geq N \Rightarrow \int |\phi_n - 1_F|^p d\mu \leq \epsilon^p$$

In particular, taking $\phi = \phi_N$, $\phi \in C^b_k(\Omega)$ and $\|\phi - 1_F\|_p \leq \epsilon$.

6. Let $\nu$ be a complex measure on $(\Omega, \mathcal{B}(\Omega))$. From theorem (57), the total variation $|\nu|$ of $\nu$ is a finite measure on $(\Omega, \mathcal{B}(\Omega))$. It follows that $C^b_k(\Omega) \subseteq L^1_\mathbb{C}(\Omega, \mathcal{B}(\Omega), |\nu|) = L^1_\mathbb{C}(\Omega, \mathcal{B}(\Omega), \nu)$. Let $h \in L^1_\mathbb{C}(\Omega, \mathcal{B}(\Omega), |\nu|)$ be such that $|h| = 1$ and $\nu = \int h d|\nu|$. Then:

$$\left| \int \phi_n d\nu - \nu(F) \right| = \left| \int \phi_n d\nu - \int 1_F d\nu \right| = \left| \int (\phi_n - 1_F) h d|\nu| \right|$$
Solutions to Exercises 20

\[ \leq \int |\phi_n - 1_F|d\nu \]

where the second equality stems from definition (97), and the last inequality from theorem (24). We conclude from 4. applied to \( \mu = |\nu| \) and \( p = 1 \), that:

\[ \nu(F) = \lim_{n \to +\infty} \int \phi_n d\nu \]

7. Let \((\Omega, \mathcal{T})\) be a metrizable topological space, and \( \mu, \nu \) be two complex measures on \((\Omega, \mathcal{B}(\Omega))\). We assume that:

\[ \forall \phi \in \mathcal{C}_c^\infty(\Omega), \quad \int \phi d\mu = \int \phi d\nu \]  \hspace{1cm} (8)

and we claim that \( \mu = \nu \). We define:

\[ \mathcal{D} = \{ E \in \mathcal{B}(\Omega) : \mu(E) = \nu(E) \} \]

Let \( F \) be a closed subset of \( \Omega \). From 6. and (8) we have:

\[ \mu(F) = \lim_{n \to +\infty} \int \phi_n d\mu = \lim_{n \to +\infty} \int \phi_n d\nu = \nu(F) \]

So \( F \in \mathcal{D} \). Hence, any closed subset of \( \Omega \) is an element of \( \mathcal{D} \). In particular, \( \Omega \in \mathcal{D} \). Furthermore, if \( A, B \in \mathcal{D} \) with \( A \subseteq B \), then:

\[ \mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B \setminus A) \]

So \( B \setminus A \in \mathcal{D} \). Finally, if \( (E_n)_{n \geq 1} \) is a sequence of elements of \( \mathcal{D} \) with \( E_n \uparrow E \), then using exercise (13) of Tutorial 12 we have:

\[ \mu(E) = \lim_{n \to +\infty} \mu(E_n) = \lim_{n \to +\infty} \nu(E_n) = \nu(E) \]

So \( E \in \mathcal{D} \), and we have proved that \( \mathcal{D} \) is a Dynkin system on \( \Omega \). In particular, \( \mathcal{D} \) is closed under complementation, and since it contains every closed subset of \( \Omega \), it also contains every open subset of \( \Omega \). So \( \mathcal{T} \subseteq \mathcal{D} \) and finally, since \( \mathcal{T} \) is closed under finite intersection, from the Dynkin system theorem (1) we conclude that \( \mathcal{B}(\Omega) = \sigma(\mathcal{T}) \subseteq \mathcal{D} \). It follows that \( \mathcal{B}(\Omega) = \mathcal{D} \) and consequently \( \mu = \nu \), which completes the proof of theorem (69).

Exercise 7

Exercise 8.

1. Let \( \epsilon > 0 \) and \( i \in \mathbb{N}_n \). Since \( A_i \in \mathcal{B}(\Omega) \), \( \mu \) is a finite measure on \((\Omega, \mathcal{B}(\Omega))\) and \((\Omega, \mathcal{T})\) is metrizable, from theorem (68) there exist \( F_i, G_i \) respectively closed and open subsets of \( \Omega \), such that \( F_i \subseteq A_i \subseteq G_i \) and \( \mu(G_i \setminus F_i) \leq \epsilon \). In particular, \( A_i \setminus F_i \subseteq G_i \setminus F_i \) and we have \( \mu(A_i \setminus F_i) \leq \epsilon \).
2. From $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ and $s' = \sum_{i=1}^{n} \alpha_i 1_{F_i}$ we obtain:

$$\|s - s'\|_p = \left\| \sum_{i=1}^{n} \alpha_i (1_{A_i} - 1_{F_i}) \right\|_p$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \cdot \|1_{A_i} - 1_{F_i}\|_p$$

$$= \sum_{i=1}^{n} |\alpha_i| \left( \int |1_{A_i} - 1_{F_i}|^p d\mu \right)^{\frac{1}{p}}$$

$$= \sum_{i=1}^{n} |\alpha_i| \left( \int 1_{A_i \setminus F_i} d\mu \right)^{\frac{1}{p}}$$

$$= \sum_{i=1}^{n} |\alpha_i| \mu(A_i \setminus F_i)^{\frac{1}{p}}$$

$$\leq \left( \sum_{i=1}^{n} |\alpha_i| \right)^{\frac{1}{p}} \epsilon^{1/p}$$

3. Let $\epsilon > 0$. Choosing $\epsilon' > 0$ sufficiently small such that:

$$\left( \sum_{i=1}^{n} |\alpha_i| \right)^{\epsilon'/p} \leq \epsilon/2$$

and applying 2. to $\epsilon'$, there exist closed subsets $F_1, \ldots, F_n$ of $\Omega$, such that $\|s - s'\|_p \leq \epsilon/2$, where $s'$ is defined as:

$$s' = \sum_{i=1}^{n} \alpha_i 1_{F_i}$$

Furthermore for all $i \in \mathbb{N}$, from 5. of exercise (7) there exists $\phi_i \in C^b_{\mathbb{R}}(\Omega)$ such that $|\alpha_i| \cdot \|\phi_i - 1_{F_i}\|_p \leq \epsilon/2n$. We Define:

$$\phi = \sum_{i=1}^{n} \alpha_i \phi_i$$

Then $\phi \in C^b_C(\Omega)$ (in fact $\phi \in C^b_{\mathbb{R}}(\Omega)$ if $\alpha_i \in \mathbb{R}$ for all $i$'s), and:

$$\|\phi - s'\|_p = \left\| \sum_{i=1}^{n} \alpha_i (\phi_i - 1_{F_i}) \right\|_p$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \cdot \|\phi_i - 1_{F_i}\|_p$$

$$\leq \epsilon/2$$

Finally, we obtain $\|\phi - s\|_p \leq \|\phi - s'\|_p + \|s - s'\|_p \leq \epsilon$. 

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4. Suppose $(\Omega, \mathcal{T})$ is a metrizable topological space, and $\mu$ is a finite measure on $(\Omega, \mathcal{B}(\Omega))$. For all $p \in [1, +\infty[$, we clearly have $C^b_K(\Omega) \subseteq L^p_K(\Omega, \mathcal{B}(\Omega), \mu)$ and we claim that $C^b_K(\Omega)$ is in fact dense in $L^p_K(\Omega, \mathcal{B}(\Omega), \mu)$. Given $f \in L^p_K(\Omega, \mathcal{B}(\Omega), \mu)$ and $\epsilon > 0$, we have to prove the existence of $\phi \in C^b_K(\Omega)$ such that $\|f - \phi\|_p \leq \epsilon$. From theorem (67), the set $S_K(\Omega, \mathcal{B}(\Omega))$ (which is a subset of $L^p_K(\Omega, \mathcal{B}(\Omega), \mu)$ since $\mu$ is finite) is dense in $L^p_K(\Omega, \mathcal{B}(\Omega), \mu)$. There exists $s \in S_K(\Omega, \mathcal{B}(\Omega))$ such that $\|f - s\|_p \leq \epsilon/2$. Applying 3. to the $K$-valued simple function $s$, there exists $\phi \in C^b_K(\Omega)$ ($\phi$ can indeed be chosen $\mathbb{R}$-valued if $K = \mathbb{R}$), such that $\|\phi - s\|_p \leq \epsilon/2$. It follows that:

$$\|f - \phi\|_p \leq \|f - s\|_p + \|\phi - s\|_p \leq \epsilon$$

which completes the proof of theorem (70).

Exercise 8

1. $F_n = \phi^{-1}([1/n, +\infty[)$ where $\phi$ is the continuous map defined by $\phi(x) = d(x, \Omega^\mathbb{C})$. Since $[1/n, +\infty[$ is a closed subset of $\mathbb{R}$, we conclude that $F_n$ is a closed subset of $\Omega$.

2. For all $n \geq 1$ it is clear that $F_n \subseteq F_{n+1}$. Let $x \in \Omega'$. Since $\Omega'$ is an open subset of $\Omega$, $\Omega^\mathbb{C}$ is a closed subset of $\Omega$ and $x \notin \Omega^\mathbb{C}$. It follows from exercise (4) that $d(x, \Omega^\mathbb{C}) > 0$. Hence, there exists $n \geq 1$ such that $d(x, \Omega^\mathbb{C}) \geq 1/n$. So $x \in F_n$ and we have proved that $\Omega' \subseteq \cup_{n \geq 1} F_n$. To prove the reverse inclusion, suppose $x \in F_n$ for a some $n \geq 1$. Then in particular $d(x, \Omega^\mathbb{C}) > 0$ and $x$ cannot be an element of $\Omega^\mathbb{C}$. So $x \in \Omega'$. This shows that $F_n \subseteq \Omega'$ for all $n \geq 1$, and we have proved that $F_n \uparrow \Omega'$.

3. Since $F_n \subseteq F_{n+1}$ and $K_n \subseteq K_{n+1}$, $F_n \cap K_n \subseteq F_{n+1} \cap K_{n+1}$. Furthermore, it is clear that $\cup_{n \geq 1} F_n \cap K_n \subseteq \Omega'$ since $F_n \subseteq \Omega'$ for all $n \geq 1$. Finally if $x \in \Omega'$, since $F_n \uparrow \Omega'$ there exists $p \geq 1$ such that $x \in F_p$. Since $K_n \uparrow \Omega$ there exists $q \geq 1$ such that $x \in K_q$. Taking $n = \max(p, q)$, we have $x \in F_n \cap K_n$. So $\Omega' \subseteq \cup_{n \geq 1} F_n \cap K_n$ and we have proved that $F_n \cap K_n \uparrow \Omega'$.

4. Let $n \geq 1$. Since $F_n$ is closed in $\Omega$, $F_n^\mathbb{C}$ is open in $\Omega$. By the very definition of the induced topology on $K_n$, $K_n \setminus F_n = K_n \cap F_n^\mathbb{C}$ is an open subset of $K_n$. We conclude that $F_n \cap K_n$ is a closed subset of $K_n$.

5. By assumption, each $K_n$ is a compact subset of $\Omega$. Equivalently, the induced topological space $(K_n, \mathcal{T}_{K_n})$ is compact. Having proved that $F_n \cap K_n$ is a closed subset of $K_n$, from exercise (2) of Tutorial 8, $F_n \cap K_n$ is a compact subset of $K_n$, or equivalently a compact subset of $\Omega'$.

6. We have found a sequence $(F_n \cap K_n)_{n \geq 1}$ of compact subsets of $\Omega'$, such that $F_n \cap K_n \uparrow \Omega'$. This shows that the induced topological space $(\Omega', \mathcal{T}_{\Omega'})$ is $\sigma$-compact. From theorem (12), it is also metrizable, which completes the proof of theorem (71).
Exercise 10.

1. Let \( x \in K \). Since \( \mu \) is locally finite, there exists \( U_x \) open subset of \( \Omega \), such that \( x \in U_x \) and \( \mu(U_x) < +\infty \). It is clear that \( K \subseteq \bigcup_{x \in K} U_x \), and \( K \) being a compact subset of \( \Omega \), there exists a finite subset \( \{x_1, \ldots, x_n\} \) of \( K \) such that \( K \subseteq U_{x_1} \cup \ldots \cup U_{x_n} \). Taking \( V_i = U_{x_i}, \) we have found \( V_1, \ldots, V_n \) open subsets of \( \Omega \), such that \( \mu(V_i) < +\infty \) for all \( i \in \mathbb{N} \) and:

\[
K \subseteq V_1 \cup \ldots \cup V_n
\]

(9)

Note that if \( n = 0 \), \( K = \emptyset \) and it is always possible to assume \( n = 1 \) by taking \( V_1 = \emptyset \) (not a very important comment).

2. From (9) and exercise (13) of Tutorial 5, we obtain:

\[
\mu(K) \leq \mu(V_1 \cup \ldots \cup V_n) \leq \sum_{i=1}^{n} \mu(V_i) < +\infty
\]

Exercise 10

Exercise 11.

1. Let \( \epsilon > 0 \). Since \((\Omega, T)\) is metrizable and \( \mu \) is a finite measure, from theorem (68) there exist \( F, G \) respectively closed and open subsets of \( \Omega \), such that \( F \subseteq B \subseteq G \) and \( \mu(G \setminus F) \leq \epsilon \). In particular, there exists \( F \) closed with \( F \subseteq B \) and \( \mu(B \setminus F) \leq \epsilon \).

2. Since \( K_n \subseteq K_{n+1}, F \setminus (K_{n+1} \cap F) \subseteq F \setminus (K_n \cap F) \) for all \( n \geq 1 \). Moreover, we have:

\[
\bigcap_{n=1}^{+\infty} F \setminus (K_n \cap F) = \bigcap_{n=1}^{+\infty} F \cap (K_n^c \cup F^c) = F \cap \left( \bigcup_{n=1}^{+\infty} K_n \right)^c = \emptyset
\]

It follows that \( F \setminus (K_n \cap F) \downarrow \emptyset \).

3. \( F \) being a closed subset of \( \Omega \), \( K_n \cap F \) is closed with respect to the induced topology on \( K_n \). In other words, \( K_n \cap F \) is a closed subset of \( K_n \).

4. Since \( K_n \) is compact, and \( K_n \cap F \) is closed in \( K_n \), from exercise (2) of Tutorial 8, \( K_n \cap F \) is itself compact.

5. Since \( F \setminus (K_n \cap F) \downarrow \emptyset \) and \( \mu \) is a finite measure, from theorem (8) we have \( \mu(F \setminus (K_n \cap F)) \to 0 \) as \( n \to +\infty \). In particular, there exists \( n \geq 1 \) such that \( \mu(F \setminus (K_n \cap F)) \leq \epsilon \). Taking \( K = K_n \cap F, \) from 4. \( K \) is a compact subset of \( K_n \), or equivalently a compact subset of \( \Omega \). Hence, we have found a compact subset \( K \) of \( \Omega \), such that \( K \subseteq F \) and \( \mu(F \setminus K) \leq \epsilon \).
6. Since $\mu(B \setminus F) \leq \epsilon$ and $\mu(F \setminus K) \leq \epsilon$, we have:

$$\mu(B) = \mu(B \setminus F) + \mu(F)$$
$$= \mu(B \setminus F) + \mu(F \setminus K) + \mu(K)$$
$$\leq \mu(K) + 2\epsilon$$

7. We have proved in 6. that for all $B \in \mathcal{B}(\Omega)$, there exists $K$ compact with $K \subseteq B$ and $\mu(B) \leq \mu(K) + 2\epsilon$. $\alpha$ being an upper bound of all $\mu(K)$, as $K$ ranges through all compacts subsets with $K \subseteq B$, we have $\mu(K) \leq \alpha$. So $\mu(B) \leq \alpha + 2\epsilon$. This being true for all $\epsilon > 0$, it follows that $\mu(B) \leq \alpha$. Moreover, for all $K$ compact with $K \subseteq B$, we have $\mu(K) \leq \mu(B)$. So $\mu(B)$ is an upper bound of all $\mu(K)$, as $K$ ranges through compacts with $K \subseteq B$. $\alpha$ being the smallest of such upper bounds, we have $\alpha \leq \mu(B)$ and finally:

$$\mu(B) = \alpha = \sup\{\mu(K) : K \subseteq B, K \text{ compact}\}$$

This being true for all $B \in \mathcal{B}(\Omega)$, from definition (103), $\mu$ is inner-regular. We have proved that any finite measure on a metrizable, $\sigma$-compact topological space is inner-regular.

Exercise 12.

1. Since $K_n \uparrow \Omega$, we have $K_n \cap B \uparrow B$. From theorem (7), it follows that $\mu(K_n \cap B) \uparrow \mu(B)$.

2. Since $\alpha < \mu(B)$ and $\mu(K_n \cap B) \to \mu(B)$, there exists $n \geq 1$ such that $\alpha < \mu(K_n \cap B)$. Taking $K = K_n$, we have found $K$ compact subset of $\Omega$ such that $\alpha < \mu(K \cap B)$.

3. From exercise (10), $\mu$ being a locally finite measure and $K$ being compact, we have $\mu(K) < +\infty$. Hence, for all $A \in \mathcal{B}(\Omega)$:

$$\mu^K(A) = \mu(K \cap A) \leq \mu(K) < +\infty$$

So $\mu^K$ is a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Since $(\Omega, \mathcal{T})$ is metrizable and $\sigma$-compact, from exercise (11) it follows that $\mu^K$ is inner-regular. In particular:

$$\mu^K(B) = \sup\{\mu^K(K^*) : K^* \subseteq B, K^* \text{ compact}\}$$

4. It appears from 3. that $\mu^K(B)$ is the smallest upper bound of all $\mu^K(K^*)$, as $K^*$ ranges through compacts with $K^* \subseteq B$. Since $\alpha < \mu^K(B)$, $\alpha$ cannot be such an upper bound. Hence, there exists $K^*$ compact with $K^* \subseteq B$, such that $\alpha < \mu(K \cap K^*)$.

5. $(\Omega, \mathcal{T})$ being metrizable, it is a Hausdorff topological space. $K^*$ being a compact subset of $\Omega$, we conclude from theorem (35) that $K^*$ is a closed subset of $\Omega$.
6. Having proved that \( K^* \) is a closed subset of \( \Omega \), \( K \cap K^* \) is closed relative to the induced topology on \( K \). In other words, \( K \cap K^* \) is a closed subset of \( K \).

7. \( K \cap K^* \) being a closed subset of \( K \), and \( K \) being compact, from exercise (2) of Tutorial 8 we conclude that \( K \cap K^* \) is itself compact.

8. We have shown that \( \alpha < \mu(K \cap K^*) \) and that \( K \cap K^* \) is a compact subset of \( \Omega \). Since \( K^* \subseteq B \), we have \( K \cap K^* \subseteq B \) and we conclude that:
\[
\alpha < \mu(K \cap K^*) \leq \sup\{\mu(K') : K' \subseteq B, K' \text{ compact}\}
\] (10)

9. For all \( \alpha \in \mathbb{R} \) with \( \alpha < \mu(B) \), inequality (10) holds. Hence:
\[
\mu(B) \leq \sup\{\mu(K') : K' \subseteq B, K' \text{ compact}\}
\]

10. It is clear that:
\[
\sup\{\mu(K') : K' \subseteq B, K' \text{ compact}\} \leq \mu(B)
\]
We conclude that:
\[
\mu(B) = \sup\{\mu(K') : K' \subseteq B, K' \text{ compact}\}
\]
This being true for all \( B \in \mathcal{B}(\Omega) \), from definition (103), \( \mu \) is inner-regular.
We have proved that any locally finite measure on a metrizable and \( \sigma \)-compact topological space, is inner-regular.

Exercise 12

Exercise 13.

1. Let \((\Omega, \mathcal{T})\) be a metrizable topological space. Suppose \((\Omega, \mathcal{T})\) is separable. From definition (58), there exists a sequence \((x_n)_{n \geq 1}\) of elements of \(\Omega\), which are dense in \(\Omega\). The set of open balls:
\[
\mathcal{H} = \{B(x_n, 1/p) : n \geq 1, p \geq 1\}
\]
is easily seen to be a countable base of \((\Omega, \mathcal{T})\). Indeed, it is a subset of the topology \(\mathcal{T}\) which is at most countable, and for any open set \(U\) and any \(x \in U\), on can easily find \(n \geq 1\) and \(p \geq 1\) such that:
\[
x \in B(x_n, 1/p) \subseteq U
\]
So \(U\) is a union of elements of \(\mathcal{H}\). We have proved that if \((\Omega, \mathcal{T})\) is separable, then it has a countable base. Conversely, suppose \((\Omega, \mathcal{T})\) has a countable base, say \(\mathcal{H}\). For all \(V \in \mathcal{H}, V \neq \emptyset\), let \(x_V\) be an arbitrary element of \(V\). Then, the set:
\[
A = \{x_V : V \in \mathcal{H}, V \neq \emptyset\}
\]
is at most countable, and is easily seen to be dense in \(\Omega\). Indeed, for all \(x \in \Omega\) and \(\epsilon > 0\), the open ball \(B(x, \epsilon)\) being a union of elements of \(\mathcal{H}\) (see definition (57) of a countable base), we have \(x \in V \subseteq B(x, \epsilon)\) for
some $V \in \mathcal{H}$, $V \neq \emptyset$. In particular, we have found $x_V \in A$, such that $d(x, x_V) < \epsilon$. This shows that $(\Omega, T)$ is separable, and we have proved the equivalence between the separability of $(\Omega, T)$, and the fact that it has a countable base. This equivalence was already proved in slightly more detail, as part of exercise (19) of Tutorial 6.

2. We assume that $(\Omega, T)$ is not only metrizable, but also compact. Let $n \geq 1$. Then $(B(x, 1/n))_{x \in \Omega}$ is a family of open sets whose union is equal to $\Omega$ itself. In other words, it is an open covering of $\Omega$. Since $(\Omega, T)$ is compact, this open covering has a finite sub-covering. In other words, there exists an integer $p \geq 1$ and $x_1, \ldots, x_p$ in $\Omega$, such that:

$$\Omega = B(x_1, 1/n) \cup \ldots \cup B(x_p, 1/n)$$

We have proved that $\Omega$ can be covered by a finite number of open balls with radius $1/n$.

3. We assume that $(\Omega, T)$ is not only metrizable but also compact. From 2, given $n \geq 1$, $\Omega$ can be covered by a finite number, say $p_n \geq 1$, of open balls with radius $1/n$. Let $x_{1,n}, \ldots, x_{p_n,n}$ be the centers of such open balls. Then, the set $A = \{x_{k,n} : n \geq 1, k = 1, \ldots, p_n\}$ is at most countable, and we claim that it is dense in $\Omega$. Let $x \in \Omega$. We have to show that there exists $y \in A$, i.e. that given $U$ open containing $x$, we have $U \cap A \neq \emptyset$. $(\Omega, T)$ being metrizable, it is sufficient to show that given $\epsilon > 0$, $B(x, \epsilon) \cap A \neq \emptyset$. Let $n \geq 1$ be such that $1/n \leq \epsilon$. Since $x$ belongs to an open ball $B(x_{k,n}, 1/n)$ for some $k = 1, \ldots, p_n$, in particular we have $d(x, x_{k,n}) < \epsilon$. This shows that $B(x, \epsilon) \cap A \neq \emptyset$ and we have proved that $A$ is dense in $\Omega$. This shows that $(\Omega, T)$ is separable. The purpose of this exercise is to show that a metrizable compact topological space is also separable.

Exercise 14.

1. From theorem (12), the induced metric $d|_{K_n}$ induces the induced topology $T|_{K_n}$ on $K_n$.

2. By assumption, each $K_n$ is a compact subset of $\Omega$. In other words, the topological space $(K_n, T|_{K_n})$ is compact. However from 1, it is also metrizable. It follows from exercise (13) that $(K_n, T|_{K_n})$ is separable.

3. Let $A = \{x^p_n : n \geq 1, p \geq 1\}$. Then $A$ is an at most countable set, and we claim that $A$ is dense in $\Omega$. Since $(\Omega, T)$ is metrizable, given $x \in \Omega$ and $\epsilon > 0$, it is sufficient to show that $A \cap B(x, \epsilon) \neq \emptyset$. Since $\Omega = \cup_{n \geq 1} K_n$, there is $n \geq 1$ such that $x \in K_n$. By assumption, the sequence $(x^p_n)_{p \geq 1}$ is dense in $K_n$. Hence, there exists $p \geq 1$ such that $d_{K_n}(x, x^p_n) \geq \epsilon$. Equivalently, we have $d(x, x^p_n) < \epsilon$. It follows that $A \cap B(x, \epsilon) \neq \emptyset$ and we have proved that $A$ is dense in $\Omega$. This shows that $(\Omega, T)$ is separable. The purpose of this exercise is to prove that a metrizable and $\sigma$-compact topological space, is also separable. This is the objective of theorem (72).
Exercise 15.

1. Let \( U \) be open in \( \Omega \) and \( x \in U \). The measure \( \mu \) being locally finite, there exists some open set \( W_x \) such that \( x \in W_x \) and \( \mu(W_x) < +\infty \). Defining \( U_x = U \cap W_x \), \( U_x \) is an open set in \( \Omega \) such that \( x \in U_x \subseteq U \) and \( \mu(U_x) < +\infty \).

2. Since \( U_x \) is open, and \( \mathcal{H} \) is a countable base of \((\Omega, \mathcal{T})\), \( U_x \) can be expressed as a union of elements of \( \mathcal{H} \). In particular, since \( x \in U_x \), there exists some \( V_x \in \mathcal{H} \) such that \( x \in V_x \subseteq U_x \).

3. \( \mathcal{H}' \) being a subset of \( \mathcal{H} \), and \( \mathcal{H} \) being a countable base of \((\Omega, \mathcal{T})\), \( \mathcal{H}' \) is an at most countable set of open sets in \( \Omega \). Furthermore, given \( U \) open in \( \Omega \) and \( x \in U \), it follows from 1. and 2. that there exists \( \mathcal{V}_x \subset \mathcal{H} \) such that \( x \in \mathcal{V}_x \subseteq U \) and \( \mu(\mathcal{V}_x) < +\infty \). In other words, there exists \( \mathcal{V}_x \in \mathcal{H}' \) such that \( x \in \mathcal{V}_x \subseteq U \). Consequently, \( U \) can be expressed as \( U = \bigcup_{x \in U} \mathcal{V}_x \) and we have proved that any open set in \( \Omega \) can be written as a union of elements of \( \mathcal{H}' \). This shows that \( \mathcal{H}' \) is a countable base of \((\Omega, \mathcal{T})\).

4. Since \( \Omega \) is an open set in \( \Omega \), and \( \mathcal{H}' \) is a countable base of \((\Omega, \mathcal{T})\), \( \Omega \) can be written as a union of elements of \( \mathcal{H}' \). In other words, there exists a subset \( \mathcal{G} \subset \mathcal{H} \) such that \( \Omega = \bigcup_{x \in \mathcal{G}} \mathcal{V}_x \). \( \mathcal{H}' \) being at most countable, \( \mathcal{G} \) is itself at most countable. There exists a map \( \phi : \mathbb{N}^* \to \mathcal{G} \) which is surjective. So \( \Omega = \bigcup_{n \in \mathbb{N}} \phi(n) \), and defining \( V_n = \phi(n) \) we obtain \( \Omega = \bigcup_{n \in \mathbb{N}} V_n \) where each \( V_n \) is an element of \( \mathcal{G} \). In particular, each \( V_n \) is an open set in \( \Omega \) with \( \mu(V_n) < +\infty \).

Exercise 16.

1. Let \( \mu^n = \mu(V_n \cap \cdot) \). Since \( \mu(V_n) < +\infty \), \( \mu^n \) is a finite measure on \((\Omega, \mathcal{B}(\Omega)) \). Furthermore, \((\Omega, \mathcal{T})\) is a metrizable topological space. Applying theorem (68), since \( B \in \mathcal{B}(\Omega) \), there exist \( F_n \) closed and \( G_n \) open such that \( F_n \subseteq B \subseteq G_n \) and \( \mu^n(G_n \setminus F_n) \leq \epsilon/2^n \). In particular, since \( G_n \setminus B \subseteq G_n \setminus F_n \), there exists \( G_n \) open such that \( B \subseteq G_n \) and \( \mu^n(G_n \setminus B) \leq \epsilon/2^n \).

2. Let \( G = \bigcup_{n \geq 1} V_n \cap G_n \). Each \( V_n \) and \( G_n \) is an open set in \( \Omega \). So \( G \) is a union of open sets in \( \Omega \). It follows that \( G \) is an open set in \( \Omega \). Furthermore, since \( \Omega = \bigcup_{n \geq 1} V_n \) and \( B \subseteq G_n \) for all \( n \geq 1 \), we have:

\[
B = \bigcup_{n=1}^{+\infty} V_n \cap B \subseteq \bigcup_{n=1}^{+\infty} V_n \cap G_n = G
\]

3. We have:

\[
G \setminus B = G \cap B^c = \bigcup_{n=1}^{+\infty} V_n \cap G_n \cap B^c = \bigcup_{n=1}^{+\infty} V_n \cap (G_n \setminus B)
\]
4. From 3. and 1. we obtain:

\[ \mu(G \setminus B) \leq \sum_{n=1}^{+\infty} \mu(V_n \cap (G_n \setminus B)) = \sum_{n=1}^{+\infty} \mu^*_{n}(G_n \setminus B) \leq \epsilon \]

Since \( B \subseteq G \), we have \( \mu(G) = \mu(B) + \mu(G \setminus B) \) and consequently \( \mu(G) \leq \mu(B) + \epsilon \).

5. Since \( G \) is open and \( B \subseteq G \), we have \( \alpha \leq \mu(G) \). Using 4. it follows that \( \alpha \leq \mu(B) + \epsilon \). This being true for all \( \epsilon > 0 \), we conclude that \( \alpha \leq \mu(B) \).

6. For all \( G \) open with \( B \subseteq G \), we have \( \mu(B) \leq \mu(G) \). It follows that \( \mu(B) \) is a lower bound of all \( \mu(G) \)'s where \( G \) is open with \( B \subseteq G \). \( \alpha \) being the greatest of such lower bounds, we have \( \mu(B) \leq \alpha \). However, from 5. we have \( \alpha \leq \mu(B) \). It follows that \( \alpha = \mu(B) \). We have proved that for all \( B \in \mathcal{B}(\Omega) \):

\[ \mu(B) = \inf \{ \mu(G) : B \subseteq G, G \text{ open} \} \]

This shows that \( \mu \) is outer-regular.

7. In this exercise, we proved that a locally finite measure on a metrizable and \( \sigma \)-compact topological space is outer-regular. However, in exercise (12), we proved that it is also inner-regular. It follows that a locally finite measure on a metrizable and \( \sigma \)-compact topological space is regular. This proves theorem (73).

Exercise 17

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \), and \( \mu \) be a locally finite measure in \( (\Omega, \mathcal{B}(\Omega)) \). \( \mathbb{R}^n \) is a metrizable topological space, and furthermore from theorem (48) any closed and bounded subset of \( \mathbb{R}^n \) is compact. In particular, \( K_p = [-p, p]^n \) is a compact subset of \( \mathbb{R}^n \) for all \( p \geq 1 \). So \( \mathbb{R}^n \) is both metrizable and \( \sigma \)-compact. From theorem (71) it follows that the induced topological space \( (\Omega, (\mathcal{R}^n)_{|\Omega}) \) is also metrizable and \( \sigma \)-compact. Applying theorem (73), we conclude that \( \mu \) being locally finite, is a regular measure. We have proved that any locally finite measure on an open subset of \( \mathbb{R}^n \) is regular. This is the objective of theorem (74).

Exercise 18

1. Since \( (\Omega, T) \) is locally compact, for all \( x \in \Omega \), there exists \( W_x \) open in \( \Omega \) such that \( x \in W_x \) and \( \bar{W}_x \) is compact. Let \( n \geq 1 \). \( K_n \) is a compact subset of \( \Omega \). Furthermore, \( (K_n \cap W_x)_{x \in K_n} \) is an open covering of \( K_n \), from which therefore we can extract a finite sub-covering. There exists an integer \( p_n \geq 1 \) and \( x_{1}^{p_n}, \ldots, x_{p_n}^{p_n} \) elements of \( K_n \), such that:

\[ K_n = (K_n \cap W_{x_{1}^{p_n}}) \cup \ldots \cup (K_n \cap W_{x_{p_n}^{p_n}}) \]

Setting \( V_{k}^{p_n} = W_{x_{k}^{p_n}} \) for \( k = 1, \ldots, p_n \), we have found \( V_{1}^{p_n}, \ldots, V_{p_n}^{p_n} \) open subsets of \( \Omega \) such that \( K_n \subseteq V_{1}^{p_n} \cup \ldots \cup V_{p_n}^{p_n} \) and \( \bar{V}_{1}^{p_n}, \ldots, \bar{V}_{p_n}^{p_n} \) are compact subsets of \( \Omega \).

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Exercise 19.

1. Since $A \subseteq \Omega'$ and $A \subseteq \bar{A}$, we have $A \subseteq \Omega' \cap \bar{A}$. 

2. Let $W_n = V^n_1 \cup \ldots \cup V^n_{p_n}$ and $V_n = \bigcup_{k=1}^{p_n} W_k$ for $n \geq 1$. Since $V^n_1, \ldots, V^n_{p_n}$ are open, each $W_n$ is open, and consequently each $V_n$ is open. So $(V_n)_{n \geq 1}$ is a sequence of open sets in $\Omega$, and it is clear that $V_n \subseteq V_{n+1}$ for all $n \geq 1$. Let $x \in \Omega$. Since $K_n \uparrow \Omega$, in particular $\Omega = \bigcup_{n \geq 1} K_n$ and there exists $n \geq 1$ such that $x \in K_n$. From 1, we have $K_n \supseteq W_n$, and since $W_n \subseteq V_n$, it follows that $x \in V_n$. This shows that $\Omega = \bigcup_{n \geq 1} V_n$ and we have proved that $(V_n)_{n \geq 1}$ is a sequence of open sets such that $V_n \uparrow \Omega$.

3. In order to show that $\check{W}_n = \check{V}^n_1 \cup \ldots \cup \check{V}^n_{p_n}$ it is sufficient to prove that for all $A, B$ subsets of $\Omega$, we have $\check{A} \cup \check{B} = \check{A} \cup \check{B}$. Recall from exercise (21) of Tutorial 4 that the closure in $\Omega$ of any set $A$, is the smallest closed set containing $A$ (in the sense of inclusion). In particular, we have $A \subseteq \check{A}$ and $B \subseteq \check{B}$ and consequently $A \cup B \subseteq \check{A} \cup \check{B}$. However, $\check{A} \cup \check{B}$ being closed, this implies that $\check{A} \cup \check{B} \subseteq \check{A} \cup \check{B}$. Furthermore since $A \subseteq A \cup B \subseteq \check{A} \cup \check{B}$ and $\check{A} \cup \check{B}$ is closed, we have $A \subseteq \check{A} \cup \check{B}$ and likewise $B \subseteq \check{A} \cup \check{B}$. It follows that $\check{A} \cup \check{B} \subseteq \check{A} \cup \check{B}$ and we have proved the equality $\check{A} \cup \check{B} = \check{A} \cup \check{B}$.

4. Since $\check{W}_n = \check{V}^n_1 \cup \ldots \cup \check{V}^n_{p_n}$ and each $\check{V}^n_k$ is a compact subset of $\Omega$, in order to prove that $\check{W}_n$ is compact, it is sufficient to show that if $A$ and $B$ are compact subsets of $\Omega$, then $A \cup B$ is also a compact subset of $\Omega$. For that purpose we shall use the characterization of compact subsets proved in exercise (2) of Tutorial 8. Let $(U_i)_{i \in I}$ be a family of open sets in $\Omega$ such that $A \cup B \subseteq \bigcup_{i \in I} U_i$. Then in particular $A \subseteq \bigcup_{i \in I} U_i$ and $A$ being a compact subset of $\Omega$, there exists $I_1$ finite subset of $I$ such that $A \subseteq \bigcup_{i \in I_1} U_i$. Similarly, there exists $I_2$ finite subset of $I$ such that $B \subseteq \bigcup_{i \in I_2} U_i$. It follows that $A \cup B \subseteq \bigcup_{i \in I_1 \cup I_2} U_i$ and $I_1 \cup I_2$ being finite, we conclude that $A \cup B$ is a compact subset of $\Omega$.

5. Let $n \geq 1$. From 2, we have $V_n = \bigcup_{k=1}^{p_n} W_k$. Using a similar argument as in 3, we see that $V_n = \bigcup_{k=1}^{p_n} W_k$. Using a similar argument as in 4., each $W_k$ being compact by virtue of 4. itself, we conclude that $V_n$ is itself compact.

6. Let $(\Omega, T)$ be a topological space. If $(\Omega, T)$ is $\sigma$-compact and locally compact, we have been able to construct a sequence $(V_n)_{n \geq 1}$ of open sets in $\Omega$, such that $V_n \uparrow \Omega$ and $V_n$ is compact for all $n \geq 1$. So $(\Omega, T)$ is strongly $\sigma$-compact. Conversely, suppose that $(\Omega, T)$ is strongly $\sigma$-compact, and let $(V_n)_{n \geq 1}$ be a sequence of open sets in $\Omega$, such that $V_n \uparrow \Omega$ and each $V_n$ is compact. Then $V_n \uparrow \Omega$ and $\Omega$ is therefore $\sigma$-compact. Furthermore, for all $x \in \Omega$, there exists $n \geq 1$ such that $x \in V_n$. Since $V_n$ is open and $V_n$ is compact, this shows that $\Omega$ is locally compact. This completes the proof of theorem (75).

Exercise 18

1. Since $A \subseteq \Omega'$ and $A \subseteq \bar{A}$, we have $A \subseteq \Omega' \cap \bar{A}$.
2. The complement of $\Omega' \cap \bar{A}$ in $\Omega'$ is:

$$\Omega' \setminus (\Omega' \cap \bar{A}) = \Omega' \setminus (\Omega^c \cup \bar{A}^c) = \Omega' \setminus \bar{A}^c$$

Since $\bar{A}$ is closed in $\Omega$, $\bar{A}^c$ is open in $\Omega$ and consequently by definition of the induced topology, $\Omega' \cap \bar{A}^c$ is open in $\Omega'$. It follows that $\Omega' \cap \bar{A}$ is closed in $\Omega'$. Note more generally that if $F$ is closed in $\Omega$, then $\Omega' \setminus F$ is closed in $\Omega'$.

3. The closure $\bar{A}'$ of $A$ in $\Omega'$ being the smallest closed subset of $\Omega'$ containing $A$, we conclude from $A \subseteq \Omega' \cap \bar{A}$ and $\Omega' \cap \bar{A}$ closed in $\Omega'$, that $\bar{A}' \subseteq \Omega' \cap \bar{A}$.

4. Let $x \in \Omega' \cap \bar{A}$. Suppose $U' \in \mathcal{T}_{\Omega'}$ and $x \in U'$. There exists $U \in \mathcal{T}$ such that $U' = U \cap \Omega'$. From $x \in U'$, we have $x \in U$ and since $x \in \bar{A}$, we obtain that $A \cap U \neq \emptyset$. However by assumption, $A$ is a subset of $\Omega'$. Hence:

$$A \cap U' = A \cap (U \cap \Omega') = (A \cap \Omega') \cap U = A \cap U \neq \emptyset$$

So we have proved that $A \cap U' \neq \emptyset$.

5. It follows from 4. that $\Omega' \cap \bar{A} \subseteq \bar{A}'$. However from 3. we have $\bar{A}' \subseteq \Omega' \cap \bar{A}$.

We conclude that $\bar{A}' = \Omega' \cap \bar{A}$.

Exercise 19

Exercise 20.

1. Let $x \in \Omega$ and $\epsilon > 0$. Let $y \in \overline{B(x, \epsilon)}$. For all $U$ open in $\Omega$ such that $y \in U$, we have $U \cap B(x, \epsilon) \neq \emptyset$. In particular, for all $\eta > 0$, we have $B(y, \eta) \cap B(x, \epsilon) \neq \emptyset$. Let $z \in \Omega$ be such that $d(y, z) < \eta$ and $d(x, z) < \epsilon$.

From the triangle inequality:

$$d(x, y) \leq d(x, z) + d(y, z) < \epsilon + \eta$$

This being true for all $\eta > 0$, it follows that $d(x, y) \leq \epsilon$. We have proved that:

$$\overline{B(x, \epsilon)} \subseteq \{ y \in \Omega : d(x, y) \leq \epsilon \}$$

2. Let $\Omega = [0, 1/2] \cup \{1\}$ together with its usual metric. Then, the open ball $B(0, 1)$ is given by:

$$B(0, 1) = \{ x \in \Omega : |x| < 1 \} = [0, 1/2]$$

3. The complement of $[0, 1/2]$ in $\Omega$ is $\{1\}$, which can be written as $]1/2, 2[ \cap \Omega$ and is therefore open in $\Omega$, since $]1/2, 2[$ is open in $\mathbb{R}$. It follows that $[0, 1/2]$ is closed in $\Omega$.

4. From 2. we have $B(0, 1) = [0, 1/2]$ and from 3. $[0, 1/2]$ is a closed subset of $\Omega$, and is therefore equal to its closure. Hence:

$$\overline{B(0, 1)} = [0, 1/2] = [0, 1/2]$$
5. Since \( \Omega = \{ y \in \Omega : |y| \leq 1 \} \) and \([0, 1/2] \neq \Omega\), we conclude that:

\[
\overline{B(0, 1)} \neq \{ y \in \Omega : |y| \leq 1 \}
\]

The purpose of this exercise is to provide a counter-example to the belief that the inclusion proved in 1.

\[
\overline{B(x, \epsilon)} \subseteq \{ y \in \Omega : d(x, y) \leq \epsilon \}
\]

can be shown to be an equality.

Exercise 20

1. \( \Omega \) being locally compact, there exists \( U \) open with compact closure such that \( x \in U \).

2. Since \( x \in \Omega' \) and \( x \in U \), we have \( x \in U \cap \Omega' \). Furthermore, both \( U \) and \( \Omega' \) being open in \( \Omega \), \( U \cap \Omega' \) is open in \( \Omega \). The topology on \( \Omega \) being metric, there exists \( \epsilon > 0 \) such that \( B(x, \epsilon) \subseteq U \cap \Omega' \).

3. From \( B(x, \epsilon/2) \subseteq B(x, \epsilon) \subseteq U \cap \Omega' \subseteq U \) we conclude that \( \overline{B(x, \epsilon/2)} \subseteq \overline{U} \).

4. From 3. we have \( \overline{B(x, \epsilon/2)} = \overline{B(x, \epsilon/2)} \cap \overline{U} \) and \( \overline{B(x, \epsilon/2)} \) being closed in \( \Omega \), we conclude that it is also closed in \( \overline{U} \).

5. Since \( \overline{U} \) is compact and \( \overline{B(x, \epsilon/2)} \) is a closed subset of \( \overline{U} \), it follows from exercise (2) of Tutorial 8 that \( \overline{B(x, \epsilon/2)} \) is a compact subset of \( \overline{U} \), and consequently also a compact subset of \( \Omega \).

6. Let \( y \in \overline{B(x, \epsilon/2)} \). From 1. of exercise (20), \( d(x, y) \leq \epsilon/2 \) and in particular \( d(x, y) < \epsilon \). From 2, we have \( B(x, \epsilon) \subseteq \Omega' \) and consequently \( y \in \Omega' \). This shows that \( \overline{B(x, \epsilon/2)} \subseteq \Omega' \).

7. Let \( U' = B(x, \epsilon/2) \cap \Omega' = B(x, \epsilon/2) \). It is clear that \( x \in U' \) and furthermore \( B(x, \epsilon/2) \) being open in \( \Omega, U' \) is open in \( \Omega' \), i.e. \( U' \in T_{\Omega'} \). Using 6. and exercise (19), we obtain:

\[
\overline{U'} \cap \Omega' = \overline{B(x, \epsilon/2)} \cap \Omega' = \overline{B(x, \epsilon/2)}
\]

In particular \( \overline{U'} \cap \Omega' \) is compact, as can be seen from 5.

8. Given \( x \in \Omega' \), we have found \( U' \) open in \( \Omega' \) such that \( x \in U' \) and \( \overline{U'} \) is compact. This shows that \((\Omega', T_{\Omega'})\) is locally compact.

9. Let \((\Omega, T)\) be a metrizable and strongly \( \sigma \)-compact topological space. Let \( \Omega' \) be an open subset of \( \Omega \). From theorem (75), \((\Omega, T)\) is metrizable, \( \sigma \)-compact and locally compact. Since \( \Omega' \) is open, it follows from theorem (71) that the induced topological space \((\Omega', T_{\Omega'})\) is itself metrizable and \( \sigma \)-compact. Furthermore, we have proved in this exercise that \((\Omega', T_{\Omega'})\) is also locally compact. So \((\Omega', T_{\Omega'})\) is metrizable, \( \sigma \)-compact and locally compact. Using theorem (75) once more, we conclude that
(Ω’, T_{Ω’}) is metrizable and strongly σ-compact. This completes the proof of theorem (76).

Exercise 21

Exercise 22.

1. The constant map \( \phi : x \to 0 \) is continuous. Indeed for any \( U \) open in \( K \), \( \phi^{-1}(U) \) is either equal to \( \emptyset \) or to \( \Omega \) itself. In any case \( \phi^{-1}(U) \) is an open subset of \( \Omega \). Furthermore, \( \text{supp}(\phi) = \emptyset \) and is therefore compact (see exercise (2) of Tutorial 8). This shows that \( \phi \in C^c_K(\Omega) \).

2. \( C^c_K(\Omega) \) being a non-empty subset of the set of all maps \( \phi : \Omega \to K \), to show that \( C^c_K(\Omega) \) is a \( K \)-vector space, it is sufficient to show that given \( \phi, \psi \in C^c_K(\Omega) \) and \( \lambda \in K \), the map \( \phi + \lambda \psi \) is also an element of \( C^c_K(\Omega) \). To show that \( \phi + \lambda \psi \) is continuous, one may proceed as follows: define \( \Phi : K^2 \to K \) by \( \Phi(x, y) = x + \lambda y \), and \( \Psi : \Omega \to K^2 \) by \( \Psi(\omega) = (\phi(\omega), \psi(\omega)) \). Then \( \phi + \lambda \psi = \Phi \circ \Psi \) and \( \Phi \) being continuous, it is sufficient to show that \( \Psi \) is itself a continuous map. However, the continuity of \( \Psi \) follows from the fact that each coordinate mapping \( \phi \) and \( \psi \) is continuous. Indeed if \( U \times V \) is an open rectangle in \( K^2 \), then \( \Psi^{-1}(U \times V) = \phi^{-1}(U) \cap \psi^{-1}(V) \) and is therefore open in \( \Omega \). Any open set \( W \) in \( K^2 \) being a union of open rectangles, it is clear that \( \Psi^{-1}(W) \) is open in \( \Omega \). So much for the continuity of \( \phi + \lambda \psi \). From the inclusion:

\( \{\phi + \lambda \psi \neq 0\} \subseteq \{\phi \neq 0\} \cup \{\psi \neq 0\} \)

and the fact that given \( A, B \) subsets of \( \Omega, \overline{A \cup B} = \overline{A} \cup \overline{B} \) (see the proof of 3. in exercise (18)), we obtain:

\( \text{supp}(\phi + \lambda \psi) \subseteq \text{supp}(\phi) \cup \text{supp}(\psi) \)

Since \( \phi \) and \( \psi \) lie in \( C^c_K(\Omega) \), both \( \text{supp}(\phi) \) and \( \text{supp}(\psi) \) are compact and consequently \( A = \text{supp}(\phi) \cup \text{supp}(\psi) \) is itself compact (see the proof of 4. in exercise (18)). Furthermore, \( \text{supp}(\phi + \lambda \psi) \) being closed in \( \Omega \) while being a subset of \( A \), it is also closed in \( A \). From exercise (2) of Tutorial 8, \( \text{supp}(\phi + \lambda \psi) \) is therefore compact. We have proved that \( \phi + \lambda \psi \in C^c_K(\Omega) \).

3. Let \( \phi \in C^c_K(\Omega) \). If \( \phi = 0 \) then \( \phi \in C^b_K(\Omega) \). We assume that \( \phi \neq 0 \). Let \( A = \text{supp}(\phi) \). Then \( |\phi|_A \) is a continuous map defined on the non-empty compact topological space \( (A, \mathcal{T}_A) \). From theorem (37), \( |\phi|_A \) attains its maximum, i.e. there exists \( x_M \in A \) such that:

\[ |\phi(x_M)| = \sup_{x \in A} |\phi(x)| \]

Since \( \phi(x) = 0 \) for all \( x \in A^c \), we have:

\[ |\phi(x_M)| = \sup_{x \in \Omega} |\phi(x)| \]

which shows in particular that \( \sup_{x \in \Omega} |\phi(x)| < +\infty \). So \( \phi \in C^b_K(\Omega) \) and we have proved that \( C^c_K(\Omega) \subseteq C^b_K(\Omega) \).
Exercise 23.

1. Since \( \Omega \) is locally compact, for all \( x \in \Omega \) there exists an open set \( W_x \) such that \( x \in W_x \) and \( W_x \) is compact. From \( K \subseteq \bigcup_{x \in K} W_x \) and the fact that \( K \) is a compact subset of \( \Omega \), we deduce the existence of \( n \geq 1 \) and \( x_1, \ldots, x_n \in K \) such that \( K \subseteq \bigcup_{k=1}^n W_{x_k} \). Setting \( V_k = W_{x_k} \) for all \( k = 1, \ldots, n \), we have found open sets \( V_1, \ldots, V_n \) such that:

\[
K \subseteq V_1 \cup \ldots \cup V_n \tag{11}
\]

and each \( V_k \) is compact.

2. An arbitrary union of open sets is open. A finite intersection of open sets \( A;B \) is also open. By assumption, \( K \subseteq G \) and it therefore follows from (11) that \( K \subseteq V \). The fact that \( V \subseteq G \) is clear. We have proved that \( V \) is open and \( K \subseteq V \subseteq G \).

3. Given \( A, B \) subsets of \( \Omega \), \( A \cup B = \bar{A} \cup \bar{B} \) (see proof of 3. in exercise (18)). From \( V \subseteq V_1 \cup \ldots \cup V_n \) we obtain:

\[
\bar{V} \subseteq \bar{V}_1 \cup \ldots \cup \bar{V}_n = \bar{V}_1 \cup \ldots \cup \bar{V}_n
\]

4. If \( A, B \) are compact subsets of \( \Omega \), \( A \cup B \) is a compact subset of \( \Omega \) (see proof of 4. in exercise (18)). It follows that \( K' = \bar{V}_1 \cup \ldots \cup \bar{V}_n \) is a compact subset of \( \Omega \). Furthermore from 3. \( \bar{V} \) is a subset of \( K' \). Being closed in \( \Omega \), \( \bar{V} \) is also closed in \( \Omega \). Using exercise (2) of Tutorial 8, it follows that \( \bar{V} \) is compact.

5. Given \( A \) subset of \( \Omega \), \( d(x, A) \) is well defined for all \( x \in \Omega \) as:

\[
d(x, A) = \inf \{ d(x, y) : y \in A \}
\]

where it is understood that \( \inf \emptyset = +\infty \). Since \( K \neq \emptyset \) and \( V \neq \Omega \), \( d(x, K) \) and \( d(x, V^c) \) are well-defined real numbers for all \( x \in \Omega \). Furthermore, for all \( A \) closed in \( \Omega \), \( d(x, A) = 0 \) is equivalent to \( x \in A \) (see exercise (22) of Tutorial 4). \( V \) being open in \( \Omega \), \( V^c \) is a closed subset of \( \Omega \). So \( d(x, V^c) = 0 \) is equivalent to \( x \in V^c \). \( K \) being a compact subset of \( \Omega \) and \( \Omega \) a Hausdorff topological space (it is metric), \( K \) is a closed subset of \( \Omega \) (see theorem (35)). So \( d(x, K) = 0 \) is equivalent to \( x \in K \). It follows that \( d(x, V^c) + d(x, K) = 0 \) is equivalent to \( x \in K \cap V^c \), which can never happen since \( K \subseteq V \). We have proved that for all \( x \in \Omega \), \( \phi(x) \) is a well-defined real number. So \( \phi : \Omega \to \mathbb{R} \) is well-defined. For all \( A \) subsets of \( \Omega \), the map \( x \to d(x, A) \) is continuous (see exercise (22) of Tutorial 4). We conclude that \( \phi \) is also continuous.

6. \( \phi(x) \neq 0 \) is equivalent to \( d(x, V^c) \neq 0 \) which is itself equivalent to \( x \not\in V^c \) (since \( V^c \) is closed), i.e. \( x \in V \). We have proved that \( \{ \phi \neq 0 \} = V \).
7. From 7, \( \{ \phi \neq 0 \} = V \) and consequently \( \text{supp}(\phi) = \bar{V} \). Having proved in 4. that \( \bar{V} \) is compact, it follows that \( \phi \) has compact support. So \( \phi : \Omega \to \mathbb{R} \) is continuous with compact support, i.e. \( \phi \in \mathcal{C}^c(\Omega) \).

8. To show that \( 1_K \leq \phi \) it is sufficient to show that \( x \in K \) implies \( 1 \leq \phi(x) \). However, \( K \) being closed in \( \Omega \), \( x \in K \) is equivalent to \( d(x, K) = 0 \). In particular, \( x \in K \) implies that \( \phi(x) = 1 \). It is clear that \( \phi(x) \leq 1 \) for all \( x \in \Omega \). To show that \( \phi \leq 1_G \), it is sufficient to show that \( x \notin G \) implies \( \phi(x) = 0 \). But \( V \subseteq G \) and consequently \( x \notin V \), i.e. \( x \in V^c \). And \( V^c \) being closed, \( x \in V^c \) is equivalent to \( d(x, V^c) = 0 \). In particular, we see that \( x \notin G \) implies \( \phi(x) = 0 \). So \( 1_K \leq \phi \leq 1_G \).

9. Suppose \( K = \emptyset \). With \( \phi = 0 \), \( \phi \in \mathcal{C}^c(\Omega) \) and \( 1_K \leq \phi \leq 1_G \).

10. Suppose \( V = \Omega \). Then \( \bar{V} = \bar{\Omega} = \Omega \). \( \bar{V} \) being compact (see 4.), it follows that \( \Omega \) is compact.

11. Suppose \( V = \Omega \). Since \( V \subseteq G \), we have \( G = \Omega \), i.e. \( 1_G = 1 \). Take \( \phi = 1 \). Then \( \phi \) is continuous and \( \text{supp}(\phi) = \Omega \) is compact (see 10.). So \( \phi \in \mathcal{C}^c(\Omega) \) and \( 1_K \leq \phi \leq 1_G \). This proves theorem (77).

Exercise 23

Exercise 24.

1. Let \( \phi \in \mathcal{C}^c_K(\Omega) \). Then \( \phi \) is continuous and from exercise (13) of Tutorial 4, the map \( \phi : (\Omega, \mathcal{B}(\Omega)) \to (K, \mathcal{B}(K)) \) is therefore measurable. Furthermore from exercise (22), \( \mathcal{C}^c_K(\Omega) \subseteq \mathcal{C}^b_K(\Omega) \). So \( \phi \) is also bounded. There exists \( m \in \mathbb{R}^+ \) such that \( |\phi| \leq m \). Let \( A = \text{supp}(\phi) \). Then \( A \) is a compact subset of \( \Omega \), and from exercise (10), \( \mu \) being locally finite, \( \mu(A) < +\infty \). Since \( \{ \phi \neq 0 \} \subseteq A \), we have \( A^c \subseteq \{ \phi = 0 \} \) and consequently \( \phi = \phi 1_A \). Hence:

\[
\int |\phi|^p d\mu = \int 1_A |\phi|^p d\mu \leq m^p \mu(A) < +\infty
\]

So \( \phi \in L^p_K(\Omega, \mathcal{B}(\Omega), \mu) \) and finally \( \mathcal{C}^c_K(\Omega) \subseteq L^p_K(\Omega, \mathcal{B}(\Omega), \mu) \).

2. Let \( \epsilon > 0 \). Since \( (\Omega, T) \) is metrizable and strongly \( \sigma \)-compact, in particular from theorem (75), it is metrizable and \( \sigma \)-compact. Since \( \mu \) is a locally finite measure on \( (\Omega, \mathcal{B}(\Omega)) \), from theorem (73) \( \mu \) is regular. Having assumed that \( \mu(B) < +\infty \), we have \( \mu(B) < \mu(B) + \epsilon/2 \). From the outer-regularity of \( \mu \), \( \mu(B) \) is the greatest lower-bound of all \( \mu(G) \)'s where \( G \) is open with \( B \subseteq G \). So \( \mu(B) + \epsilon/2 \) cannot be such lower-bound. There exists \( G \) open with \( B \subseteq G \) such that:

\[
\mu(G) < \mu(B) + \frac{\epsilon}{2}
\]

Likewise, \( \mu(B) - \epsilon/2 < \mu(B) \) and from the inner-regularity of \( \mu \), \( \mu(B) \) is the lowest upper-bound of all \( \mu(K) \)'s where \( K \) is compact with \( K \subseteq B \).
So $\mu(B) - \varepsilon/2$ cannot be such upper-bound, and consequently, there exists $K$ compact with $K \subseteq B$ such that:

$$\mu(B) - \frac{\varepsilon}{2} < \mu(K)$$

(13)

Hence, we have found $K$ compact and $G$ open with $K \subseteq B \subseteq G$, and furthermore from (12) and (13) we have:

$$\mu(G) < \mu(B) + \frac{\varepsilon}{2} < \mu(K) + \varepsilon$$

and consequently:

$$\mu(K) + \mu(G \setminus K) = \mu(G) < \mu(K) + \varepsilon$$

$K$ being compact and $\mu$ locally finite, from exercise (10) we have $\mu(G \setminus K) < +\infty$, and we conclude that $\mu(G \setminus K) < \varepsilon$. In particular $\mu(G \setminus K) \leq \varepsilon$.

3. The fact that $\mu(B) < +\infty$ was used when writing the inequalities $\mu(B) < \mu(B) + \varepsilon/2$ and $\mu(B) - \varepsilon/2 < \mu(B)$. Without this assumption, these inequalities would not be strict, and the argument developed in 2. would fail.

4. Since $(\Omega, T)$ is metrizable and strongly $\sigma$-compact, in particular from theorem (75), it is metrizable and locally compact. $K$ being compact and $G$ open with $K \subseteq G$, from theorem (77), there exists $\phi \in C_R^\infty(\Omega)$ such that $1_K \leq \phi \leq 1_G$.

5. Since $1_K \leq \phi \leq 1_G$, in particular $0 \leq \phi \leq 1$ and consequently we have $|\phi - 1_B|^p \leq 1$. Suppose $x \notin G$. Then $1_G(x) = 0$ and therefore $\phi(x) = 0$. Since $B \subseteq G$, we also have $1_B(x) = 0$ and consequently $|\phi(x) - 1_B(x)|^p = 0$. Suppose $x \in K$. Then $1_K(x) = 1$ and therefore $\phi(x) = 1$. Since $K \subseteq B$ we also have $1_B(x) = 1$ and consequently $|\phi(x) - 1_B(x)|^p = 0$. We have proved that $x \notin G \setminus K$ implies that $|\phi(x) - 1_B(x)|^p = 0$. It follows that $|\phi - 1_B|^p \leq 1_{G \setminus K}$ and finally:

$$\int |\phi - 1_B|^p d\mu \leq \int 1_{G \setminus K} d\mu = \mu(G \setminus K)$$

6. Let $\varepsilon > 0$. Applying 2. to $\varepsilon^p$ instead of $\varepsilon$ itself, we can find $K$ and $G$ such that $\mu(G \setminus K) \leq \varepsilon^p$. From 4. and 5. there exists $\phi \in C_R^\infty(\Omega)$ such that:

$$\int |\phi - 1_B|^p d\mu \leq \mu(G \setminus K) \leq \varepsilon^p$$

from which we conclude that $\|\phi - 1_B\|_p \leq \varepsilon$.

7. Let $s \in S_C(\Omega, B(\Omega)) \cap L^p_C(\Omega, B(\Omega), \mu)$ and $\varepsilon > 0$. From 3. of exercise (1) there exists an integer $n \geq 1$, together with $\alpha_1, \ldots, \alpha_n \in C$ and $A_1, \ldots, A_n \in B(\Omega)$ such that:

$$s = \sum_{i=1}^n \alpha_i 1_{A_i}$$
and $\mu(A_i) < +\infty$ for all $i \in \mathbb{N}$. Without loss of generality, we may assume that $\alpha_i \neq 0$ for all $i$'s (if $s = 0$ then $s \in C_c^\infty(\Omega)$ and finding $\phi \in C_c^\infty(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon$ is trivial). Applying 6. to $B = A_i$ (recall that $A_i \in \mathcal{B}(\Omega)$ and $\mu(A_i) < +\infty$) and $\epsilon_n/|\alpha_i|$ instead of $\epsilon$, there exists $\phi \in C_c^\infty(\Omega)$ such that $\|\phi - 1_{A_i}\|_p \leq \epsilon_n/|\alpha_i|$. Since $C_c^\infty(\Omega)$ is a vector space, the map $\phi = \sum_{i=1}^n \alpha_i \phi_i$ is an element of $C_c^\infty(\Omega)$ and we have:

$$\|\phi - s\|_p = \left\| \sum_{i=1}^n \alpha_i \phi_i - \sum_{i=1}^n \alpha_i 1_{A_i} \right\|_p \leq \sum_{i=1}^n |\alpha_i| \cdot \|\phi_i - 1_{A_i}\|_p \leq \sum_{i=1}^n |\alpha_i| \cdot \left( \frac{\epsilon}{n|\alpha_i|} \right) = \frac{\epsilon}{\epsilon}.$$  

We have found $\phi \in C_c^\infty(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon$. Note that if $s \in \mathcal{S}_R(\Omega, \mathcal{B}(\Omega))$ then $\alpha_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, and $\phi = \sum_{i=1}^n \alpha_i \phi_i$ is in fact an element of $C_c^\infty(\Omega)$.

8. To show that $C_c^\infty(\Omega)$ is dense in $L^p_k(\Omega, \mathcal{B}(\Omega), \mu)$, it is sufficient to show that given $f \in L^p_k(\Omega, \mathcal{B}(\Omega), \mu)$ and $\epsilon > 0$, there exists $\phi \in C_c^\infty(\Omega)$ such that $\|f - \phi\|_p \leq \epsilon$. However, from theorem (67) there exists $s \in \mathcal{S}_K(\Omega, \mathcal{B}(\Omega)) \cap L^p_k(\Omega, \mathcal{B}(\Omega), \mu)$ such that $\|f - s\|_p \leq \epsilon/2$. Applying 7. to $s$ and $\epsilon/2$ instead of $\epsilon$, there exists $\phi \in C_c^\infty(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon/2$. It follows that we have found $\phi \in C_c^\infty(\Omega)$ such that $\|f - \phi\|_p \leq \|f - s\|_p + \|\phi - s\|_p \leq \epsilon$. This completes the proof of theorem (78).

Exercise 24

**Exercise 25.** Let $\Omega$ be an open subset of $\mathbb{R}^n$ where $n \geq 1$. Let $\mu$ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$ and $p \in [1, +\infty[$. For $k \geq 1$, $V_k = ]-k, k[^n$ is an open subset of $\mathbb{R}^n$ with compact closure, and $V_k \uparrow \mathbb{R}^n$. From definition (104), $\mathbb{R}^n$ is strongly $\sigma$-compact. Furthermore, it is metrizable. It follows from theorem (76) that $\Omega$ being an open subset of $\mathbb{R}^n$, is also metrizable and strongly $\sigma$-compact. Applying theorem (78), we conclude that $C_c^\infty(\Omega)$ is dense in $L^p_k(\Omega, \mathcal{B}(\Omega), \mu)$. This completes the proof of theorem (79).

Exercise 25