6. Product Spaces

In the following, $I$ is a non-empty set.

**Definition 50** Let $(\Omega_i)_{i \in I}$ be a family of sets, indexed by a non-empty set $I$. We call **Cartesian product** of the family $(\Omega_i)_{i \in I}$ the set, denoted $\Pi_{i \in I} \Omega_i$, and defined by:

$$\prod_{i \in I} \Omega_i \triangleq \{ \omega : I \to \bigcup_{i \in I} \Omega_i, \ \omega(i) \in \Omega_i, \ \forall i \in I \}$$

In other words, $\Pi_{i \in I} \Omega_i$ is the set of all maps $\omega$ defined on $I$, with values in $\bigcup_{i \in I} \Omega_i$, such that $\omega(i) \in \Omega_i$ for all $i \in I$.

**Theorem 25 (Axiom of choice)** Let $(\Omega_i)_{i \in I}$ be a family of sets, indexed by a non-empty set $I$. Then, $\Pi_{i \in I} \Omega_i$ is non-empty, if and only if $\Omega_i$ is non-empty for all $i \in I$.

**Exercise 1.**

1. Let $\Omega$ be a set and suppose that $\Omega_i = \Omega, \forall i \in I$. We use the notation $\Omega^I$ instead of $\Pi_{i \in I} \Omega_i$. Show that $\Omega^I$ is the set of all maps $\omega : I \to \Omega$.

2. What are the sets $\mathbb{R}^{\mathbb{R}^+}$, $\mathbb{R}^{\mathbb{N}}$, $[0,1]^\mathbb{N}$, $\mathbb{R}^{\mathbb{R}^?}$?

3. Suppose $I = \mathbb{N}^*$, We sometimes use the notation $\Pi_{n=1}^{\infty} \Omega_n$ instead of $\Pi_{n \in \mathbb{N}} \Omega_n$. Let $S$ be the set of all sequences $(x_n)_{n \geq 1}$ such that $x_n \in \Omega_n$ for all $n \geq 1$. Is $S$ the same thing as the product $\Pi_{n=1}^{\infty} \Omega_n$?

4. Suppose $I = \{1, \ldots, n\}$, $n \geq 1$. We use the notation $\Omega_1 \times \ldots \times \Omega_n$ instead of $\Pi_{i \in \{1, \ldots, n\}} \Omega_i$. For $\omega \in \Omega_1 \times \ldots \times \Omega_n$, it is customary to write $(\omega_1, \ldots, \omega_n)$ instead of $\omega$, where we have $\omega_i = \omega(i)$. What is your guess for the definition of sets such as $\mathbb{R}^n$, $\mathbb{R}^n$, $\mathbb{Q}^n$, $\mathbb{C}^n$.

5. Let $E, F, G$ be three sets. Define $E \times F \times G$.

**Definition 51** Let $I$ be a non-empty set. We say that a family of sets $(I_\lambda)_{\lambda \in \Lambda}$, where $\Lambda \neq \emptyset$, is a **partition** of $I$, if and only if:

(i) $\forall \lambda \in \Lambda, \ I_\lambda \neq \emptyset$

(ii) $\forall \lambda, \lambda' \in \Lambda, \ \lambda \neq \lambda' \Rightarrow I_\lambda \cap I_{\lambda'} = \emptyset$

(iii) $I = \bigcup_{\lambda \in \Lambda} I_\lambda$

**Exercise 2.** Let $(\Omega_i)_{i \in I}$ be a family of sets indexed by $I$, and $(I_\lambda)_{\lambda \in \Lambda}$ be a partition of the set $I$.

1. For each $\lambda \in \Lambda$, recall the definition of $\Pi_{i \in I_\lambda} \Omega_i$.

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1 When $I$ is finite, this theorem is traditionally derived from other axioms.
2. Recall the definition of $\Pi_{\lambda \in \Lambda}(\Omega_i)$. 

3. Define a natural bijection $\Phi: \Pi_{i \in I} \Omega_i \rightarrow \Pi_{\lambda \in \Lambda}(\Omega_i)$. 

4. Define a natural bijection $\psi: R^p \times R^n \rightarrow R^{p+n}$, for all $n, p \geq 1$. 

**Definition 52** Let $(\Omega_i)_{i \in I}$ be a family of sets, indexed by a non-empty set $I$. For all $i \in I$, let $\mathcal{E}_i$ be a set of subsets of $\Omega_i$. We define a rectangle of the family $(\mathcal{E}_i)_{i \in I}$, as any subset $A$ of $\pi_{i \in I} \Omega_i$, of the form $A = \pi_{i \in I} \Omega_i$ where $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$ for all $i \in I$, and such that $A_i = \Omega_i$ except for a finite number of indices $i \in I$. Consequently, the set of all rectangles, denoted $\pi_{i \in I} \mathcal{E}_i$, is defined as:

$$\prod_{i \in I} \mathcal{E}_i \triangleq \left\{ \prod_{i \in I} A_i : A_i \in \mathcal{E}_i \cup \{\Omega_i\}, \ A_i \neq \Omega_i \text{ for finitely many } i \in I \right\}$$

**Exercise 3.** $(\Omega_i)_{i \in I}$ and $(\mathcal{E}_i)_{i \in I}$ being as above:

1. Show that if $I = N_n$ and $\Omega_i \in \mathcal{E}_i$ for all $i = 1, \ldots, n$, then $\mathcal{E}_1 \ldots \mathcal{E}_n = \{A_1 \times \ldots \times A_n : A_i \in \mathcal{E}_i, \ \forall i \in I\}$.

2. Let $A$ be a rectangle. Show that there exists a finite subset $J$ of $I$ such that: $A = \{\omega \in \pi_{i \in I} \Omega_i : \omega(j) \in A_j, \ \forall j \in J\}$ for some $A_j$’s such that $A_j \in \mathcal{E}_j$, for all $j \in J$.

**Definition 53** Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces, indexed by a non-empty set $I$. We call measurable rectangle, any rectangle of the family $(\mathcal{F}_i)_{i \in I}$. The set of all measurable rectangles is given by:

$$\prod_{i \in I} \mathcal{F}_i \triangleq \left\{ \prod_{i \in I} A_i : A_i \in \mathcal{F}_i, \ A_i \neq \Omega_i \text{ for finitely many } i \in I \right\}$$

**Definition 54** Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces, indexed by a non-empty set $I$. We define the product $\sigma$-algebra of $(\mathcal{F}_i)_{i \in I}$, as the $\sigma$-algebra on $\pi_{i \in I} \Omega_i$, denoted $\otimes_{i \in I} \mathcal{F}_i$, and generated by all measurable rectangles, i.e.

$$\otimes_{i \in I} \mathcal{F}_i \triangleq \sigma \left( \prod_{i \in I} \mathcal{F}_i \right)$$

**Exercise 4.**

1. Suppose $I = N_n$. Show that $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$ is generated by all sets of the form $A_1 \times \ldots \times A_n$, where $A_i \in \mathcal{F}_i$ for all $i = 1, \ldots, n$.

2. Show that $\mathcal{B}(R) \otimes \mathcal{B}(R) \otimes \mathcal{B}(R)$ is generated by sets of the form $A \times B \times C$ where $A, B, C \in \mathcal{B}(R)$. 

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3. Show that if \((\Omega, \mathcal{F})\) is a measurable space, \(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}\) is the \(\sigma\)-algebra on \(\mathbb{R}^+ \times \Omega\) generated by sets of the form \(B \times F\) where \(B \in \mathcal{B}(\mathbb{R}^+)\) and \(F \in \mathcal{F}\).

**Exercise 5.** Let \((\Omega_i)_{i \in I}\) be a family of non-empty sets and \(\mathcal{E}_i\) be a subset of the power set \(\mathcal{P}(\Omega_i)\) for all \(i \in I\).

1. Give a generator of the \(\sigma\)-algebra \(\bigotimes_{i \in I} \sigma(\mathcal{E}_i)\) on \(\Pi_{i \in I} \Omega_i\).

2. Show that:
   \[
   \sigma \left( \prod_{i \in I} \mathcal{E}_i \right) \subseteq \bigotimes_{i \in I} \sigma(\mathcal{E}_i)
   \]

3. Let \(A\) be a rectangle of the family \((\sigma(\mathcal{E}_i))_{i \in I}\). Show that if \(A\) is not empty, then the representation \(A = \Pi_{i \in I} A_i\) with \(A_i \in \sigma(\mathcal{E}_i)\) is unique. Define \(J_A = \{i \in I : A_i \neq \Omega_i\}\). Explain why \(J_A\) is a well-defined finite subset of \(I\).

4. If \(A \in \Pi_{i \in I} \sigma(\mathcal{E}_i)\), Show that if \(A = \emptyset\), or \(A \neq \emptyset\) and \(J_A = \emptyset\), then \(A \in \sigma(\Pi_{i \in I} \mathcal{E}_i)\).

**Exercise 6.** Everything being as before, Let \(n \geq 0\). We assume that the following induction hypothesis has been proved:

\[A \in \prod_{i \in I} \sigma(\mathcal{E}_i), A \neq \emptyset, \text{card}J_A = n \Rightarrow A \in \sigma \left( \prod_{i \in I} \mathcal{E}_i \right)\]

We assume that \(A\) is a non empty measurable rectangle of \((\sigma(\mathcal{E}_i))_{i \in I}\) with \(\text{card}J_A = n + 1\). Let \(J_A = \{i_1, \ldots, i_{n+1}\}\) be an extension of \(J_A\). For all \(B \subseteq \Omega_{i_1}\), we define:

\[A^B \triangleq \prod_{i \in I} \tilde{A}_i\]

where each \(\tilde{A}_i\) is equal to \(A_i\) except \(\tilde{A}_{i_1} = B\). We define the set:

\[\Gamma \triangleq \left\{ B \subseteq \Omega_{i_1} : A^B \in \sigma \left( \prod_{i \in I} \mathcal{E}_i \right) \right\}\]

1. Show that \(A^{\Omega_{i_1}} \neq \emptyset\), \(\text{card}J_A^{\Omega_{i_1}} = n\) and that \(A^{\Omega_{i_1}} \in \Pi_{i \in I} \sigma(\mathcal{E}_i)\).

2. Show that \(\Omega_{i_1} \in \Gamma\).

3. Show that for all \(B \subseteq \Omega_{i_1}\), we have \(A^{\Omega_{i_1} \setminus B} = A^{\Omega_{i_1}} \setminus A^B\).

4. Show that \(B \in \Gamma \Rightarrow \Omega_{i_1} \setminus B \in \Gamma\).

5. Let \(B_n \subseteq \Omega_{i_1}, n \geq 1\). Show that \(A^{\cup B_n} = \cup_{n \geq 1} A^{B_n}\).

6. Show that \(\Gamma\) is a \(\sigma\)-algebra on \(\Omega_{i_1}\).
7. Let \( B \in \mathcal{E}_i \), and for \( i \in I \) define \( B_i = \Omega_i \) for all \( i \)'s except \( B_{i_1} = B \). Show that \( A^B = A^{\Omega_i} \cap (\Pi_{i \in I} B_i) \).

8. Show that \( \sigma(\mathcal{E}_i) \subseteq \Gamma \).

9. Show that \( A = A^{\Omega_i} \) and \( A \in \sigma(\Pi_{i \in I} \mathcal{E}_i) \).

10. Show that \( \Pi_{i \in I} \sigma(\mathcal{E}_i) \subseteq \sigma(\Pi_{i \in I} \mathcal{E}_i) \).

11. Show that \( \sigma(\Pi_{i \in I} \mathcal{E}_i) = \otimes_{i \in I} \sigma(\mathcal{E}_i) \).

**Theorem 26** Let \((\Omega_i)_{i \in I}\) be a family of non-empty sets indexed by a non-empty set \( I \). For all \( i \in I \), let \( \mathcal{E}_i \) be a set of subsets of \( \Omega_i \). Then, the product \( \sigma \)-algebra \( \otimes_{i \in I} \sigma(\mathcal{E}_i) \) on the Cartesian product \( \Pi_{i \in I} \Omega_i \) is generated by the rectangles of \((\mathcal{E}_i)_{i \in I} \), i.e.:

\[
\bigotimes_{i \in I} \sigma(\mathcal{E}_i) = \sigma \left( \prod_{i \in I} \mathcal{E}_i \right)
\]

**Exercise 7.** Let \( \mathcal{T}_R \) denote the usual topology in \( R \). Let \( n \geq 1 \).

1. Show that \( \mathcal{T}_R \cup \ldots \cup \mathcal{T}_R = \{ A_1 \times \ldots \times A_n : A_i \in \mathcal{T}_R \} \).

2. Show that \( B(\mathbb{R}) \otimes \ldots \otimes B(\mathbb{R}) = \sigma(\mathcal{T}_R \cup \ldots \cup \mathcal{T}_R) \).

3. Define \( C_2 = \{ [a_1, b_1] \times \ldots \times [a_n, b_n] : a_i, b_i \in \mathbb{R} \} \). Show that \( C_2 \subseteq \mathcal{S} \cup \ldots \cup \mathcal{S} \), where \( \mathcal{S} = \{ [a, b] : a, b \in \mathbb{R} \} \), but that the inclusion is strict.

4. Show that \( \mathcal{S} \cup \ldots \cup \mathcal{S} \subseteq \sigma(C_2) \).

5. Show that \( B(\mathbb{R}) \otimes \ldots \otimes B(\mathbb{R}) = \sigma(C_2) \).

**Exercise 8.** Let \( \Omega \) and \( \Omega' \) be two non-empty sets. Let \( A \) be a subset of \( \Omega \) such that \( \emptyset \neq A \neq \Omega \). Let \( \mathcal{E} = \{ A \} \subseteq \mathcal{P}(\Omega) \) and \( \mathcal{E}' = \emptyset \subseteq \mathcal{P}(\Omega') \).

1. Show that \( \sigma(\mathcal{E}) = \{ \emptyset, A, A^c, \Omega \} \).

2. Show that \( \sigma(\mathcal{E}') = \{ \emptyset, \Omega' \} \).

3. Define \( \mathcal{C} = \{ E \times F : E \in \mathcal{E}, F \in \mathcal{E}' \} \) and show that \( \mathcal{C} = \emptyset \).

4. Show that \( \mathcal{E} \cup \mathcal{E}' = \{ A \times \Omega', \Omega \times \Omega' \} \).

5. Show that \( \sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') = \{ \emptyset, A \times \Omega', A^c \times \Omega', \Omega \times \Omega' \} \).

6. Conclude that \( \sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') \neq \sigma(\mathcal{C}) = \{ \emptyset, \Omega \times \Omega' \} \).

**Exercise 9.** Let \( n \geq 1 \) and \( p \geq 1 \) be two positive integers.

1. Define \( \mathcal{F} = B(\mathbb{R}) \otimes \ldots \otimes B(\mathbb{R}) \) and \( \mathcal{G} = B(\mathbb{R}) \otimes \ldots \otimes B(\mathbb{R}) \). Explain why \( \mathcal{F} \otimes \mathcal{G} \) can be viewed as a \( \sigma \)-algebra on \( \mathbb{R}^{n+p} \).

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2. Show that \( \mathcal{F} \otimes \mathcal{G} \) is generated by sets of the form \( A_1 \times \ldots \times A_{n+p} \) where \( A_i \in \mathcal{B}(\mathbb{R}), i = 1, \ldots, n + p \).

3. Show that:
\[
\bigcap_{n+p} \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = \left( \bigcap_{n} \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) \right) \otimes \left( \bigcap_{p} \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) \right)
\]

**Exercise 10.** Let \((\Omega_i, \mathcal{F}_i)_{i \in I}\) be a family of measurable spaces. Let \((I_\lambda)_{\lambda \in \Lambda}\), where \( \Lambda \neq \emptyset \), be a partition of \( I \). Let \( \Omega = \Pi_{i \in I} \Omega_i \) and \( \Omega' = \Pi_{\lambda \in \Lambda} (\Pi_{i \in I_\lambda} \Omega_i) \).

1. Define a *natural* bijection between \( \mathcal{P}(\Omega) \) and \( \mathcal{P}(\Omega') \).
2. Show that through such bijection, \( A = \Pi_{i \in I} A_i \subseteq \Omega \), where \( A_i \subseteq \Omega_i \), is identified with \( A' = \Pi_{\lambda \in \Lambda} (\Pi_{i \in I_\lambda} A_i) \subseteq \Omega' \).
3. Show that \( \Pi_{i \in I} \mathcal{F}_i = \Pi_{\lambda \in \Lambda} (\Pi_{i \in I_\lambda} \mathcal{F}_i) \).
4. Show that \( \otimes_{i \in I} \mathcal{F}_i = \otimes_{\lambda \in \Lambda} (\otimes_{i \in I_\lambda} \mathcal{F}_i) \).

**Definition 55** Let \( \Omega \) be set and \( \mathcal{A} \) be a set of subsets of \( \Omega \). We call **topology generated** by \( \mathcal{A} \), the topology on \( \Omega \), denoted \( T(\mathcal{A}) \), equal to the intersection of all topologies on \( \Omega \), which contain \( \mathcal{A} \).

**Exercise 11.** Let \( \Omega \) be a set and \( \mathcal{A} \subseteq \mathcal{P}(\Omega) \).

1. Explain why \( T(\mathcal{A}) \) is indeed a topology on \( \Omega \).
2. Show that \( T(\mathcal{A}) \) is the smallest topology \( T \) such that \( \mathcal{A} \subseteq T \).
3. Show that the metric topology on a metric space \((E,d)\) is generated by the open balls \( \mathcal{A} = \{ B(x, \epsilon) : x \in E, \epsilon > 0 \} \).

**Definition 56** Let \((\Omega_i, T_i)_{i \in I}\) be a family of topological spaces, indexed by a non-empty set \( I \). We define the **product topology** of \((T_i)_{i \in I}\), as the topology on \( \Pi_{i \in I} \Omega_i \), denoted \( \otimes_{i \in I} T_i \), and generated by all rectangles of \((T_i)_{i \in I}\), i.e.
\[
\bigotimes_{i \in I} T_i \triangleq T \left( \prod_{i \in I} T_i \right)
\]

**Exercise 12.** Let \((\Omega_i, T_i)_{i \in I}\) be a family of topological spaces.

1. Show that \( U \in \bigotimes_{i \in I} T_i \), if and only if:
\[
\forall x \in U, \ \exists V \in \Pi_{i \in I} T_i, \ x \in V \subseteq U
\]
2. Show that \( \Pi_{i \in I} T_i \subseteq \bigotimes_{i \in I} T_i \).
3. Show that \( \otimes_{i \in I} \mathcal{B}(\Omega_i) = \sigma(\Pi_{i \in I} T_i) \).
4. Show that \( \otimes_{i \in I} \mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Pi_{i \in I} \Omega_i) \).

**Exercise 13.** Let \( n \geq 1 \) be a positive integer. For all \( x, y \in \mathbb{R}^n \), let:

\[
(x, y) \triangleq \sum_{i=1}^{n} x_i y_i
\]

and we put \( \|x\| = \sqrt{(x, x)} \).

1. Show that for all \( t \in \mathbb{R} \), \( \|x + ty\|^2 = \|x\|^2 + t^2\|y\|^2 + 2t(x, y) \).
2. From \( \|x + ty\|^2 \geq 0 \) for all \( t \), deduce that \( |(x, y)| \leq \|x\|\|y\| \).
3. Conclude that \( \|x + y\| \leq \|x\| + \|y\| \).

**Exercise 14.** Let \( (\Omega_1, \tau_1), \ldots, (\Omega_n, \tau_n), n \geq 1, \) be metrizable topological spaces. Let \( d_1, \ldots, d_n \) be metrics on \( \Omega_1, \ldots, \Omega_n \), inducing the topologies \( \tau_1, \ldots, \tau_n \) respectively. Let \( \Omega = \Omega_1 \times \ldots \times \Omega_n \) and \( \tau \) be the product topology on \( \Omega \). For all \( x, y \in \Omega \), we define:

\[
d(x, y) \triangleq \sqrt{\sum_{i=1}^{n} (d_i(x_i, y_i))^2}
\]

1. Show that \( d : \Omega \times \Omega \to \mathbb{R}^+ \) is a metric on \( \Omega \).
2. Show that \( U \subseteq \Omega \) is open in \( \Omega \), if and only if, for all \( x \in U \) there are open sets \( U_1, \ldots, U_n \) in \( \Omega_1, \ldots, \Omega_n \) respectively, such that:\n
\[
x \in U_1 \times \ldots \times U_n \subseteq U
\]
3. Let \( U \in \tau \) and \( x \in U \). Show the existence of \( \epsilon > 0 \) such that:

\[(\forall i = 1, \ldots, n) \ d_i(x_i, y_i) < \epsilon \implies y \in U \]
4. Show that \( \tau \subseteq \tau_{\Omega}^d \).
5. Let \( U \in \tau_{\Omega}^d \) and \( x \in U \). Show the existence of \( \epsilon > 0 \) such that:

\[x \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon) \subseteq U\]
6. Show that \( \tau_{\Omega}^d \subseteq \tau \).
7. Show that the product topological space \( (\Omega, \tau) \) is metrizable.
8. For all \( x, y \in \Omega \), define:

\[
d'(x, y) \triangleq \sum_{i=1}^{n} d_i(x_i, y_i)
\]

\[
d''(x, y) \triangleq \max_{i=1,...,n} d_i(x_i, y_i)
\]

Show that \( d', d'' \) are metrics on \( \Omega \).
9. Show the existence of $\alpha', \beta', \alpha''$ and $\beta'' > 0$, such that we have $\alpha' d' \leq d \leq \beta' d'$ and $\alpha'' d'' \leq d \leq \beta'' d''$.

10. Show that $d'$ and $d''$ also induce the product topology on $\Omega$.

**Exercise 15.** Let $(\Omega_n, T_n)_{n \geq 1}$ be a sequence of metrizable topological spaces. For all $n \geq 1$, let $d_n$ be a metric on $\Omega_n$ inducing the topology $T_n$. Let $\Omega = \prod_{n=1}^{+\infty} \Omega_n$ be the Cartesian product and $T$ be the product topology on $\Omega$. For all $x, y \in \Omega$, we define:

$$d(x, y) \triangleq \sum_{n=1}^{+\infty} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n))$$

1. Show that for all $a, b \in \mathbb{R}^+$, we have $1 \wedge (a + b) \leq 1 \wedge a + 1 \wedge b$.

2. Show that $d$ is a metric on $\Omega$.

3. Show that $U \subseteq \Omega$ is open in $\Omega$, if and only if, for all $x \in U$, there is an integer $N \geq 1$ and open sets $U_1, \ldots, U_N$ in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

4. Show that $d(x, y) < 1/2^n \Rightarrow d_n(x_n, y_n) \leq 2^n d(x, y)$.

5. Show that for all $U \in T$ and $x \in U$, there exists $\epsilon > 0$ such that $d(x, y) < \epsilon \Rightarrow y \in U$.

6. Show that $T \subseteq T_{\Omega}^d$.

7. Let $U \in T_{\Omega}^d$ and $x \in U$. Show the existence of $\epsilon > 0$ and $N \geq 1$, such that:

$$\sum_{n=1}^{N} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n)) < \epsilon \Rightarrow y \in U$$

8. Show that for all $U \in T_{\Omega}^d$ and $x \in U$, there is $\epsilon > 0$ and $N \geq 1$ such that:

$$x \in B(x_1, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

9. Show that $T_{\Omega}^d \subseteq T$.

10. Show that the product topological space $(\Omega, T)$ is metrizable.

**Definition 57** Let $(\Omega, T)$ be a topological space. A subset $\mathcal{H}$ of $T$ is called a **countable base** of $(\Omega, T)$, if and only if $\mathcal{H}$ is at most countable, and has the property:

$$\forall U \in T, \exists \mathcal{H} \subseteq \mathcal{H}, U = \bigcup_{V \in \mathcal{H}} V$$

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EXERCISE 16.

1. Show that $\mathcal{H} = \{ r, q : r, q \in \mathbb{Q} \}$ is a countable base of $\langle \mathbb{R}, T_{\mathbb{R}} \rangle$.

2. Show that if $(\Omega, T)$ is a topological space with countable base, and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', T|_{\Omega'})$ also has a countable base.

3. Show that $[-1, 1]$ has a countable base.

4. Show that if $(\Omega, T)$ and $(S, T_S)$ are homeomorphic, then $(\Omega, T)$ has a countable base if and only if $(S, T_S)$ has a countable base.

5. Show that $\langle \mathbb{R}, T_{\mathbb{R}} \rangle$ has a countable base.

EXERCISE 17. Let $(\Omega_n, T_n)_{n \geq 1}$ be a sequence of topological spaces with countable base. For $n \geq 1$, let $\{ V_{n,k} : k \in I_n \}$ be a countable base of $(\Omega_n, T_n)$ where $I_n$ is a finite or countable set. Let $\Omega = \prod_{n=1}^{\infty} \Omega_n$ be the Cartesian product and $T$ be the product topology on $\Omega$. For all $p \geq 1$, we define:

$$H^p \triangleq \left\{ V_{1,k_1} \times \ldots \times V_{p,k_p} \times \prod_{n=p+1}^{\infty} \Omega_n : (k_1, \ldots, k_p) \in I_1 \times \ldots \times I_p \right\}$$

and we put $\mathcal{H} = \cup_{p \geq 1} H^p$.

1. Show that for all $p \geq 1$, $H^p \subseteq T$.

2. Show that $\mathcal{H} \subseteq T$.

3. For all $p \geq 1$, show the existence of an injection $j_p : H^p \rightarrow N^p$.

4. Show the existence of a bijection $\phi_2 : N^2 \rightarrow N$.

5. For $p \geq 1$, show the existence of an bijection $\phi_p : N^p \rightarrow N$.

6. Show that $H^p$ is at most countable for all $p \geq 1$.

7. Show the existence of an injection $j : \mathcal{H} \rightarrow N^2$.

8. Show that $\mathcal{H}$ is a finite or countable set of open sets in $\Omega$.

9. Let $U \in T$ and $x \in U$. Show that there is $p \geq 1$ and $U_1, \ldots, U_p$ open sets in $\Omega_1, \ldots, \Omega_p$ such that:

$$x \in U_1 \times \ldots \times U_p \times \prod_{n=p+1}^{\infty} \Omega_n \subseteq U$$

10. Show the existence of some $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U$.

11. Show that $\mathcal{H}$ is a countable base of the topological space $(\Omega, T)$.

12. Show that $\otimes_{n=1}^{\infty} B(\Omega_n) \subseteq B(\Omega)$.
13. Show that $\mathcal{H} \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$.

14. Show that $\mathcal{B}(\Omega) = \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$

**Theorem 27** Let $(\Omega_n, T_n)_{n \geq 1}$ be a sequence of topological spaces with countable base. Then, the product space $(\Pi_{n=1}^{+\infty} \Omega_n, \otimes_{n=1}^{+\infty} T_n)$ has a countable base and:

$$\mathcal{B} \left( \prod_{n=1}^{+\infty} \Omega_n \right) = \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$$

**Exercise 18.**

1. Show that if $(\Omega, T)$ has a countable base and $n \geq 1$:

$$\mathcal{B}(\Omega^n) = \mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)$$

2. Show that $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$.

3. Show that $\mathcal{B}(C) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

**Definition 58** We say that a metric space $(E, d)$ is separable, if and only if there exists a finite or countable dense subset of $E$, i.e. a finite or countable subset $A$ of $E$ such that $E = \bar{A}$, where $\bar{A}$ is the closure of $A$ in $E$.

**Exercise 19.** Let $(E, d)$ be a metric space.

1. Suppose that $(E, d)$ is separable. Let $\mathcal{H} = \{B(x_n, \frac{1}{p}) : n, p \geq 1\}$, where $\{x_n : n \geq 1\}$ is a countable dense subset in $E$. Show that $\mathcal{H}$ is a countable base of the metric topological space $(E, T_E^d)$.

2. Suppose conversely that $(E, T_E^d)$ has a countable base $\mathcal{H}$. For all $V \in \mathcal{H}$ such that $V \neq \emptyset$, take $x_V \in V$. Show that the set $\{x_V : V \in \mathcal{H}, V \neq \emptyset\}$ is at most countable and dense in $E$.

3. For all $x, y, x', y' \in E$, show that:

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')$$

4. Let $T_{E \times E}$ be the product topology on $E \times E$. Show that the map $d : (E \times E, T_{E \times E}) \to (\mathbb{R}^+, T_{\mathbb{R}^+})$ is continuous.

5. Show that $d : (E \times E, \mathcal{B}(E \times E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable.

6. Show that $d : (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, whenever $(E, d)$ is a separable metric space.
7. Let $(\Omega, \mathcal{F})$ be a measurable space and $f, g : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ be measurable maps. Show that $\Phi : (\Omega, \mathcal{F}) \to E \times E$ defined by $\Phi(\omega) = (f(\omega), g(\omega))$ is measurable with respect to the product $\sigma$-algebra $\mathcal{B}(E) \otimes \mathcal{B}(E)$.

8. Show that if $(E, d)$ is separable, then $\Psi : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by $\Psi(\omega) = d(f(\omega), g(\omega))$ is measurable.

9. Show that if $(E, d)$ is separable then $\{ f = g \} \in \mathcal{F}$.

10. Let $(E_n, d_n)_{n \geq 1}$ be a sequence of separable metric spaces. Show that the product space $\prod_{n=1}^{\infty} E_n$ is metrizable and separable.

EXERCISE 20. Prove the following theorem.

**Theorem 28** Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces and $(\Omega, \mathcal{F})$ be a measurable space. For all $i \in I$, let $f_i : \Omega \to \Omega_i$ be a map, and define $f : \Omega \to \prod_{i \in I} \Omega_i$ by $f(\omega) = (f_i(\omega))_{i \in I}$. Then, the map:

$$f : (\Omega, \mathcal{F}) \to \left( \prod_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{F}_i \right)$$

is measurable, if and only if each $f_i : (\Omega, \mathcal{F}) \to (\Omega_i, \mathcal{F}_i)$ is measurable.

EXERCISE 21.

1. Let $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}$ with $\phi(x, y) = x + y$ and $\psi(x, y) = x \cdot y$. Show that both $\phi$ and $\psi$ are continuous.

2. Show that $\phi, \psi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are measurable.

3. Let $(\Omega, \mathcal{F})$ be a measurable space, and $f, g : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable maps. Using the previous results, show that $f + g$ and $f \cdot g$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$. 
Solutions to Exercises

Exercise 1.

1. If $\Omega_i = \Omega$ for all $i \in I$, then $\cup_{i \in I} \Omega_i = \Omega$. For any map $f : I \to \Omega$, the condition $f(i) \in \Omega_i$ for all $i \in I$, is automatically satisfied. Hence, $\Omega^I$ is the set of all maps $f : I \to \Omega$.

2. $\mathbb{R}^{\mathbb{R}^+}$ is the set of all maps $f : \mathbb{R}^+ \to \mathbb{R}$. The set $\mathbb{R}^\mathbb{N}$ is that of all maps $f : \mathbb{N} \to \mathbb{R}$, or in other words, the set of all sequences $(u_n)_{n \geq 0}$ with values in $\mathbb{R}$. As for $[0,1]^\mathbb{N}$, it is the set of all sequences $(u_n)_{n \geq 0}$ with values in $[0,1]$. Finally, $\mathbb{R}^\mathbb{R}$ etc.

3. Yes. Maps defined on $\mathbb{N}^*$ or sequences are the same thing.

4. For any set $E$, $E^n$ is the set of all maps $f : \mathbb{N}_n \to E$.

5. $E \times F \times G$ is the set of all maps $\omega : \mathbb{N}_3 \to E \cup F \cup G$ such that $\omega_1 \in E$, $\omega_2 \in F$ and $\omega_3 \in G$.

Exercise 2.

1. $\Pi_{i \in I_\lambda} \Omega_i$ is the set of all maps $f$ defined on $I_\lambda$, with $f(i) \in \Omega_i$ for all $i \in I_\lambda$.

2. $\Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} \Omega_i)$ is the set of all maps $x$ defined on $\Lambda$, such that $x(\lambda) \in \Pi_{i \in I_\lambda} \Omega_i$, for all $\lambda \in \Lambda$.

3. Given $\omega \in \Pi_{i \in I} \Omega_i$ and $\lambda \in \Lambda$, let $\omega|_{I_\lambda}$ be the restriction of $\omega$ to $I_\lambda \subseteq I$.

Exercise 1

So $\omega = \omega'$, and $\Phi$ is an injective map. We have found a natural bijection from $\Pi_{i \in I} \Omega_i$ to $\Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} \Omega_i)$.

Given a map $\omega \in \Pi_{i \in I} \Omega_i$, it is customary to regard $\omega$ as the family $(\omega_i)_{i \in I}$ where $\omega_i = \omega(i)$ for all $i \in I$. (A map defined on $I$ is nothing but a family indexed by $I$). Hence, the restriction $\omega|_{I_\lambda}$ is nothing but the family $(\omega_i)_{i \in I_\lambda}$, and the map $\Phi(\omega)$ can be written as:

$$\Phi((\omega_i)_{i \in I}) = ((\omega_i)_{i \in I_\lambda})_{\lambda \in \Lambda}$$

The mapping $\Phi$ looks like a pretty natural mapping, given the partition $(I_\lambda)_{\lambda \in \Lambda}$ of the set $I$.
Exercise 3.

1. Let $A = A_1 \times \ldots \times A_n$ be such that $A_i \in \mathcal{E}_i$ for all $i = 1, \ldots, n$. Then $A$ is of the form $A = \Pi_{i \in \mathbb{N}_n} A_i$ with $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$, and the condition $A_i \neq \Omega_i$ for finitely many $i \in \mathbb{N}_n$, is obviously satisfied. So $A$ is a rectangle of the family $(\mathcal{E}_i)_{i \in \mathbb{N}_n}$, that is $A \in \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_n$. Conversely, Let $A = \Pi_{i \in \mathbb{N}_n} A_i$ be a rectangle of the family $(\mathcal{E}_i)_{i \in \mathbb{N}_n}$. Then, each $A_i$ is an element of $\mathcal{E}_i \cup \{\Omega_i\}$. Since $\Omega_i \in \mathcal{E}_i$ for all $i \in \mathbb{N}_n$, each $A_i$ is in fact an element of $\mathcal{E}_i$. So $A$ is of the form $A = A_1 \times \ldots \times A_n$, with $A_i \in \mathcal{E}_i$. We have proved that the set of rectangles of $(\mathcal{E}_i)_{i \in \mathbb{N}_n}$ is given by:

$$\mathcal{E}_1 \cap \ldots \cap \mathcal{E}_n = \{A_1 \times \ldots \times A_n : A_i \in \mathcal{E}_i, \forall i \in \mathbb{N}_n\}$$

2. Let $A$ be a rectangle of the family $(\mathcal{E}_i)_{i \in I}$. Then $A = \Pi_{i \in I} A_i$, where $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$, and $A_i \neq \Omega_i$ for finitely many $i \in I$. Let $J$ be the set $J = \{i \in I : A_i \neq \Omega_i\}$. Then $J$ is a finite subset of $I$. Moreover, for all $j \in J$, $A_j \neq \Omega_j$, yet $A_j \in \mathcal{E}_j \cup \{\Omega_j\}$. So $A_j \in \mathcal{E}_j$. Let $\omega \in A = \Pi_{i \in I} A_i$. Then $\omega$ is a map defined on $I$ such that $\omega(i) \in A_i \subseteq \Omega_i$ for all $i \in I$. In particular, $\omega \in \Pi_{i \in I} \Omega_i$, and $\omega(j) \in A_j$ for all $j \in J$. Conversely, suppose $\omega \in \Pi_{i \in I} \Omega_i$ is such that $\omega(j) \in A_j$ for all $j \in J$. Then $\omega$ is a map defined on $J$ such that $\omega(i) \in A_i$ for all $i \in I$, and furthermore, $\omega(j) \in A_j$ for all $j \in J$. However, for all $i \in I \setminus J$, we have $A_i = \Omega_i$. It follows that $\omega$ is a map defined on $I$ such that $\omega(i) \in A_i$ for all $i \in I$. So $\omega \in \Pi_{i \in I} A_i = A$. We have proved that there exists a finite subset $J$ of $I$, and a family $(A_j)_{j \in J}$ with $A_j \in \mathcal{E}_j$, such that $A = \{\omega \in \Pi_{i \in I} \Omega_i : \omega(j) \in A_j, \forall j \in J\}$.  

Exercise 4.

1. By definition, $\mathcal{F}_1 \cap \ldots \cap \mathcal{F}_n$ is generated by the set of measurable rectangles $\mathcal{F}_1 \cap \ldots \cap \mathcal{F}_n$. Since $\Omega_i \in \mathcal{F}_i$ for all $i \in \mathbb{N}_n$, and since $\mathbb{N}_n$ is finite, these rectangles are of the form $A_1 \times \ldots \times A_n$ where $A_i \in \mathcal{F}_i$, for all $i \in \mathbb{N}_n$.

2. $\mathcal{B}(\mathbb{R}) \cap \mathcal{B}(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$ is generated by the set of measurable rectangles $\mathcal{B}(\mathbb{R}) \cap \mathcal{B}(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$. These rectangles are of the form $A \times B \times C$, where $A, B, C \in \mathcal{B}(\mathbb{R})$.

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4We view ordered pairs as maps defined on $\mathbb{N}_2$...
3. Since \( \mathbb{R}^+ \in \mathcal{B}(\mathbb{R}^+) \) and \( \Omega \in \mathcal{F} \), the set of measurable rectangles \( \mathcal{B}(\mathbb{R}^+) \Pi \mathcal{F} \) is the set of all \( B \times F \), where \( B \in \mathcal{B}(\mathbb{R}^+) \) and \( F \in \mathcal{F} \). Such sets generate the \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \) on \( \mathbb{R}^+ \otimes \Omega \).

Exercise 5.

1. By definition, a generator of \( \otimes_{i \in I} \sigma(\mathcal{E}_i) \) is the set of measurable rectangles of the family \( \{ \sigma(\mathcal{E}_i) \}_{i \in I} \), i.e. \( \Pi_{i \in I} \sigma(\mathcal{E}_i) \).

2. Let \( A = \Pi_{i \in I} A_i \) be a rectangle in \( \Pi_{i \in I} \mathcal{E}_i \). Then, each \( A_i \) is an element of \( \mathcal{E}_i \cup \{ \Omega_i \} \), and \( A_i \neq \Omega_i \) for finitely many \( i \in I \). In particular, \( A \) is also a rectangle in \( \Pi_{i \in I} \sigma(\mathcal{E}_i) \). Hence, we have:

\[
\prod_{i \in I} \mathcal{E}_i \subseteq \prod_{i \in I} \sigma(\mathcal{E}_i) \subseteq \sigma \left( \prod_{i \in I} \sigma(\mathcal{E}_i) \right) \triangleq \otimes_{i \in I} \sigma(\mathcal{E}_i)
\]

and consequently, \( \sigma(\Pi_{i \in I} \mathcal{E}_i) \subseteq \otimes_{i \in I} \sigma(\mathcal{E}_i) \).

3. Let \( A \neq \emptyset \) be a rectangle of the family \( \{ \sigma(\mathcal{E}_i) \}_{i \in I} \). Suppose that \( A = \Pi_{i \in I} A_i = \Pi_{i \in I} B_i \) are two representations of \( A \). Since \( A \) is non-empty, there exists \( f \in A \). The mapping \( f \) defined on \( I \) is such that \( f(i) \in A_i \cap B_i \) for all \( i \in I \). Let \( j \in I \) be given. Suppose \( x \in A_j \). Define \( g \) on \( I \), by \( g(i) = f(i) \) if \( i \neq j \), and \( g(j) = x \). Then, \( g(i) \in A_i \) for all \( i \in I \). So \( g \in \Pi_{i \in I} A_i = A = \Pi_{i \in I} B_i \), and in particular, \( x \in g(j) \in B_j \). Hence, we see that \( A_j \subseteq B_j \), and similarly \( B_j \subseteq A_j \), \( j \in I \) being arbitrary, we have proved that \( A_i = B_i \) for all \( i \in I \). The set \( J_A = \{ i \in I : A_i \neq \Omega_i \} \) is therefore well-defined, as the \( A_i \)'s are uniquely determined. Furthermore, \( A \) being a rectangle, the set \( J_A \) is finite.

4. Let \( A = \Pi_{i \in I} \sigma(\mathcal{E}_i) \). If \( A = \emptyset \), then \( A \) is an element of the \( \sigma \)-algebra \( \sigma(\Pi_{i \in I} \mathcal{E}_i) \). If \( A \neq \emptyset \) but \( J_A = \emptyset \), then \( A_i = \Omega_i \) for all \( i \in I \), and \( A = \Pi_{i \in I} A_i = \Pi_{i \in I} \Omega_i \) is also an element of the \( \sigma \)-algebra \( \sigma(\Pi_{i \in I} \mathcal{E}_i) \).

Exercise 6.

1. By assumption, \( A \neq \emptyset \). There exists a map \( f \) defined on \( I \), such that \( f(i) \in A_i \) for all \( i \in I \). Since \( A_i \subseteq \Omega_1 \), \( f \) is also an element of \( A^{\Omega_1} \). So \( A^{\Omega_1} \neq \emptyset \). By definition, \( J_A^{\Omega_1} = \{ i \in I : A_i \neq \Omega_i \} \), where each \( A_i \) is equal to \( A_i \), except \( A_i = \Omega_i \). It follows that \( J_A^{\Omega_i} = \{ i \in I \setminus \{ i_1 \} : A_i \neq \Omega_i \} \). Since by assumption, \( i_1 \in J_A \), and \( \text{card} J_A = n+1 \), \( \text{card} J_A^{\Omega_1} = n \). Finally, \( A \) being a rectangle of the family \( \{ \sigma(\mathcal{E}_i) \}_{i \in I} \), each \( A_i \) is an element of \( \sigma(\mathcal{E}_i) \cup \{ \Omega_i \} = \sigma(\mathcal{E}_i) \). It follows that \( A_i \in \sigma(\mathcal{E}_i) \) for all \( i \in I \). Since \( A_i \neq \Omega_i \) for finitely many \( i \in I \), we conclude that \( A^{\Omega_1} = \Pi_{i \in I} A_i \in \Pi_{i \in I} \sigma(\mathcal{E}_i) \).
2. Our induction hypothesis is that if $A$ is a non-empty rectangle of the family $(\sigma(\mathcal{E}_i))_{i \in I}$ with $\text{card} I_A = n$, then $A \in \sigma(\cap_{i \in I} \mathcal{E}_i)$. Since from 1., $A^{\Omega_{\alpha_i}}$ satisfies such properties, $A^{\Omega_{\alpha_i}} \in \sigma(\cap_{i \in I} \mathcal{E}_i)$. It follows that $\Omega_{\alpha_i} \in \Gamma$.

3. Let $B \subseteq \Omega_{\alpha_i}$. Let $f \in A^{\Omega_{\alpha_i} \setminus B}$. Then, $f$ is a map defined on $I$, such that $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$, and $f(i_1) \in \Omega_{\alpha_i} \setminus B$. In particular, $f \in A^{\Omega_{\alpha_i}}$ and $f \notin A^B$. So $f \in A^{\Omega_{\alpha_i}} \setminus A^B$, and $A^{\Omega_{\alpha_i} \setminus B} \subseteq A^{\Omega_{\alpha_i}} \setminus A^B$. Conversely, suppose $f \in A^{\Omega_{\alpha_i}} \setminus A^B$. Then, $f$ is an element of $A^{\Omega_{\alpha_i}}$, $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$. Since $f \notin A^B$, $f(i_1)$ cannot be an element of $B$. It follows that $f(i_1) \in \Omega_{\alpha_i} \setminus B$, and $f \in A^{\Omega_{\alpha_i} \setminus B}$. We have proved that $A^{\Omega_{\alpha_i} \setminus B} = A^{\Omega_{\alpha_i}} \setminus A^B$.

4. Let $B \in \Gamma$. Then, $A^B \in \sigma(\cap_{i \in I} \mathcal{E}_i)$. All $\sigma$-algebras being closed under complementation, we have $(A^B)^c \in \sigma(\cap_{i \in I} \mathcal{E}_i)$. Moreover, from 2., $A^{\Omega_{\alpha_i}} \in \sigma(\cap_{i \in I} \mathcal{E}_i)$. It follows that:

\[ A^{\Omega_{\alpha_i} \setminus B} = A^{\Omega_{\alpha_i}} \setminus A^B = A^{\Omega_{\alpha_i}} \cap (A^B)^c \in \sigma(\cap_{i \in I} \mathcal{E}_i) \]

We conclude that $\Omega_{\alpha_i} \setminus B \in \Gamma$.

5. Let $(B_n)_{n \geq 1}$ be a sequence of subsets of $\Omega_{\alpha_i}$. If $f \in A^{\cup_{1}^{B_n}}$, then $f$ is a map defined on $I$, such that $f(i) \in A_i$ for all $i \neq i_1$, and $f(i_1) \in \cap_{n \geq 1} B_n$. There exists $n \geq 1$ such that $f(i_1) \in B_n$, which implies that $f \in A^{B_n}$. So $f \in \cap_{n \geq 1} A^{B_n}$, and we see that $A^{\cup_{1}^{B_n}} \subseteq \cap_{n \geq 1} A^{B_n}$. Conversely, suppose that $f \in \cap_{n \geq 1} A^{B_n}$. There exists $n \geq 1$, such that $f \in A^{B_n}$. In particular, $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$, and $f(i_1) \in B_n \subseteq \cap_{n \geq 1} B_n$. So $f \in A^{B_n}$. We have proved that $A^{\cup_{1}^{B_n}} = \cap_{n \geq 1} A^{B_n}$.

6. From 2., $\Omega_{\alpha_i} \in \Gamma$. From 4., $\Gamma$ is closed under complementation. To show that $\Gamma$ is a $\sigma$-algebra on $\Omega_{\alpha_i}$, it remains to show that $\Gamma$ is closed under countable union. Let $(B_n)_{n \geq 1}$ be a sequence of elements of $\Gamma$. Then, for all $n \geq 1$, $A^{B_n} \in \sigma(\cap_{i \in I} \mathcal{E}_i)$. It follows that:

\[ A^{\cup_{1}^{B_n}} = \cup_{n \geq 1} A^{B_n} \in \sigma(\cap_{i \in I} \mathcal{E}_i) \]

So $\cup_{n \geq 1} B_n \in \Gamma$, and $\Gamma$ is indeed closed under countable union. We have proved that $\Gamma$ is a $\sigma$-algebra on $\Omega_{\alpha_i}$.

7. Let $B \in \mathcal{E}_i$, $\bar{B}_i = \Omega_{\alpha_i}$ for all $i \neq i_1$, and $\bar{B}_1 = B$. Let $f \in A^B$. Then, $f$ is a map defined on $I$, such that $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$, and $f(i_1) \in B$. In particular, $f \in A^{\Omega_{\alpha_i}}$ and $f(i) \in \bar{B}_i$ for all $i \in I$, i.e. $f \in \cap_{i \in I} \bar{B}_i$. Hence, $A^B \subseteq A^{\Omega_{\alpha_i}} \cap (\cap_{i \in I} \bar{B}_i)$. Conversely, suppose that $f \in A^{\Omega_{\alpha_i}} \cap (\cap_{i \in I} \bar{B}_i)$. Then, $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$ and $f(i) \in \bar{B}_i$ for all $i \in I$. In particular, $f(i_1) \in \bar{B}_1 = B$. It follows that $f \in A^B$. We have proved that $A^B = A^{\Omega_{\alpha_i}} \cap (\cap_{i \in I} \bar{B}_i)$.

8. Let $B \in \mathcal{E}_i$, and $\bar{B}_i = \Omega_{\alpha_i}$ for all $i \in I \setminus \{i_1\}$, and $\bar{B}_1 = B$. Then, $\cap_{i \in I} \bar{B}_i \in \cap_{i \in I} \mathcal{E}_i$, and in particular, $\cap_{i \in I} \bar{B}_i \in \sigma(\cap_{i \in I} \mathcal{E}_i)$. From 2., $\Omega_{\alpha_i} \in \Gamma$, i.e. $A^{\Omega_{\alpha_i}}$ is also an element of $\sigma(\cap_{i \in I} \mathcal{E}_i)$. It follows from 7. that:

\[ A^B = A^{\Omega_{\alpha_i}} \cap (\cap_{i \in I} \bar{B}_i) \in \sigma(\cap_{i \in I} \mathcal{E}_i) \]
We conclude that \( B \in \Gamma \). This being true for all \( B \in \mathcal{E}_i \), we have \( \mathcal{E}_i \subseteq \Gamma \). However, since \( \Gamma \) is a \( \sigma \)-algebra on \( \Omega_i \), we finally see that \( \sigma(\mathcal{E}_i) \subseteq \Gamma \).

9. Let \( f \in A = \Pi_{i \in I} A_i \). Then, \( f(i) \in A_i \) for all \( i \in I \setminus \{i_1\} \), and \( f(i_1) \in A_{i_1} \). So \( f \in A^{A_{i_1}} \). Conversely, if \( f \in A^{A_{i_1}} \), then \( f \in A \). So \( A = A^{A_{i_1}} \). Since \( A \) is a rectangle of the family \( (\sigma(\mathcal{E}_i))_{i \in I} \), \( A_{i_1} \in \sigma(\mathcal{E}_i) \). From 8., \( \sigma(\mathcal{E}_i) \subseteq \Gamma \). It follows that \( A_{i_1} \in \Gamma \), and consequently \( A = A^{A_{i_1}} \in \sigma(\Pi_{i \in I} \mathcal{E}_i) \). This proves our induction hypothesis for \( \text{card} J_A = n + 1 \).

10. Let \( A \in \Pi_{i \in I} \sigma(\mathcal{E}_i) \). If \( A = \emptyset \), then \( A \) is an element of \( \sigma(\Pi_{i \in I} \mathcal{E}_i) \). Let \( A \neq \emptyset \). If \( \text{card} J_A = 0 \), then \( A = \Pi_{i \in I} I_i \in \sigma(\Pi_{i \in I} \mathcal{E}_i) \). Using an induction argument on \( \text{card} J_A \), we have proved that for all \( n \geq 0 \):

\[
\text{card} J_A = n \Rightarrow A \in \sigma(\Pi_{i \in I} \mathcal{E}_i)
\]

Since \( A \) is a rectangle of the family \( (\sigma(\mathcal{E}_i))_{i \in I} \), \( J_A \) is a finite set. It follows that \( A \in \sigma(\Pi_{i \in I} \mathcal{E}_i) \). Finally, we conclude that \( \Pi_{i \in I} \sigma(\mathcal{E}_i) \subseteq \sigma(\Pi_{i \in I} \mathcal{E}_i) \).

11. From 10., we have \( \otimes_{i \in I} \sigma(\mathcal{E}_i) = \sigma(\Pi_{i \in I} \sigma(\mathcal{E}_i)) \subseteq \sigma(\Pi_{i \in I} \mathcal{E}_i) \). However, from exercise (5), \( \sigma(\Pi_{i \in I} \mathcal{E}_i) \subseteq \otimes_{i \in I} \sigma(\mathcal{E}_i) \). It follows that \( \otimes_{i \in I} \sigma(\mathcal{E}_i) = \sigma(\Pi_{i \in I} \mathcal{E}_i) \). The purpose of this difficult exercise is to prove theorem (26). Congratulations!

Exercise 7.

1. Since \( R \in T_R \) and \( N_n \) is finite, from definition (52), the set of rectangles \( T_R \Pi \ldots T_R \) reduces to all sets of the form \( \Pi_{i \in N_n} A_i \), where \( A_i \in T_R \) for all \( i \in N_n \). In other words:

\[
T_R \Pi \ldots T_R = \{ A_1 \times \ldots \times A_n : A_i \in T_R, \forall i \in N_n \}
\]

2. By definition of the Borel \( \sigma \)-algebra, \( B(R) \) is generated by the topology \( T_R \), i.e. \( B(R) = \sigma(T_R) \). From theorem (26), we have:

\[
B(R) \otimes \ldots \otimes B(R) = \sigma(T_R \Pi \ldots T_R)
\]

3. Let \( C_2 = \{[a_1, b_1] \times \ldots \times [a_n, b_n] : a_i, b_i \in R \} \), and let \( S \) be the semi-ring on \( R \), \( S = \{[a, b] : a, b \in R \} \). Since \( N_n \) is finite, from definition (52), the set of rectangles \( S \Pi \ldots S \Pi S \) is made of all sets of the form \( \Pi_{i \in N_n} A_i \), where \( A_i \in S \cup \{R \} \). Hence, each element of \( C_2 \) is an element of \( S \Pi \ldots S \Pi S \), i.e. \( C_2 \subseteq S \Pi \ldots S \Pi S \). However, \( R^{\infty} \) is an element of \( S \Pi \ldots S \), but do not belong to \( C_2 \). So the inclusion \( C_2 \subseteq S \Pi \ldots S \Pi S \) is strict.

4. Let \( A \in S \Pi \ldots S \). Then \( A \) is of the form \( A = A_1 \times \ldots \times A_n \), where each \( A_i \) is an element of \( S \), or \( A_i = R \). If all \( A_i \)'s lie in \( S \), then \( A \in C_2 \subseteq \sigma(C_2) \). Let \( J'_A = \{ k \in N_n : A_k = R \} \). We have just seen that if \( J'_A = \emptyset \), or equivalently if \( \text{card} J'_A = 0 \), then \( A \in \sigma(C_2) \). Suppose we have proved the induction hypothesis, for \( k = 0, \ldots, n - 1 \):

\[
A \in S \Pi \ldots \Pi S, \; \text{card} J'_A = k \Rightarrow A \in \sigma(C_2)
\]

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and let \( A \in \mathcal{S} \Pi \ldots \Pi \mathcal{S} \) be such that \( \text{card} J^*_A = k + 1 \). Let \( i_1 \) be an arbitrary element of \( J^*_A \). Then, \( A_{i_1} = R = \bigcup_{p=1}^{+\infty} \] \(-p, p]\). Hence, \( A \) can be written as:

\[
A = A_1 \times \ldots \times A_n = \bigcup_{p=1}^{+\infty} A_1 \times \ldots \times \] \(-p, p]\ \times \ldots \times A_n \tag{1}
\]

where \( A_1 \times \ldots \times \] \(-p, p]\ \times \ldots \times A_n = B_p \) is a notation for \( \Pi_{i \in \mathbb{N}_n} A_i \) where \( A_i = A_i \) for all \( i \neq i_1 \), and \( A_{i_1} \] \(-p, p]\). Since for all \( p \geq 1 \), \( \] \(-p, p]\ \in \mathcal{S} \), \( B_p \) is an element of \( \mathcal{S} \Pi \ldots \Pi \mathcal{S} \), and more importantly \( \text{card} J^*_B = k \). From our induction hypothesis, it follows that \( B_p \in \sigma(\mathcal{C}_2) \). Hence, we see from equation (1) that \( A \in \sigma(\mathcal{C}_2) \), and we have proved our induction hypothesis for \( \text{card} J^*_A = k + 1 \). We conclude that for all \( A \in \mathcal{S} \Pi \ldots \Pi \mathcal{S} \), we have \( A \in \sigma(\mathcal{C}_2) \), i.e. \( \mathcal{S} \Pi \ldots \Pi \mathcal{S} \subseteq \sigma(\mathcal{C}_2) \).

5. From theorem (6)\(^4\), we know that the semi-ring \( \mathcal{S} \) generates the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}) \) on \( \mathbb{R} \), i.e. \( \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{S}) \). Applying theorem (26), we have:

\[
\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{S} \Pi \ldots \Pi \mathcal{S}) \tag{2}
\]

However, from 3., \( \mathcal{C}_2 \subseteq \mathcal{S} \Pi \ldots \Pi \mathcal{S} \), hence \( \sigma(\mathcal{C}_2) \subseteq \sigma(\mathcal{S} \Pi \ldots \Pi \mathcal{S}) \). Moreover, from 4., \( \mathcal{S} \Pi \ldots \Pi \mathcal{S} \subseteq \sigma(\mathcal{C}_2) \), and consequently, we have \( \sigma(\mathcal{S} \Pi \ldots \Pi \mathcal{S}) \subseteq \sigma(\mathcal{C}_2) \). It follows that \( \sigma(\mathcal{S} \Pi \ldots \Pi \mathcal{S}) = \sigma(\mathcal{C}_2) \). Finally, from equation (2), \( \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_2) \).

Exercise 7

Exercise 8

1. Let \( \Sigma = \sigma(\mathcal{E}) \) be the \( \sigma \)-algebra generated by \( \mathcal{E} = \{ A \} \). Let \( \mathcal{F} \) be the set of subsets of \( \Omega \) defined by \( \mathcal{F} = \{ \emptyset, A, A^c, \Omega \} \). Note that \( \Omega \in \mathcal{F} \), \( \mathcal{F} \) is closed under complementation and countable union, so \( \mathcal{F} \) is a \( \sigma \)-algebra on \( \Omega \). Since \( \mathcal{E} \subseteq \mathcal{F} \), we have \( \Sigma = \sigma(\mathcal{E}) \subseteq \mathcal{F} \). However, since \( \mathcal{E} \subseteq \sigma(\mathcal{E}) \), \( A \in \Sigma \). So \( A^c \in \Sigma \). Furthermore, \( \Omega \in \Sigma \) and \( \emptyset \in \Sigma \). Finally, \( \mathcal{F} \subseteq \Sigma \). We have proved that \( \mathcal{F} = \Sigma \).

2. Since \( \{ \emptyset, \Omega \} \) is a \( \sigma \)-algebra on \( \Omega' \) with \( \mathcal{E}' \subseteq \{ \emptyset, \Omega' \} \), we have \( \sigma(\mathcal{E}') \subseteq \{ \emptyset, \Omega' \} \). However, \( \sigma(\mathcal{E}') \) being a \( \sigma \)-algebra on \( \Omega' \), we have \( \Omega' \in \sigma(\mathcal{E}') \) and \( \emptyset \in \sigma(\mathcal{E}') \). Finally, \( \sigma(\mathcal{E}') = \{ \emptyset, \Omega' \} \).

3. Since \( \mathcal{E}' = \emptyset \), \( \mathcal{C} = \{ E \times F : E \in \mathcal{E}, F \in \mathcal{E}' \} = \emptyset \).

4. The rectangles in \( \mathcal{E} \Pi \mathcal{E}' \) are the sets of the form \( A_1 \times A_2 \), where \( A_1 \in \mathcal{E} \cup \{ \Omega \} \) and \( A_2 \in \mathcal{E}' \cup \{ \Omega' \} \). Since \( \mathcal{E}' = \emptyset \), the only possible value for \( A_2 \) is \( \Omega' \). Since \( \mathcal{E} = \{ A \} \), \( A_1 \) can be equal to \( A \) or \( \Omega \). It follows that \( \mathcal{E} \Pi \mathcal{E}' = \{ A \times \Omega', \Omega \times \Omega' \} \).

5. From theorem (26), \( \sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') = \sigma(\mathcal{E} \Pi \mathcal{E}') \). Let \( \mathcal{F} \) be defined by \( \mathcal{F} = \{ \emptyset, A \times \Omega', A^c \times \Omega', \Omega \times \Omega' \} \). Note that the complement of \( A \times \Omega' \) in

\(^4\)Beware of external links!
Exercise 10.

$\Omega \times \Omega'$ is $(A \times \Omega')^c = A^c \times \Omega'$. So $F$ is closed under complementation, and in fact, $F$ is a $\sigma$-algebra on $\Omega \times \Omega'$. However, from 4., $\mathcal{E} \cap \mathcal{E}' = A \times \Omega', \Omega \times \Omega'$. So $\mathcal{E} \cap \mathcal{E}' \subseteq F$, and consequently $\sigma(\mathcal{E} \cap \mathcal{E}') \subseteq F$. Since all elements of $F$ have to be in $\sigma(\mathcal{E} \cap \mathcal{E}')$, we also have $F \subseteq \sigma(\mathcal{E} \cap \mathcal{E}')$. We have proved that $F = \sigma(\mathcal{E} \cap \mathcal{E}')$. We conclude that $\sigma(\mathcal{E}) \cap \sigma(\mathcal{E}') = F$.

6. Since $\mathcal{C} = \emptyset$, we have $\sigma(\mathcal{C}) = \{\emptyset, \Omega \times \Omega'\}$. It follows from 5. that $\sigma(\mathcal{C}) \neq \sigma(\mathcal{E}) \cap \sigma(\mathcal{E}')$. The purpose of this exercise is to emphasize an easy mistake to make, when applying theorem (26). This theorem states that $\sigma(\mathcal{E}) \cap \sigma(\mathcal{E}') = \sigma(\mathcal{E} \cap \mathcal{E}')$. It is very tempting to conclude that:

$$\sigma(\mathcal{E}) \cap \sigma(\mathcal{E}') = \sigma\{(E \times F : E \in \mathcal{E}, F \in \mathcal{E}')\}$$

But this is wrong! The reason being that the set of rectangles $\mathcal{E} \cap \mathcal{E}'$ is larger than the set of all $E \times F$, where $E \in \mathcal{E}$ and $F \in \mathcal{E}'$. The elements of $\mathcal{E} \cap \mathcal{E}'$ are indeed of the form $E \times F$, but with $E \in \mathcal{E} \cup \{\Omega\}$ and $F \in \mathcal{E}' \cup \{\Omega'\}$. (Do not forget the ´∪´). So $\sigma(\mathcal{E}) \cap \sigma(\mathcal{E}') = \sigma\{(E \times F : E \in \mathcal{E} \cup \{\Omega\}, F \in \mathcal{E}' \cup \{\Omega'\})\}$. You have been warned...

Exercise 9.

1. Strictly speaking, $F \otimes G$ is a $\sigma$-algebra on $\mathbb{R}^n \times \mathbb{R}^p$. However, $\mathbb{R}^n \times \mathbb{R}^p$ and $\mathbb{R}^{n+p}$ can be identified, through the bijection $\psi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n+p}$, defined by $\psi(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_p)$. Hence, $F \otimes G$ can be viewed as a $\sigma$-algebra on $\mathbb{R}^{n+p}$.

2. By definition, $F = \sigma(\mathcal{C}_1)$, where $\mathcal{C}_1$ is the set of measurable rectangles $\mathcal{C}_1 = \{A_1 \times \ldots \times A_n : A_i \in \mathcal{B}(\mathbb{R}), \forall i \in \mathbb{N}_n\}$. Similarly, if $\mathcal{C}_2 = \{A_{n+1} \times \ldots \times A_{n+p} : A_{n+i} \in \mathcal{B}(\mathbb{R}), \forall i \in \mathbb{N}_p\}$, then $G = \sigma(\mathcal{C}_2)$. From theorem (26), we have $F \otimes G = \sigma(\mathcal{C}_1 \cap \mathcal{C}_2)$. Furthermore, since $\mathbb{R}^n \in \mathcal{C}_1$ and $\mathbb{R}^p \in \mathcal{C}_2$, the set of rectangles $\mathcal{C}_1 \cap \mathcal{C}_2$ is given by $\mathcal{C}_1 \cap \mathcal{C}_2 = \{A \times A' : A \in \mathcal{C}_1, A' \in \mathcal{C}_2\}$. If we identify sets of the form $(A_1 \times \ldots \times A_n) \times (A_{n+1} \times \ldots \times A_{n+p})$ with $A_1 \times \ldots \times A_{n+p}$, then $\mathcal{C}_1 \cap \mathcal{C}_2$ can be written as:

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \{A_1 \times \ldots \times A_{n+p} : A_i \in \mathcal{B}(\mathbb{R}), \forall i \in \mathbb{N}_{n+p}\}$$

We conclude that $F \otimes G$ is generated by the sets of the form $A_1 \times \ldots \times A_{n+p}$, where $A_i \in \mathcal{B}(\mathbb{R})$ for all $i \in \mathbb{N}_{n+p}$.

3. Let $C = \{A_1 \times \ldots \times A_{n+p} : A_i \in \mathcal{B}(\mathbb{R}), \forall i \in \mathbb{N}_{n+p}\}$. From 2., $F \otimes G = \sigma(C)$. However, $C$ is the set of measurable rectangles in $\mathbb{R}^{n+p}$. Consequently, $\sigma(C) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$ ($n+p$ terms). We conclude that $\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = F \otimes G$, i.e.

$$\mathcal{B}(\mathbb{R})^{n+p} \otimes \ldots \otimes \mathcal{B}(\mathbb{R}) = (\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})) \otimes (\mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R}))$$

Exercise 9

Exercise 10.

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1. In exercise (2), we defined a natural bijection $\Phi : \Omega \rightarrow \Omega'$, by:

$$
\Phi((\omega_i)_{i \in I}) \triangleq ((\omega_i)_{i \in I})_{\lambda \in \Lambda}
$$

This allows us to define $\tilde{\Phi} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega')$, by:

$$
\tilde{\Phi}(A) \triangleq \Phi(A) \triangleq \{ \Phi(\omega) : \omega \in A \}
$$

for all $A \subseteq \Omega$. In other words, $\tilde{\Phi}$ maps every subset $A$ of $\Omega$, with its direct image $\Phi(A)$ by the bijection $\Phi : \Omega \rightarrow \Omega'$. Let $A' \subseteq \Omega'$. Since $\Phi$ is a bijection, we have $A' = \Phi(\Phi^{-1}(A'))$, i.e. the direct image of $A'$ by $\Phi$ is equal to $A'$. So $A' = \Phi(\Phi^{-1}(A'))$, and $\Phi$ is a surjective map. If $A, B \subseteq \Omega$ are such that $\tilde{\Phi}(A) = \tilde{\Phi}(B)$, taking the inverse images of both sides, we have $A = B$. So $\Phi$ is an injective map. We have proved that $\tilde{\Phi}$ is a bijection from $\mathcal{P}(\Omega)$ to $\mathcal{P}(\Omega')$. Informally, $\Phi$ is a bijection allowing us to identify an element of $\Pi_{i \in I} \Omega_i$ with an element of $\Pi_{\lambda \in \Lambda}(\Pi_{i \in I} \Omega_i)$. The bijection $\tilde{\Phi}$ allows us to identify a subset of $\Pi_{i \in I} \Omega_i$ with a subset of $\Pi_{\lambda \in \Lambda}(\Pi_{i \in I} \Omega_i)$.

2. Let $A$ be a subset of $\Omega$ of the form $A = \Pi_{i \in I} A_i$. Let $A'$ be the corresponding set $A' = \Pi_{\lambda \in \Lambda}(\Pi_{i \in I} A_i)$. Saying that $A$ and $A'$ are identified through the bijection $\Phi$, is just another way of saying that $A' = \Phi(A)$. Suppose $y \in \Phi(A)$. There exists $x \in A$ such that $y = \Phi(x)$. For all $\lambda \in \Lambda$, we have $y(\lambda) = \Phi(x)(\lambda) = x|_{I_\lambda}$. Since $x \in A$, each $x|_{I_\lambda}$ is an element of $\Pi_{i \in I_\lambda} A_i$. So $y(\lambda) \in \Pi_{i \in I_\lambda} A_i$ for all $\lambda \in \Lambda$. It follows that $y \in \Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} A_i) = A'$. So $\Phi(A) \subseteq A'$. Conversely, suppose $y \in A'$. $y$ is a map defined on $\Lambda$, such that $y(\lambda) \in \Pi_{i \in I_\lambda} A_i$ for all $\lambda \in \Lambda$. Each $y(\lambda)$ is a map defined on $I_\lambda$, such that $y(\lambda)(i) \in A_i$ for all $i \in I_\lambda$. Let $x$ be the map defined on $I$ by $x(i) = y(\lambda)(i)$, where given $i \in I$, $\lambda$ is the unique element of $\Lambda$ such that $i \in I_\lambda$. Then, $x$ is such that $x(i) \in A_i$ for all $i \in I$, so $x \in \Pi_{i \in I} A_i = A$. Moreover, by construction, for all $\lambda \in \Lambda$, $x|_{I_\lambda} = y(\lambda)$. So $y(\lambda) = \Phi(x)(\lambda)$ for all $\lambda \in \Lambda$, i.e. $y = \Phi(x)$. We have found $x \in A$, such that $y = \Phi(x)$. So $y \in \Phi(A) = \tilde{\Phi}(A)$. We have proved that $A' \subseteq \tilde{\Phi}(A)$. Finally, $A' = \Phi(A)$. We have proved that the sets $\Pi_{i \in I} A_i$ and $\Pi_{\lambda \in \Lambda}(\Pi_{i \in I} A_i)$ are indeed identified through the bijection $\Phi$.

3. Let $\Pi_{i \in I} A_i \in \Pi_{i \in I} \mathcal{F}_i$. Then, for all $i \in I$, $A_i \in \mathcal{F}_i$, and $A_i \neq \Omega_i$ for finitely many $i \in I$. For each $\lambda \in \Lambda$, $\Pi_{i \in I_\lambda} A_i$ is therefore such that $A_i \in \mathcal{F}_i$ for all $i \in I_\lambda$, and $A_i \neq \Omega_i$ for finitely many $i \in I_\lambda$. So $\Pi_{i \in I_\lambda} A_i \in \Pi_{i \in I_\lambda} \mathcal{F}_i$. It follows that $\Pi_{i \in I} A_i$ can be written as (through identification):

$$
\Pi_{i \in I} A_i = \Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} A_i) = \Pi_{\lambda \in \Lambda} B_\lambda
$$

where $B_\lambda \in \Pi_{i \in I_\lambda} \mathcal{F}_i$ for all $\lambda \in \Lambda$. Moreover, the set of all $\lambda \in \Lambda$ for which $B_\lambda \neq \Pi_{i \in I_\lambda} \Omega_i$, is necessarily finite. It follows that $\Pi_{i \in I} A_i \in \Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} \mathcal{F}_i)$. So $\Pi_{i \in I} \mathcal{F}_i \subseteq \Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} \mathcal{F}_i)$. Conversely, let $\Pi_{\lambda \in \Lambda} B_\lambda \in \Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} \mathcal{F}_i)$. For all $\lambda \in \Lambda$, we have $B_\lambda \in \Pi_{i \in I_\lambda} \mathcal{F}_i$, and $B_\lambda \neq \Pi_{i \in I_\lambda} \Omega_i$ for finitely many $\lambda \in \Lambda$. Hence, each $B_\lambda$ is of the form $\Pi_{i \in I_\lambda} A_i$, where
Exercise 11.

1. Let \( T(\mathcal{A}) \) be the set of all topologies \( T \) on \( \Omega \), which contain \( \mathcal{A} \), i.e. such that \( \mathcal{A} \subseteq T \). Note that \( T(\mathcal{A}) \) is not the empty set, as the power set \( \mathcal{P}(\Omega) \) is clearly a topology on \( \Omega \) (called the discrete topology) which satisfies \( \mathcal{A} \subseteq \mathcal{P}(\Omega) \). By definition (55), the topology \( T(\mathcal{A}) \) generated by \( \mathcal{A} \), is equal to \( \cap_{T \in T(\mathcal{A})} T \). In order to show that \( T(\mathcal{A}) \) is a topology on \( \Omega \), it is sufficient to prove that an arbitrary intersection of topologies on \( \Omega \), is also a topology on \( \Omega \). Let \( \{T_i\}_{i \in I} \) be an arbitrary family of topologies on \( \Omega \), and let \( T = \cap_{i \in I} T_i \). Since \( \emptyset \) and \( \Omega \) belong to \( T_i \) for all \( i \in I \), \( \emptyset \) and \( \Omega \) are elements of \( T \). If \( A, B \in T \), then \( A, B \in T_i \) for all \( i \in I \), and therefore \( A \cap B \in T \), \( A \cap B \in T_i \) for all \( i \in I \). It follows that \( A \cap B \in T \), and \( T \) is closed under finite intersection. If \( \{A_j\}_{j \in J} \) is an arbitrary family of elements of \( T \), then for all \( i \in I \), \( \{A_j\}_{j \in J} \) is an arbitrary family of elements of \( T_i \), and consequently \( \cup_{j \in J} A_j \in T_i \). This being true for all \( i \in I \), \( \cup_{j \in J} A_j \in T \), and \( T \) is closed under arbitrary union. We have proved that \( T \) is a topology on \( \Omega \). An arbitrary intersection of topologies on \( \Omega \), is a topology on \( \Omega \). In particular, the topology \( T(\mathcal{A}) \) is a topology on \( \Omega \).

2. Given \( T(\mathcal{A}) = \{T : T \text{ topology on } \Omega, \mathcal{A} \subseteq T \} \), the topology \( T(\mathcal{A}) \) generated by \( \mathcal{A} \) is given by \( T(\mathcal{A}) = \cap_{T \in T(\mathcal{A})} T \). Hence, we have \( \mathcal{A} \subseteq T(\mathcal{A}) \). Suppose \( T \) is another topology on \( \Omega \), such that \( \mathcal{A} \subseteq T \). Then, \( T \in T(\mathcal{A}) \). It follows that \( T(\mathcal{A}) \subseteq T \). We have proved that \( T(\mathcal{A}) \) is the smallest topology on \( \Omega \), such that \( \mathcal{A} \subseteq T(\mathcal{A}) \).

3. Let \((E, d)\) be a metric space, and \( \mathcal{A} \) be the set of all open balls:
\[
\mathcal{A} = \{B(x, \epsilon) : x \in E, \epsilon > 0\}
\]
Let \( \mathcal{T}_d \) be the metric topology on \( E \). Since any open ball in \( E \) is open with respect to the metric topology, i.e. belongs to \( \mathcal{T}_d \), we have \( \mathcal{A} \subseteq \mathcal{T}_d \) and therefore \( T(\mathcal{A}) \subseteq \mathcal{T}_d \). Conversely, let \( U \in \mathcal{T}_d \). Define \( \Gamma = \{B(x, \epsilon) : x \in E, \epsilon > 0\} \), i.e. let \( \Gamma \) be the set of all open balls in \( E \) which are contained in \( U \). Since \( U \) is open for the metric topology, from definition (30), for all \( x \in U \), there exists \( \epsilon > 0 \) such that \( B(x, \epsilon) \subseteq U \).
In particular, there exists $B \in \Gamma$ such that $x \in B$. Hence, $U \subseteq \bigcup_{B \in \Gamma} B$. Conversely, for all $x \in \bigcup_{B \in \Gamma} B$, there exists $B \in \Gamma$ such that $x \in B$. But $B \subseteq U$. So $x \in U$. Hence, we see that $U = \bigcup_{B \in \Gamma} B$. However, $\Gamma$ is a subset of $A \subseteq T(A)$. It follows that $\bigcup_{B \in \Gamma} B$ is an element of $T(A)$. We have proved that $U \in T(A)$. Hence $T_E^d \subseteq T(A)$. Finally, $T_E^d = T(A)$, i.e. the metric topology on $E$ is generated by the set of all open balls in $E$.

Exercise 12.

1. Let $U$ be a subset of $\Pi_{i \in I} \Omega_i$ with the property:

$$\forall x \in U, \exists V \in \Pi_{i \in I} T_i : x \in V \subseteq U$$

(3)

Define $\Gamma = \{ V \in \Pi_{i \in I} T_i : V \subseteq U \}$. Given $x \in U$, since property (3) holds, there exists $V \in \Gamma$ such that $x \in V$. So $U \subseteq \bigcup V \in \Gamma$. Conversely, if $x \in \bigcup V \in \Gamma$, there exists $V \in \Gamma$ such that $x \in V$. But $V \subseteq U$. So $x \in U$. Hence, we see that $U = \bigcup V \in \Gamma$. Since $\Gamma \subseteq \Pi_{i \in I} T_i \subseteq \bigcirc_{i \in I} T_i$, each $V \in \Gamma$ is an element of the product topology $\bigcirc_{i \in I} T_i$. So $\bigcup V \in \Gamma$ is also an element of $\bigcirc_{i \in I} T_i$. We have proved that $U \in \bigcirc_{i \in I} T_i$, and therefore, any subset of $\Pi_{i \in I} \Omega_i$ with property (3), belongs to the product topology $\bigcirc_{i \in I} T_i$. Let $T$ be the set of all $U$ subset of $\Pi_{i \in I} \Omega_i$ which satisfy property (3). We claim that in fact, $T$ is a topology on $\Pi_{i \in I} \Omega_i$. Indeed, $\emptyset$ satisfies property (3) vacuously. So $\emptyset \in T$. The set of all rectangles $\Pi_{i \in I} T_i$ is a subset of $T$. In particular, $\Pi_{i \in I} \Omega_i \in T$. Suppose $A, B \in T$. Let $x \in A \cap B$. Since $A$ satisfies property (3), there exists $V \in \Pi_{i \in I} T_i$ such that $x \in V \subseteq A$. Similarly, there exists $W \in \Pi_{i \in I} T_i$ such that $x \in W \subseteq B$. It follows that $x \in V \cap W \subseteq A \cap B$. However, $V$ and $W$ being rectangles of $(T_i)_{i \in I}$, they can be written as $V = \Pi_{i \in I} A_i$ and $W = \Pi_{i \in I} B_i$, where $A_i, B_i \in T_i \cup \{ \Omega_i \} = T_i$ and $A_i \neq \Omega_i$ or $B_i \neq \Omega_i$ for finitely many $i \in I$. It follows that $V \cap W = \Pi_{i \in I} (A_i \cap B_i)$, where each $A_i \cap B_i$ lie in $T_i$ (it is a topology), and $A_i \cap B_i \neq \Omega_i$ for finitely many $i \in I$. So $V \cap W$ is a rectangle of $(T_i)_{i \in I}$, i.e. $V \cap W \in \Pi_{i \in I} T_i$, and $x \in V \cap W \subseteq A \cap B$. We have proved that $A \cap B$ satisfies property (3), i.e. $A \cap B \in T$. So $T$ is closed under finite intersection. Finally, let $(A_j)_{j \in J}$ be a family of elements of $T$. Let $x \in \bigcup_{j \in J} A_j$. There exists $j \in J$ such that $x \in A_j$. Since $A_j \in T$, there exists $V \in \Pi_{i \in I} T_i$ such that $x \in V \subseteq A_j$. In particular, $x \in V \subseteq \bigcup_{j \in J} A_j$. Hence, we see that $\bigcup_{j \in J} A_j$ satisfies property (3), i.e. $\bigcup_{j \in J} A_j \in T$. So $T$ is closed under arbitrary union. We have proved that $T$ is a topology on $\Pi_{i \in I} \Omega_i$. Since $\Pi_{i \in I} T_i \subseteq T$, we conclude that $\bigcirc_{i \in I} T_i = \bigcirc_{i \in I} (\Pi_{i \in I} T_i) \subseteq T$. It follows that any element of the product topology satisfies property (3). We have proved that a subset $U$ of $\Pi_{i \in I} \Omega_i$ is an element of $\bigcirc_{i \in I} T_i$, if and only if it satisfies property (3).

2. $\Pi_{i \in I} T_i \subseteq T(\Pi_{i \in I} T_i) = \bigcirc_{i \in I} T_i$.

3. From theorem (26), $\bigcirc_{i \in I} B(\Omega_i) = \bigcirc_{i \in I} \sigma(T_i) = \sigma(\Pi_{i \in I} T_i)$.
4. From 2., we have $\sigma(\Pi_{i \in I} T_i) \subseteq \sigma(\otimes_{i \in I} T_i) = \mathcal{B}(\Pi_{i \in I} \Omega_i)$. Using 3., we obtain $\otimes_{i \in I} \mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Pi_{i \in I} \Omega_i)$.

Exercise 13.

1. The scalar product $(x, y)$ being semi-linear and commutative:

$\|x + ty\|^2 = (x + ty, x + ty)
= (x, x) + t(y, x) + t(x, y) + t^2(y, y)
= \|x\|^2 + t^2\|y\|^2 + 2t(x, y)$

2. When $y \neq 0$, the polynomial $t \rightarrow p(t) = t^2\|y\|^2 + 2t(x, y) + \|x\|^2$ has a minimum attained at $t = -(x, y)/\|y\|^2$. The value of this minimum is $-(x, y)^2/\|y\|^2 + \|x\|^2$. Since $p(t) = \|x + ty\|^2 \geq 0$ for all $t \in \mathbb{R}$, in particular, we have $-(x, y)^2/\|y\|^2 + \|x\|^2 \geq 0$, i.e. $(x, y) \leq \|x\|\|y\|$. This inequality still holds if $y = 0$.

3. We have:

$\|x + y\|^2 = \|x\|^2 + 2(x, y) + \|y\|^2
\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$

Exercise 14.

1. Each metric $d_i$ has values in $\mathbb{R}^+$. So $d(x, y) < +\infty$ for all $x, y$, i.e. $d$ also has values in $\mathbb{R}^+$. It is clear that $d(x, y) = d(y, x)$ for all $x, y \in \Omega$. Suppose that $d(x, y) = 0$. Then, for all $i \in \mathbb{N}_n$, we have $d_i(x_i, y_i) = 0$ and consequently $x_i = y_i$. So $x = y$. Conversely, it is clear that $d(x, x) = 0$. Let $x, y, z \in \Omega$. For all $i \in \mathbb{N}_n$, we have:

$d_i(x_i, y_i) \leq d_i(x_i, z_i) + d_i(z_i, y_i)$

and therefore:

$d(x, y) \leq \sqrt{\sum_{i=1}^{n} (d_i(x_i, z_i) + d_i(z_i, y_i))^2}$

Using exercise (13), we conclude that:

$d(x, y) \leq \sqrt{\sum_{i=1}^{n} (d_i(x_i, z_i))^2 + \sum_{i=1}^{n} (d_i(z_i, y_i))^2}$

i.e. $d(x, y) \leq d(x, z) + d(z, y)$. It follows from definition (28) that $d$ is indeed a metric on $\Omega$.

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2. The set of rectangles $\Pi_{i \in N_n} T_i$ is given by:

$$\Pi_{i \in N_n} T_i = \{ U_1 \times \ldots \times U_n : U_i \in T_i, \forall i \in N_n \}$$

It follows from exercise (12) that $U \subseteq \Omega$ is open in $\Omega$, i.e. belongs to the product topology $T$, if and only if for all $x \in U$, there exist $U_1, \ldots, U_n$ open in $\Omega_1, \ldots, \Omega_n$ respectively, such that:

$$x \in U_1 \times \ldots \times U_n \subseteq U$$

3. Let $U \in T$. From 2., for all $x \in U$, there exist $U_1, \ldots, U_n$ open in $\Omega_1, \ldots, \Omega_n$ respectively, such that $x \in U_1 \times \ldots \times U_n \subseteq U$. By assumption, each topology $T_i$ is induced by the metric $d_i$, i.e. $T_i = T_{\Omega^i}$. For all $i \in N_n$, $x_i \in U_i$. Hence, there exists $\epsilon_i > 0$, such that $B(x_i, \epsilon_i) \subseteq U_i$, where $B(x_i, \epsilon_i)$ denotes the open ball in $\Omega_i$. Let $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Suppose $y \in \Omega$ is such that $d_i(x_i, y_i) < \epsilon$, for all $i \in N_n$. Then, $y_i \in B(x_i, \epsilon_i) \subseteq U_i$ for all $i \in N_n$, and consequently $y \in U_1 \times \ldots \times U_n \subseteq U$. We have found $\epsilon > 0$ such that:

$$(\forall i \in N_n, d_i(x_i, y_i) < \epsilon) \Rightarrow y \in U$$

4. Let $U \in T$, and $x \in U$. Let $\epsilon > 0$ be as in 3. Let $y \in B(x, \epsilon)$, where $B(x, \epsilon)$ denotes the open ball in $\Omega = \Omega_1 \times \ldots \times \Omega_n$, with respect to the metric $d$. Then, $d(x, y) < \epsilon$. Since for all $i \in N_n$, $d_i(x_i, y_i) \leq d(x, y)$, we have $d_i(x_i, y_i) < \epsilon$ for all $i \in N_n$. From 3., we see that $y \in U$. So $B(x, \epsilon) \subseteq U$. For all $x \in U$, we have found $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. It follows that $U$ belongs to the metric topology $T_{\Omega^d}$. We have proved that $T \subseteq T_{\Omega^d}$.

5. Let $U \in T_{\Omega^d}$ and $x \in U$. From definition (30) of the metric topology, there exists $\epsilon' > 0$ such that $B(x, \epsilon') \subseteq U$. Define $\epsilon = \epsilon'/\sqrt{n}$, and let $y \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon)$. Then, for all $i \in N_n$, $d_i(x_i, y_i) < \epsilon$. Hence, $d(x, y) < \sqrt{n}\epsilon^2 = \sqrt{n}\epsilon = \epsilon'$. So $y \in U$. We have found $\epsilon > 0$ such that:

$$x \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon) \subseteq U$$

6. Let $U \in T_{\Omega^d}$ and $x \in U$. Let $\epsilon > 0$ be as in 5. Then, we have $x \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon) \subseteq U$. Each $B(x_i, \epsilon)$ being open in $\Omega_i$, we have found $U_1, \ldots, U_n$ open in $\Omega_1, \ldots, \Omega_n$ respectively, such that $x \in U_1 \times \ldots \times U_n \subseteq U$. From 2., we conclude that $U \in T$. So $T_{\Omega^d} \subseteq T$.

7. From 4. and 6., we have $T = T_{\Omega^d}$. In other words, the product topology $T = T_1 \circ \ldots \circ T_n$ is equal to the metric topology $T_{\Omega^d}$ on $\Omega$, induced by the metric $d$. In particular, the topological space $(\Omega, T)$ is metrizable.

8. Both $d'$ and $d''$ have values in $\mathbb{R}^+$. For all $x, y \in \Omega$, we have $d'(x, y) = d'(y, x)$ and $d''(x, y) = d''(y, x)$. Moreover, it is clear that $d'(x, y) = 0$ is equivalent to each $d_i(x_i, y_i)$ being equal to 0, hence equivalent to $x_i = y_i$.

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6 Beware of external links!
for all i’s, i.e. equivalent to \( x = y \). Similarly, \( d''(x, y) = 0 \) is equivalent to \( x = y \). Given \( x, y, z \in \Omega \), for all \( i \in \mathbb{N}_n \), we have:

\[ d_i(x_i, y_i) \leq d_i(x_i, z_i) + d_i(z_i, y_i) \]

It follows immediately that \( d'(x, y) \leq d'(x, z) + d'(z, y) \), and furthermore, for all \( i = 1, \ldots, n \):

\[ d_i(x_i, y_i) \leq d''(x, z) + d''(z, y) \]

From which we conclude that \( d''(x, y) \leq d''(x, z) + d''(z, y) \). We have proved that \( d' \) and \( d'' \) are metrics on \( \Omega \).

9. Let \( x, y \in \Omega \). For all \( i \in \mathbb{N}_n \), define \( a_i = d_i(x_i, y_i) \). Let \( a, b \in \mathbb{R}^n \) be given \( a = (a_1, \ldots, a_n) \) and \( b = (1, \ldots, 1) \). From exercise (13), we have \( |(a, b)| \leq ||a|| ||b|| \), and consequently:

\[ d'(x, y) \leq \sqrt{n}d(x, y) \]

From \((\sum_{i=1}^n a_i)^2 \geq \sum_{i=1}^n a_i^2 \), we obtain:

\[ d(x, y) \leq d'(x, y) \]

Hence, \( \alpha' d' \leq d \leq \beta' d' \), where \( \alpha' = 1/\sqrt{n} \) and \( \beta' = 1 \). From \( \sum_{i=1}^n a_i^2 \leq n(\max_i a_i)^2 \), we obtain:

\[ d(x, y) \leq \sqrt{n}d''(x, y) \]

From \((\max_i a_i)^2 \leq \sum_{i=1}^n a_i^2 \) we obtain:

\[ d''(x, y) \leq d(x, y) \]

Hence, \( \alpha'' d'' \leq d \leq \beta'' d'' \), where \( \alpha'' = 1 \) and \( \beta'' = \sqrt{n} \).

10. From 9., there exist \( \beta' > 0 \) such that \( d \leq \beta' d' \). Let \( U \in T_\Omega^d \), and \( x \in U \). There exists \( \epsilon > 0 \) such that \( B_d(x, \epsilon) \subseteq U \), where \( B_d(x, \epsilon) \) denotes the open ball in \( \Omega \), relative to the metric \( d \). Suppose \( y \in \Omega \) is such that \( d'(x, y) < \epsilon/\beta' \). Then, we have \( d(x, y) \leq \beta' d'(x, y) < \epsilon \), and it follows that \( y \in U \). So \( B_d(x, \epsilon/\beta') \subseteq U \). For all \( x \in U \), we have found \( \epsilon' = \epsilon/\beta' > 0 \) such that \( B_{d'}(x, \epsilon') \subseteq U \). It follows that \( U \in T_{d'} \). We have proved that \( T_\Omega^d \subseteq T_{d'}^d \). Using 9., from \( d' \leq (1/\alpha')d \), we conclude similarly that \( T_{d'}^d \subseteq T_{d'}^d \). Hence, \( T_{d'}^d = T_{d'}^d \). Similarly, from \( \alpha'' d'' \leq d \leq \beta'' d'' \), we have \( T_{d''}^d = T_{d''}^d \). We have proved that \( T_{d''}^d = T_{d''}^d = T_{d''}^d \). Since \( T_{d''}^d = T \) is the product topology on \( \Omega \), we conclude that \( d' \) and \( d'' \) also induce the product topology \( T = T_1 \circ \ldots \circ T_n \) on \( \Omega \).

Exercise 14

Exercise 15.

1. For all \( a \in \mathbb{R}^+ \), \( 1 \land a = \min(1, a) \). Let \( a, b \in \mathbb{R}^+ \). Suppose \( a + b \leq 1 \). Then, both \( a \leq 1 \) and \( b \leq 1 \), and we have:

\[ 1 \land (a + b) = a + b = 1 \land a + 1 \land b \]
Suppose $a + b \geq 1$. If both $a \leq 1$ and $b \leq 1$, we have:

$$1 \land (a + b) = 1 \leq a + b = 1 \land a + 1 \land b$$

if $a \geq 1$, we have:

$$1 \land (a + b) = 1 = 1 \land a \leq 1 \land a + 1 \land b$$

In any case, we see that:

$$1 \land (a + b) \leq 1 \land a + 1 \land b$$

2. For all $x, y \in \Omega$, we have:

$$d(x, y) = \sum_{n=1}^{+\infty} \frac{1}{2^n} (1 \land d_n(x_n, y_n)) \leq \sum_{n=1}^{+\infty} \frac{1}{2^n} < +\infty$$

So $d$ has values in $\mathbb{R}^+$. It is clear that $d(x, y) = d(y, x)$. Moreover, $d(x, y) = 0$ is equivalent to $d_n(x_n, y_n) = 0$ for all $n \geq 1$, which in turn equivalent to $x = y$. For all $x, y, z \in \Omega$, and $n \geq 1$, we have:

$$d_n(x_n, y_n) = d_n(x_n, z_n) + d_n(z_n, y_n)$$

and consequently, using 1:

$$1 \land d_n(x_n, y_n) \leq 1 \land d_n(x_n, z_n) + 1 \land d_n(z_n, y_n)$$

It follows that $d(x, y) \leq d(x, z) + d(z, y)$. We have proved that $d$ is a metric on $\Omega$.

3. Let $V = \Pi_{n=1}^{+\infty} U_n$ be a rectangle of the family $(T_n)_{n \geq 1}$. The set $\{ n \geq 1 : U_n \neq \Omega_n \}$ being finite, it is either empty or has a maximal element $N \geq 1$. it follows that $V$ can be written as:

$$V = U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n$$

where $U_1, \ldots, U_N$ are open in $\Omega_1, \ldots, \Omega_N$ respectively. If the set $\{ n \geq 1 : U_n \neq \Omega_n \}$ is empty, then $V$ is also of the form (4), for any $N \geq 1$. Conversely, any set $V$ of the form (4) is a rectangle in $\Pi_{n=1}^{+\infty} T_n$. From exercise (12), $U \in T = \bigcap_{n=1}^{+\infty} T_n$, if and only if, for all $x \in U$, there exists $V \in \Pi_{n=1}^{+\infty} T_n$ such that $x \in V \subseteq U$. It follows that $U \subseteq \Omega$ is open in $\Omega$, i.e. belongs to the product topology $T$, if and only if for all $x \in U$, there exists $N \geq 1$ and open sets $U_1, \ldots, U_N$ in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

4. Suppose that $d(x, y) < 1/2^n$, for some $n \geq 1$. Then, $d_n(x_n, y_n)$ has to be less than 1. Specifically:

$$d(x, y) \geq \frac{1}{2^n} (1 \land d_n(x_n, y_n)) = \frac{1}{2^n} d_n(x_n, y_n)$$

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So $d(x, y) < 1/2^n \Rightarrow d_n(x_n, y_n) \leq 2^n d(x, y)$

5. Let $U \in T$ and $x \in U$. From 3., there exist $N \geq 1$ and $U_1, \ldots, U_N$ open in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

Let $i \in \{1, \ldots, N\}$. Then $x_i \in U_i \in T_i$. The topology $T_i$ being the metric topology $T_{\Omega_i}$, there exists $\epsilon_i > 0$ such that we have $B(x_i, \epsilon_i) \subseteq U_i$. Let $\epsilon = \min(1/2^N, \epsilon_1/2, \ldots, \epsilon_N/2^N)$ and $y \in \Omega$ be such that $d(x, y) < \epsilon$. In particular, we have $d(x, y) < 1/2^i$, for all $i = 1, \ldots, N$. Hence, from 4., we see that $d_i(x_i, y_i) \leq 2^i d(x, y) \leq 2^i \epsilon \leq \epsilon_i$. It follows that $y_i \in U_i$ for all $i = 1, \ldots, N$ and consequently $y \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$. We have found $\epsilon > 0$ such that $d(x, y) < \epsilon \Rightarrow y \in U$.

6. From 5. for all $U \in T$ and $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. It follows that $U \in T^d_T$. So $T \subseteq T^d_T$.

7. Let $U \in T^d_T$ and $x \in U$. By definition (30) of the metric topology, there exists $\epsilon' > 0$ such that $B(x, \epsilon') \subseteq U$. In other words, there exists $\epsilon' > 0$ such that for all $y \in \Omega$:

$$d(x, y) < \epsilon' \Rightarrow y \in U$$

Let $\epsilon = \epsilon' / 2$ and $N \geq 1$ be such that:

$$\sum_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \epsilon$$

Suppose $y \in \Omega$ is such that:

$$\sum_{n=1}^{N} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n)) < \epsilon$$

Then, we have:

$$d(x, y) < \epsilon + \sum_{n=N+1}^{+\infty} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n)) \leq 2 \epsilon = \epsilon'$$

It follows that $y \in U$. We have found $\epsilon > 0$ and $N \geq 1$ such that:

$$\sum_{n=1}^{N} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n)) < \epsilon \Rightarrow y \in U$$

8. Let $U \in T^d_{\Omega}$ and $x \in U$. Let $\epsilon > 0$ an $N \geq 1$ be as in 7. Let $y \in \Omega$ be such that:

$$y \in B(x_1, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \prod_{n=N+1}^{+\infty} \Omega_n$$
For all \( n \in \{1, \ldots, N\} \), \( d_n(x_n, y_n) < \epsilon \). Hence:
\[
\sum_{n=1}^{N} \frac{1}{2^n} (1 \wedge d_n(x_n, y_n)) \leq \epsilon \sum_{n=1}^{N} \frac{1}{2^n} < \epsilon
\]
From 7., we conclude that \( y \in U \). We have found \( \epsilon > 0 \) and \( N \geq 1 \) such that:
\[
x \in B(x, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \Pi_{n=N+1}^{+\infty} \Omega_n \subseteq U
\]

9. Let \( U \in T_{\Omega}^d \) and \( x \in U \). Let \( N \geq 1 \) and \( \epsilon > 0 \) be as in 8. Each open ball \( B(x_n, \epsilon) \) for \( n = 1, \ldots, N \) being open in \( \Omega_n \), we have found \( U_1, \ldots, U_N \) open in \( \Omega_1, \ldots, \Omega_N \) respectively, such that:
\[
x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U
\]
From 3., it follows that \( U \in T = \cap_{n=1}^{+\infty} T_n \). We have proved that \( T_{\Omega}^d \subseteq T \).

10. From 6. and 9., \( T_{\Omega}^d = T \). In other words, the product topology \( T = \cap_{n=1}^{+\infty} T_n \) is induced by the metric \( d \) on \( \Omega \). In particular, the topological space \((\Omega, T)\) is metrizable. The purpose of this exercise, is to show that a countable product of metrizable topological spaces, is itself a metrizable topological space.

Exercise 16.

1. \( \mathcal{H} = \{[r, q] : r, q \in \mathbb{Q}\} \) is a countable subset of \( T_{\mathbb{R}} \). Let \( U \in T_{\mathbb{R}} \). Define \( \mathcal{H}' = \{V \in \mathcal{H} : V \subseteq U\} \). For all \( x \in U \), there exists \( \epsilon > 0 \) such that \( |x - \epsilon, x + \epsilon| \subseteq U \). In fact, the set of rational numbers \( \mathbb{Q} \) being dense in \( \mathbb{R} \), there exist \( r, q \in \mathbb{Q} \) such that \( x \in [r, q] \subseteq U \). In other words, there exists \( V \in \mathcal{H}' \) such that \( x \in V \). Hence, we see that \( U \subseteq \cup_{V \in \mathcal{H}'} V \). The reverse inclusion being clearly satisfied, we have \( U = \cup_{V \in \mathcal{H}'} V \), i.e. \( U \) can be expressed as a union of elements of \( \mathcal{H} \). This being true for all open sets \( U \in T_{\mathbb{R}} \), we have proved that \( \mathcal{H} \) is a countable base of \((\mathbb{R}, T_{\mathbb{R}})\).

2. Let \( \mathcal{H} \) be a countable base of \((\Omega, T)\). Let \( \mathcal{H}_{\Omega'} \) be the trace of \( \mathcal{H} \) on \( \Omega' \), i.e. \( \mathcal{H}_{\Omega'} = \{\Omega' \cap V : V \in \mathcal{H}\} \). Since \( \mathcal{H} \) is a countable or finite subset of the topology \( T \), \( \mathcal{H}_{\Omega'} \) is a countable or finite subset of the induced topology \( T_{\Omega'} \). Let \( U' \in T_{\Omega'} \) be an open subset in \( \Omega' \). Then \( U' \) is of the form \( U' = \Omega' \cap U \) where \( U \in T \). \( \mathcal{H} \) being a countable base of \((\Omega, T)\), there exists a family \((V_i)_{i \in I} \) of elements of \( \mathcal{H} \) such that \( U = \cup_{i \in I} V_i \). It follows that \((\Omega' \cap V_i)_{i \in I} \) is a family of elements of \( \mathcal{H}_{\Omega'} \) such that \( U' = \cup_{i \in I} (\Omega' \cap V_i) \). We conclude that \( \mathcal{H}_{\Omega'} \) is a countable base of the induced topological space \((\Omega', T_{\Omega'})\).

3. From 1., \( \mathbb{R} \) has a countable base. It follows from 2. that the induced topological space \([-1, 1]\) also has a countable base.
Exercise 17.

4. Let \( h : (Ω, T) \to (S, T_S) \) be a homeomorphism, i.e. a continuous bijection such that \( h^{-1} \) is also continuous. Suppose \((Ω, T)\) has a countable base \( H \). Define \( h(H) = \{ h(V) : V \in H \} \). Since \( H \) is a countable or finite subset of \( T \), \( h^{-1} \) being continuous, \( h(H) \) is a countable or finite subset of \( T_S \). (Note that each direct image \( h(V) \) of \( V \) by \( h \) can be viewed the inverse image \((h^{-1})^{-1}(V)\) of \( V \) by \( h^{-1} \).) Let \( U' \in T_S \) have being continuous, \( h^{-1}(U') \in T \). \( H \) being a countable base of \((Ω, T)\), there exists a family \((V_i)_{i \in I} \) of elements of \( H \), such that \( h^{-1}(U') = \bigcup_{i \in I} V_i \). However, \( h(h^{-1}(U')) = U' \), and moreover:

\[
h(\bigcup_{i \in I} V_i) = (h^{-1})^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} (h^{-1})^{-1}(V_i)
\]

So \( U' = \bigcup_{i \in I} h(V_i) \). We conclude that \( U' \) can be expressed as a union of elements of \( h(H) \). So \( h(H) \) is a countable base of \((S, T_S)\). We have proved that if \((Ω, T)\) has a countable base, then \((S, T_S)\) also has a countable base. Using the same argument, switching the roles of \( h \) and \( h^{-1} \), we see that conversely, if \((S, T_S)\) has a countable base, then so does \((Ω, T)\). We have proved that given two homeomorphic topological spaces, one has a countable base, if and only if the other also has a countable base.

5. The topological spaces \((\mathbb{R}, T_{\mathbb{R}})\) and \([-1, 1], T_{[-1,1]}\) being homeomorphic, we conclude from 3. and 4. that \((\mathbb{R}, T_{\mathbb{R}})\) has a countable base.

Exercise 16

1. Let \( p \geq 1 \) and \( A \in H^p \) of the form:

\[
A = V_1^{k_1} \times \ldots \times V_p^{k_p} \times \prod_{n=p+1}^{+\infty} \Omega_n
\]

For all \( n \geq 1 \), the set \( \{ V_n^k : k \in I_n \} \) being a countable base of \( T_n \), it is a subset of \( T_n \). Hence, for all \( i \in \{1, \ldots, p\} \), we have \( V_i^{k_i} \in T_i \). It follows that \( A \) is a rectangle of the family \((T_n)_{n \geq 1} \), i.e. \( A \in \prod_{n=1}^{+\infty} T_n \). From definition (56), the product topology \( T \) on \( \prod_{n=1}^{+\infty} \Omega_n \) being generated by \( \prod_{n=1}^{+\infty} T_n \), we have \( \prod_{n=1}^{+\infty} T_n \subseteq T \). In particular, \( A \in T \). We have proved that \( H^p \subseteq T \).

2. Using 1., \( H = \bigcup_{n \geq 1} H^p \subseteq T \).

3. By assumption, for all \( n \geq 1 \), the index set \( I_n \) is finite or countable. There exists an injective map \( i_n : I_n \to \mathbb{N} \). Given \( p \geq 1 \), consider the map \( j_p : H^p \to \mathbb{N}^p \), defined in the following way: for \( A = V_1^{k_1} \times \ldots \times V_p^{k_p} \times \prod_{n=p+1}^{+\infty} \Omega_n \in H^p \), we put:

\[
j_p(A) = (i_1(k_1), \ldots, i_p(k_p))
\]

Suppose \( B = V_1^{k'_1} \times \ldots \times V_p^{k'_p} \times \prod_{n=p+1}^{+\infty} \Omega_n \) is another element of \( H^p \) such that \( j_p(A) = j_p(B) \). Then:

\[
(i_1(k_1), \ldots, i_p(k_p)) = (i_1(k'_1), \ldots, i_p(k'_p))
\]
Hence, for all $m \in \mathbb{N}_p$, $i_m(k_m) = i_m(k'_m)$, and $i_m$ being injective, we have $k_m = k'_m$. So $A = B$. We have proved the existence of an injective map $j_p : \mathcal{H}_p \to \mathbb{N}_p$.

4. The existence of a bijection $\phi_2 : \mathbb{N}^2 \to \mathbb{N}$ is a standard result, which we may have used in these tutorials before. Now is a good opportunity to give a formal proof of it. Informally, $\phi_2$ is defined as $\phi_2(0, 0) = 0$, $\phi_2(1, 0) = 1$, $\phi_2(0, 1) = 2$, $\phi_2(2, 0) = 3$, $\phi_2(1, 1) = 4$, $\phi_2(0, 2) = 5$, etc. As you can see, going through each diagonal one after the other, we are able to count the elements of $\mathbb{N}^2$, thus defining the bijection $\phi_2$. Formally, we define the map $\phi_2 : \mathbb{N}^2 \to \mathbb{N}$ as follows:

$$\forall (n, p) \in \mathbb{N}^2, \phi_2(n, p) = p + [0 + 1 + \ldots + (n + p)]$$

or equivalently, $\phi_2(n, p) = p + h(n + p)$ where:

$$h(m) = 0 + 1 + \ldots + m$$

Let $N \in \mathbb{N}$. Since $h(m) \uparrow +\infty$, the set $\{m \in \mathbb{N} : h(m) \leq N\}$ is finite and it is also non-empty. Hence, it has a maximal element $m$, and we have $h(m) \leq N < h(m + 1)$. Let $p = N - h(m)$. Then $p \in \mathbb{N}$, and we have $0 \leq p < h(m + 1) - h(m) = m + 1$. So $p \leq m$. If we define $n = m - p$, then $n$ is also an element of $\mathbb{N}$. So $(n, p)$ is an element of $\mathbb{N}^2$, such that $m = n + p$, and $N = p + h(m)$. It follows that:

$$\phi_2(n, p) = p + h(n + p) = p + h(m) = N$$

We have proved that $\phi_2$ is a surjective map. Suppose $(n, p)$ and $(n', p')$ are elements of $\mathbb{N}^2$, with $\phi_2(n, p) = \phi_2(n', p')$. Since $\phi_2(n, p) = p + h(n + p)$, in particular $h(n + p) \leq \phi_2(n, p)$. However, $h(n + p + 1) = p + h(n + p) + n + 1 < \phi_2(n, p)$. It follows that for all $(n, p) \in \mathbb{N}^2$, we have:

$$h(n + p) \leq \phi_2(n, p) < h(n + p + 1) \quad (5)$$

Since given $N \in \mathbb{N}$, any $m \in \mathbb{N}$ such that $h(m) \leq N < h(m + 1)$ is unique, it follows from $\phi_2(n, p) = \phi_2(n', p')$ and equation (5) that $n + p = n' + p'$. Hence:

$$p = \phi_2(n, p) - h(n + p) = \phi_2(n', p') - h(n' + p') = p'$$

and finally $n = (n + p) - p = (n' + p') - p' = n'$. We have proved that $\phi_2$ is an injective map. We conclude that $\phi_2 : \mathbb{N}^2 \to \mathbb{N}$ is a bijection.

5. Let $p \geq 1$. The existence of a bijection $\phi_p : \mathbb{N}^p \to \mathbb{N}$ is true for $p = 1$ and $p = 2$. Suppose the existence of $\phi_p$ has been proved, and let $\phi_2 : \mathbb{N}^2 \to \mathbb{N}$ be as in 4. Let $\phi_{p+1} : \mathbb{N}^{p+1} \to \mathbb{N}$ be defined by:

$$\phi_{p+1}(n_1, \ldots, n_{p+1}) = \phi_2[\phi_p(n_1, \ldots, n_p), n_{p+1}]$$

for all $(n_1, \ldots, n_{p+1}) \in \mathbb{N}^{p+1}$. Let $N \in \mathbb{N}$. $\phi_2$ being a surjection, there exists $(n, n_{p+1}) \in \mathbb{N}^2$ with $\phi_2(n, n_{p+1}) = N$. From our induction hypothesis, $\phi_p : \mathbb{N}^p \to \mathbb{N}$ is also a surjective map. There exists $(n_1, \ldots, n_p) \in \mathbb{N}^p$, www.probability.net
such that \( \phi_p(n_1, \ldots, n_p) = n \). It follows that \((n_1, \ldots, n_{p+1})\) is an element of \( \mathbb{N}^{p+1} \) such that \( \phi_{p+1}(n_1, \ldots, n_{p+1}) = N \). So \( \phi_{p+1} \) is itself a surjective map. Suppose \((n_1, \ldots, n_{p+1})\) and \((n'_1, \ldots, n'_{p+1})\) are elements of \( \mathbb{N}^{p+1} \) such that:
\[
\phi_{p+1}(n_1, \ldots, n_{p+1}) = \phi_{p+1}(n'_1, \ldots, n'_{p+1})
\]
Then, \( \phi_2 \) being injective, \( n_{p+1} = n'_{p+1} \), and:
\[
\phi_p(n_1, \ldots, n_p) = \phi_p(n'_1, \ldots, n'_p)
\]
\( \phi_p \) being itself injective, \((n_1, \ldots, n_p) = (n'_1, \ldots, n'_p)\), and we conclude that \((n_1, \ldots, n_{p+1}) = (n'_1, \ldots, n'_{p+1})\). So \( \phi_{p+1} \) is an injective map, and finally a bijection. This induction argument proves the existence of a bijection \( \phi_p : \mathbb{N}^p \to \mathbb{N} \), for all \( p \geq 1 \).

6. Let \( p \geq 1 \). From 3., there exists an injective map \( j_p : \mathcal{H}^p \to \mathbb{N}^p \). From 5., there exists a bijection \( \phi_p : \mathbb{N}^p \to \mathbb{N} \). It follows that \( \phi_p \circ j_p : \mathcal{H}^p \to \mathbb{N} \) is an injective map. This proves that \( \mathcal{H}^p \) is finite or countable, i.e. \( \mathcal{H}^p \) is at most countable.

7. From 6., for all \( p \geq 1 \), there exists an injection \( \psi_p : \mathcal{H}^p \to \mathbb{N} \). Let \( j : \mathcal{H} \to \mathbb{N}^2 \) be defined by \( j(A) = (p, \psi_p(A)) \), where \( p \geq 1 \) is chosen such that \( A \in \mathcal{H}^p \), (there is at least one such \( p \) for any \( A \in \mathcal{H} \)). Suppose \( j(A) = j(B) \) for some \( A, B \in \mathcal{H} \). Then, there exists \( p \geq 1 \) such that \( A, B \in \mathcal{H}^p \) and \( \psi_p(A) = \psi_p(B) \), and consequently \( A = B \). So \( j \) is an injection. We have proved the existence of an injective map \( j : \mathcal{H} \to \mathbb{N}^2 \).

8. Let \( \phi_2 : \mathbb{N}^2 \to \mathbb{N} \) be a bijection. From 7., there exists an injection \( j : \mathcal{H} \to \mathbb{N}^2 \). It follows that \( \phi_2 \circ j : \mathcal{H} \to \mathbb{N} \) is an injection. This proves that \( \mathcal{H} \) is finite or countable, i.e. it is at most countable. From 2., \( \mathcal{H} \subseteq \mathcal{T} \). Hence, all elements of \( \mathcal{H} \) are open sets in \( \Omega \), (with respect to the product topology). We conclude that \( \mathcal{H} \) is a finite or countable set of open sets in \( \Omega \).

9. From exercise (12), \( U \in \mathcal{T} = \mathcal{C}_{n=1}^{+\infty} \mathcal{T}_n \), if and only if for all \( x \in U \), there exists \( V \in \Pi_{n=1}^{+\infty} \mathcal{T}_n \) such that \( x \in V \subseteq U \). Since all elements of \( \Pi_{n=1}^{+\infty} \mathcal{T}_n \) can be written as \( U_1 \times \ldots \times U_p \times \Pi_{n=p+1}^{+\infty} \Omega_n \) for some \( p \geq 1 \) and \( U_1, \ldots, U_p \) open in \( \mathcal{T}_1, \ldots, \mathcal{T}_p \) respectively, it follows in particular that if \( U \in \mathcal{T} \) and \( x \in U \), there exist \( p \geq 1 \) and \( U_1, \ldots, U_p \) open in \( \Omega_1, \ldots, \Omega_p \) such that:
\[
x \in U_1 \times \ldots \times U_p \times \Pi_{n=p+1}^{+\infty} \Omega_n \subseteq U
\]

10. Let \( U \in \mathcal{T} \) and \( x \in U \). Let \( p \geq 1 \) and \( U_1, \ldots, U_p \) open \( \Omega_1, \ldots, \Omega_p \) respectively, such that \( x \in U_1 \times \ldots \times U_p \times \Pi_{n=p+1}^{+\infty} \Omega_n \subseteq U \). By assumption, for all \( n \geq 1 \), the set \( \{V_n^k : k \in I_n\} \) is a countable base of the topology \( \mathcal{T}_n \). Hence, for all \( n \in \mathbb{N}_p \), there exists a subset \( I'_n \) of \( I_n \), such that
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\[ U_n = \bigcup_{k \in I_n^k} V_n^k. \]  
In particular, since \( x_n \in U_n \), there exists \( k_n \in I_n^k \subseteq I_n \) such that \( x_n \in V_n^{k_n} \subseteq U_n \). We have found \( k_1, \ldots, k_p \) such that:

\[
x \in V_1^{k_1} \times \ldots \times V_p^{k_p} \times \prod_{n=p+1}^{\infty} \Omega_n \trianglelefteq V_x \subseteq U
\]

There exists \( V_x \in \mathcal{H}^p \subseteq \mathcal{H} \) such that \( x \in V_x \subseteq U \).

11. From 8., \( \mathcal{H} \) is a finite or countable subset of the topology \( \mathcal{T} \). From 10., for all \( U \in \mathcal{T} \), \( U \) can be written as \( U = \bigcup_{x \in U} V_x \), where \( V_x \in \mathcal{H} \) for all \( x \in U \). In other words, any open set \( U \) of \( \mathcal{T} \) can be written as a union of elements of \( \mathcal{H} \). It follows from definition (57) that \( \mathcal{H} \) is a countable base of \( (\Omega, \mathcal{T}) \).

12. From theorem (26), since \( \mathcal{B}(\Omega_n) = \sigma(\mathcal{T}_n) \) for all \( n \geq 1 \):

\[
\bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) = \sigma(\bigcup_{n=1}^{+\infty} \mathcal{T}_n) \subseteq \sigma(\mathcal{T}) = \mathcal{B}(\Omega)
\]

13. Let \( p \geq 1 \) and \( A \in \mathcal{H}^p \). Then \( A \) is a rectangle of the family \( (\mathcal{T}_n)_{n \geq 1} \). Hence \( A \in \bigcup_{n=1}^{+\infty} \mathcal{T}_n \subseteq \bigcup_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \subseteq \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \). So \( \mathcal{H}^p \subseteq \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \). We conclude that:

\[
\mathcal{H} = \bigcup_{p \geq 1} \mathcal{H}^p \subseteq \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)
\]

14. Since \( \mathcal{H} \) is a countable base of \( (\Omega, \mathcal{T}) \), any open set \( U \) of \( \mathcal{T} \) can be expressed as a union of elements of \( \mathcal{H} \). Furthermore, \( \mathcal{H} \) being at most countable, such union is at most countable. It follows that any open set \( U \) in \( \mathcal{T} \) is an element of \( \sigma(\mathcal{H}) \), i.e. \( T \subseteq \sigma(\mathcal{H}) \). From 13., we have \( \mathcal{H} \subseteq \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \) and consequently, we have \( \sigma(\mathcal{H}) \subseteq \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \). Hence, we see that \( T \subseteq \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \), and finally \( \mathcal{B}(\Omega) = \sigma(\mathcal{T}) \subseteq \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \). Using 12., we conclude that:

\[
\mathcal{B}(\Omega) = \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)
\]

The purpose of this exercise is to prove theorem (27).

Exercise 17

Exercise 18.

1. Since \( (\Omega, \mathcal{T}) \) has a countable base, a finite version of theorem (27) would allow us to conclude immediately that:

\[
\mathcal{B}(\Omega^n) = \mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)
\]

Since \( \mathcal{B}(\Omega) = \sigma(\mathcal{T}) \), from theorem (26), we have:

\[
\mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega) = \sigma(\mathcal{T} \prod_1^n \mathcal{T}) \subseteq \sigma(\mathcal{T}_{\Omega^n}) = \mathcal{B}(\Omega^n)
\]

Let \( U \) be open in \( \Omega^n \), and \( x \in U \). From exercise (12), there exist \( V_1, \ldots, V_n \) open in \( \Omega \), such that:

\[
x \in V_1 \times \ldots \times V_n \subseteq U
\]

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Exercise 19.

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2. Let $A = \{A_x : x \in U\}$. Since $\mathcal{H}$ is a subset of $\mathcal{T}$, each $A_x$ is an element of $\mathcal{T} \cap \mathcal{T}$. Although the set $U$ may not be countable, the set $I$ defined by $I = \{A_x : x \in U\}$ is at most countable, $\mathcal{H}$ being at most countable. So $U = \bigcup_{x \in U} A_x$ is in fact a countable (or finite) union of elements of $\mathcal{T} \cap \mathcal{T}$. So $U \in \sigma(\mathcal{T} \cap \mathcal{T})$. We have proved that:

$$T_{\Omega^n} \subseteq \sigma(\mathcal{T} \cap \mathcal{T}) \subseteq \mathcal{B}(\Omega) \otimes \mathcal{B}(\Omega)$$

We conclude that:

$$\mathcal{B}(\Omega^n) = \sigma(T_{\Omega^n}) \subseteq \mathcal{B}(\Omega) \otimes \mathcal{B}(\Omega)$$

We have proved that $\mathcal{B}(\Omega^n) = \mathcal{B}(\Omega) \otimes \mathcal{B}(\Omega)$.

2. This is an immediate consequence of 1. and exercise (16).

3. From 1., $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. $\mathbb{C}$ and $\mathbb{R}^2$ being identified, the usual topology on $\mathbb{C}$ is induced by the metric:

$$d(z, z') = \sqrt{(x-x')^2 + (y-y')^2}$$

with obvious notations. From exercise (14), such metric induces the product topology on $\mathbb{R}^2$. It follows that the usual topology on $\mathbb{C}$ and the product topology on $\mathbb{R}^2$ coincide. So $\mathcal{T}_\mathbb{C} = \mathcal{T}_\mathbb{R}^2$, and finally $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Exercise 18

Exercise 19.

1. $\mathcal{H} = \{B(x_n, 1/p) : n, p \geq 1\}$ is a finite or countable subset of $\mathcal{T}'$. Let $U \in \mathcal{T}'$ and $x \in U$. There exists $\epsilon > 0$, such that $B(x, \epsilon) \subseteq U$. By assumption, the set $\{x_n : n \geq 1\}$ is dense in $E$. $p \geq 1$ being such that $1/p \leq \epsilon/2$, there exists $n \geq 1$ such that $x_n \in B(x, 1/p)$. In particular, $x \in B(x_n, 1/p)$. Moreover, for all $y \in B(x_n, 1/p)$, we have:

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{2}{p} \leq \epsilon$$

So $y \in B(x, \epsilon) \subseteq U$. Hence, we see that $x \in B(x_n, 1/p) \subseteq U$. For all $x \in U$, we have found $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U$. It follows that $U = \bigcup_{x \in U} V_x$. So $U$ is a union of elements of $\mathcal{H}$. We have proved that $\mathcal{H}$ is a countable base of $(E, \mathcal{T}')$.

2. Let $A = \{x_V : V \in \mathcal{H}, V \neq \emptyset\}$. $\mathcal{H}$ being a countable base of $(E, \mathcal{T}')$, it is at most countable. There exists an injective map $j : \mathcal{H} \to \mathbb{N}$. Let $i : A \to \mathcal{H}$ be defined by $i(x_V) = V$. Then $i$ is clearly an injection, and $j \circ i : A \to \mathbb{N}$

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is therefore an injective map. So $A$ is a finite or countable subset of $E$. Let $x \in E$. Let $U \in T_E^d$ such that $x \in U$. Since $U$ can be written as a union of elements of $\mathcal{H}$, there exists $V \in \mathcal{H}$, such that $x \in V \subseteq U$. In particular, $V \neq \emptyset$ and $x_V$ is well-defined, with $x_V \in V \subseteq U$. So $x_V \in A \cap U \neq \emptyset$. We have proved that for all $U \in T_E^d$ such that $x \in U$, $U \cap A \neq \emptyset$. From definition (37), $x$ is an element of $A$, the closure of $A$. We have proved that $E \subseteq \bar{A}$. So $E = \bar{A}$, and $A$ is dense in $E$. Finally, $A$ is at most countable and dense in $E$. So $(E,d)$ is a separable metric space. The purpose of 1. and 2. is to show that for metric spaces, being separable, or having a countable base, are equivalent.

3. Let $x, y, x', y' \in E$. We have:

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$$
and therefore:

$$d(x, y) - d(x', y') \leq d(x, x') + d(y, y')$$

Similarly:

$$d(x', y') - d(x, y) \leq d(x, x') + d(y, y')$$

It follows that:

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')$$

4. Let $\delta : (E \times E)^2 \to \mathbb{R}^+$ be the metric on $E \times E$ defined by:

$$\delta((x, y), (x', y')) = d(x, x') + d(y, y')$$

From 3., we have:

$$|d(x, y) - d(x', y')| \leq \delta((x, y), (x', y'))$$

(6)

From exercise (14), the product topology $T_{E \times E}$ on $E \times E$ is induced by the metric $\delta$. Using exercise (4) of Tutorial 4, we conclude from equation (6) that $d : (E \times E, T_{E \times E}) \to (\mathbb{R}^+, T_{\mathbb{R}^+})$ is a continuous map.

5. From exercise (13) of Tutorial 4, and the continuity of the map $d : E \times E \to \mathbb{R}^+$ proved in 4., we conclude that:

$$d : (E \times E, B(E \times E)) \to (\mathbb{R}^+, B(\mathbb{R}^+))$$

is a measurable map. It follows that:

$$d : (E \times E, B(E \times E)) \to (\bar{\mathbb{R}}, B(\bar{\mathbb{R}}))$$

is a also a measurable map.

6. If $(E, d)$ is a separable metric space, from 1., it has a countable base. From exercise (18), $B(E \times E) = B(E) \otimes B(E)$. We conclude from 5. that $d : (E \times E, B(E) \otimes B(E)) \to (\bar{\mathbb{R}}, B(\bar{\mathbb{R}}))$ is a measurable map.

\*Beware of external links!
7. By definition (54), the product $\sigma$-algebra $\mathcal{B}(E) \otimes \mathcal{B}(E)$ is generated by the set of measurable rectangles $\mathcal{B}(E) \Pi \mathcal{B}(E)$. From theorem (14), in order to prove the measurability of:

$$\Phi : (\Omega, \mathcal{F}) \rightarrow (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E))$$

It is sufficient to prove that $\Phi^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(E) \Pi \mathcal{B}(E)$. However, any measurable rectangle $B$ of $\mathcal{B}(E) \Pi \mathcal{B}(E)$ is of the form $B = A_1 \times A_2$, where $A_1, A_2 \in \mathcal{B}(E)$. It follows that:

$$\Phi^{-1}(B) = f^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$$

since by assumption, both $f, g : (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{B}(E))$ are measurable maps. We have proved that $\Phi : \Omega \rightarrow E \times E$ is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(E) \otimes \mathcal{B}(E)$.

8. Suppose $(E, d)$ is a separable metric space. From 6., the map:

$$d : (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E)) \rightarrow (\mathcal{R}, \mathcal{B}(\mathcal{R}))$$

is measurable. However, from 7., the map:

$$\Phi : (\Omega, \mathcal{F}) \rightarrow (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E))$$

is also measurable. It follows that $\Psi = d(f, g) = d \circ \Phi$ is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathcal{R})$.

9. From 8., when $(E, d)$ is separable, the map $\Psi = d(f, g)$ is measurable. Hence:

$$\{f = g\} = \Psi^{-1}(\{0\}) \in \mathcal{F}$$

10. Let $(E_n, d_n)_{n \geq 1}$ be a sequence of separable metric spaces. From exercise (15), the product topological space $\Pi_{n=1}^{+\infty} E_n$ is metrizable. From 1., each $E_n$ has a countable base. From theorem (27), $\Pi_{n=1}^{+\infty} E_n$ also has a countable base. Being metrizable, it follows from 2., that it is in fact separable. We have proved that $\Pi_{n=1}^{+\infty} E_n$ is metrizable and separable.

Exercise 19

Exercise 20. Suppose each $f_i : (\Omega, \mathcal{F}) \rightarrow (\Omega_i, \mathcal{F}_i)$ is measurable. From theorem (14), in order to prove the measurability of:

$$f : (\Omega, \mathcal{F}) \rightarrow (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i)$$

It is sufficient to show that $f^{-1}(B) \in \mathcal{F}$, for all $B \in \Pi_{i \in I} \mathcal{F}_i$. Let $B = \Pi_{i \in I} A_i$ be a measurable rectangle of the family $(\mathcal{F}_i)_{i \in I}$. For all $i \in I$, $A_i \in \mathcal{F}_i$, and $J = \{i \in I : A_i \neq \Omega_i\}$ is a finite set. Hence:

$$f^{-1}(B) = \bigcap_{i \in I} \{f_i \in A_i\} = \bigcap_{i \in J} \{f_i \in A_i\} \in \mathcal{F}$$

since each $f_i$ is measurable. So $f$ is indeed measurable. Conversely, suppose $f = (f_i)_{i \in I}$ is measurable. Let $j \in I$ and $A_j \in \mathcal{F}_j$. We have:

$$f_j^{-1}(A_j) = f^{-1}(A_j \times \Pi_{i \neq j} \Omega_i) \in \mathcal{F}$$
since \( B = A_j \times \Pi_{i \neq j} \Omega_i \) is a measurable rectangle, and lies in \( \otimes_{i \in J} \mathcal{F}_i \). So \( f_j : (\Omega, \mathcal{F}) \rightarrow (\Omega_j, \mathcal{F}_j) \) is a measurable map.

Exercise 21.

1. Let \((x, y)\) and \((x', y')\) be elements of \(\mathbb{R}^2\). We have:

\[
|\phi(x, y) - \phi(x', y')| \leq |x - x'| + |y - y'| \tag{7}
\]

By definition (17), the usual topology on \(\mathbb{R}\) is the metric topology induced by \(d(x, y) = |x - y|\). From exercise (14), the product topology on \(\mathbb{R}^2\) is induced by:

\[
\delta[(x, y), (x', y')] = |x - x'| + |y - y'|
\]

It follows from equation (7), and exercise (4) of Tutorial 4 that:

\[
\phi : (\mathbb{R}^2, T_{\mathbb{R}^2}) \rightarrow (\mathbb{R}, T_{\mathbb{R}})
\]

is a continuous map.

Let \((x_0, y_0) \in \mathbb{R}^2\) and \(\epsilon > 0\). For all \((x, y) \in \mathbb{R}^2\), we have:

\[
|\psi(x, y) - \psi(x_0, y_0)| \leq |y|(|x - x_0| + |x_0|) + |y - y_0|
\]

Suppose \(\eta > 0\) is such that:

\[
|x - x_0| + |y - y_0| < \eta \leq 1
\]

Then in particular, \(|y| \leq 1 + |y_0|\), and consequently:

\[
|\psi(x, y) - \psi(x_0, y_0)| \leq M(|x - x_0| + |y - y_0|)
\]

where \(M = \max(|x_0|, 1 + |y_0|)\). Hence, we see that:

\[
\delta[(x, y), (x_0, y_0)] < \eta \Rightarrow |\psi(x, y) - \psi(x_0, y_0)| < \epsilon
\]

where \(\eta\) has been chosen as \(\eta = \min(\epsilon/M, 1)\). We conclude from exercise (4) of Tutorial 4 that \(\psi : (\mathbb{R}^2, T_{\mathbb{R}^2}) \rightarrow (\mathbb{R}, T_{\mathbb{R}})\) is a continuous map.

2. \(\phi\) and \(\psi\) being continuous, from exercise (13) of Tutorial 4:

\[
\phi, \psi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))
\]

are measurable maps. Since \((\mathbb{R}, T_{\mathbb{R}})\) has a countable base, from exercise (18), we have \(\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})\). We conclude that:

\[
\phi, \psi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))
\]

are measurable maps.

3. Given \(f, g : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))\) measurable, the fact that \(f + g\) and \(f \cdot g\) are measurable was already proved in Tutorial 4. The purpose of this exercise is to emphasize a more direct proof. From theorem (28), the map:

\[
h = (f, g) : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}))
\]

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is measurable, since both $f$ and $g$ are measurable. From 2:

$$\phi, \psi : (\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

are also measurable. It follows that $f + g = \phi \circ h$ and $f \cdot g = \psi \circ h$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$. Being real-valued, they are also measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$.

Exercise 21