18. The Jacobian Formula

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

Definition 125 *We call* **K-normed space***, an ordered pair* (E, N)*, where* E *is a* $\mathbf{K}$*-vector space, and* $N : E \to \mathbf{R}^+$ *is a norm on* E*.*

See definition (89) for *vector space*, and definition (95) for *norm*.

EXERCISE 1. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is a norm on $\mathcal{H}$.
2. Show that $(\mathcal{H}, \|\cdot\|)$ is a $\mathbf{K}$-normed space.

EXERCISE 2. Let $(E, \|\cdot\|)$ be a $\mathbf{K}$-normed space:

1. Show that $d(x, y) = \|x - y\|$ defines a metric on E.
2. Show that for all $x, y \in E$, we have $|\,\|x\| - \|y\|\,| \leq \|x - y\|$.

Definition 126 *Let* $(E, \|\cdot\|)$ *be a* $\mathbf{K}$*-normed space, and* d *be the metric defined by* $d(x, y) = \|x - y\|$*. We call* **norm topology** *on* E*, denoted* $\mathcal{T}_{\|\cdot\|}$*, the topology on* E *associated with* d*.*

Note that this definition is consistent with definition (82) of the norm topology associated with an inner-product.

EXERCISE 3. Let E, F be two $\mathbf{K}$-normed spaces, and $l : E \to F$ be a linear map. Show that the following are equivalent:

$$
\begin{array}{rl}
(i) & l \text{ is continuous (w.r. to the norm topologies)} \\
(ii) & l \text{ is continuous at } x = 0. \\
(iii) & \exists K \in \mathbf{R}^+ \ , \ \forall x \in E \ , \ \|l(x)\| \leq K\|x\| \\
(iv) & \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\} < +\infty
\end{array}
$$

Definition 127 *Let* E*,* F *be* $\mathbf{K}$*-normed spaces. The* $\mathbf{K}$*-vector space of all* **continuous linear maps** $l : E \to F$ *is denoted* $\mathcal{L}_{\mathbf{K}}(E, F)$*.*

EXERCISE 4. Show that $\mathcal{L}_{\mathbf{K}}(E, F)$ is indeed a $\mathbf{K}$-vector space.

EXERCISE 5. Let E, F be $\mathbf{K}$-normed spaces. Given $l \in \mathcal{L}_{\mathbf{K}}(E, F)$, let:

$$\|l\| \stackrel{\triangle}{=} \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\} < +\infty$$

1. Show that:
$$\|l\| = \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| \leq 1\}$$
2. Show that:
$$\|l\| = \sup\left\{\frac{\|l(x)\|}{\|x\|} : \ x \in E \ , \ x \neq 0\right\}$$

3. Show that $\|l(x)\| \leq \|l\|.\|x\|$, for all $x \in E$.

4. Show that $\|l\|$ is the smallest $K \in \mathbf{R}^+$, such that:

$$\forall x \in E \ , \ \|l(x)\| \leq K\|x\|$$

5. Show that $l \to \|l\|$ is a norm on $\mathcal{L}_{\mathbf{K}}(E,F)$.

6. Show that $(\mathcal{L}_{\mathbf{K}}(E,F), \|\cdot\|)$ is a $\mathbf{K}$-normed space.

Definition 128 *Let E, F be $\mathbf{R}$-normed spaces and U be an open subset of E. We say that a map $\phi : U \to F$ is* **differentiable** *at some $a \in U$, if and only if there exists $l \in \mathcal{L}_{\mathbf{R}}(E,F)$ such that, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $h \in E$:*

$$\|h\| \leq \delta \ \Rightarrow \ a+h \in U \ \textit{and} \ \|\phi(a+h) - \phi(a) - l(h)\| \leq \epsilon\|h\|$$

Exercise 6. Let E, F be two $\mathbf{R}$-normed spaces, and U be open in E. Let $\phi : U \to F$ be a map and $a \in U$.

1. Suppose that $\phi : U \to F$ is differentiable at $a \in U$, and that $l_1, l_2 \in \mathcal{L}_{\mathbf{R}}(E,F)$ satisfy the requirement of definition (128). Show that for all $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\forall h \in E \ , \ \|h\| \leq \delta \ \Rightarrow \ \|l_1(h) - l_2(h)\| \leq \epsilon\|h\|$$

2. Conclude that $\|l_1 - l_2\| = 0$ and finally that $l_1 = l_2$.

Definition 129 *Let E, F be $\mathbf{R}$-normed spaces and U be an open subset of E. Let $\phi : U \to F$ be a map and $a \in U$. If ϕ is differentiable at a, we call* **differential** *of ϕ at a, the unique element of $\mathcal{L}_{\mathbf{R}}(E,F)$, denoted $d\phi(a)$, satisfying the requirement of definition (128). If ϕ is differentiable at all points of U, the map $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E,F)$ is also called the differential of ϕ.*

Definition 130 *Let E, F be $\mathbf{R}$-normed spaces and U be an open subset of E. A map $\phi : U \to F$ is said to be of* **class** *C^1, if and only if $d\phi(a)$ exists for all $a \in U$, and the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E,F)$ is a continuous map.*

Exercise 7. Let E, F be two $\mathbf{R}$-normed spaces and U be open in E. Let $\phi : U \to F$ be a map, and $a \in U$.

1. Show that ϕ differentiable at $a \Rightarrow \phi$ continuous at a.

2. If ϕ is of class C^1, explain with respect to which topologies the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E,F)$ is said to be continuous.

3. Show that if ϕ is of class C^1, then ϕ is continuous.

4. Suppose that $E = \mathbf{R}$. Show that for all $a \in U$, ϕ is differentiable at $a \in U$, if and only if the derivative:

$$\phi'(a) \stackrel{\triangle}{=} \lim_{t\neq 0, t\to 0} \frac{\phi(a+t)-\phi(a)}{t}$$

exists in F, in which case $d\phi(a) \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}, F)$ is given by:

$$\forall t \in \mathbf{R}\ ,\ d\phi(a)(t) = t.\phi'(a)$$

In particular, $\phi'(a) = d\phi(a)(1)$.

EXERCISE 8. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. We assume that ϕ is differentiable at $a \in U$, and we put:

$$l_1 \stackrel{\triangle}{=} d\phi(a) \in \mathcal{L}_{\mathbf{R}}(E, F)$$

We assume that ψ is differentiable at $\phi(a) \in V$, and we put:

$$l_2 \stackrel{\triangle}{=} d\psi(\phi(a)) \in \mathcal{L}_{\mathbf{R}}(F, G)$$

1. Explain why $\psi \circ \phi : U \to G$ is a well-defined map.

2. Given $\epsilon > 0$, show the existence of $\eta > 0$ such that:

$$\eta(\eta + \|l_1\| + \|l_2\|) \leq \epsilon$$

3. Show the existence of $\delta_2 > 0$ such that for all $h_2 \in F$ with $\|h_2\| \leq \delta_2$, we have $\phi(a) + h_2 \in V$ and:

$$\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2(h_2)\| \leq \eta\|h_2\|$$

4. Show that if $h_2 \in F$ and $\|h_2\| \leq \delta_2$, then for all $h \in E$, we have:

$$\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| \leq \eta\|h_2\| + \|l_2\|.\|h_2 - l_1(h)\|$$

5. Show the existence of $\delta > 0$ such that for all $h \in E$ with $\|h\| \leq \delta$, we have $a + h \in U$ and $\|\phi(a+h) - \phi(a) - l_1(h)\| \leq \eta\|h\|$, together with $\|\phi(a+h) - \phi(a)\| \leq \delta_2$.

6. Show that if $h \in E$ is such that $\|h\| \leq \delta$, then $a + h \in U$ and:

$$\begin{aligned}\|\psi \circ \phi(a+h) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| &\leq \eta\|\phi(a+h) - \phi(a)\| + \eta\|l_2\|.\|h\| \\ &\leq \eta(\eta + \|l_1\| + \|l_2\|)\|h\| \\ &\leq \epsilon\|h\|\end{aligned}$$

7. Show that $l_2 \circ l_1 \in \mathcal{L}_{\mathbf{R}}(E, G)$

8. Conclude with the following:

Theorem 110 *Let E, F, G be three $\mathbf{R}$-normed spaces, U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. Let $a \in U$. Then, if ϕ is differentiable at $a \in U$, and ψ is differentiable at $\phi(a) \in V$, then $\psi \circ \phi$ is differentiable at $a \in U$, and furthermore:*

$$d(\psi \circ \phi)(a) = d\psi(\phi(a)) \circ d\phi(a)$$

Exercise 9. Let $(\Omega', \mathcal{T}')$ and $(\Omega, \mathcal{T})$ be two topological spaces, and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a set of subsets of Ω generating the topology $\mathcal{T}$, i.e. such that $\mathcal{T} = \mathcal{T}(\mathcal{A})$ as defined in (55). Let $f : \Omega' \to \Omega$ be a map, and define:

$$\mathcal{U} \stackrel{\triangle}{=} \{A \subseteq \Omega \ : \ f^{-1}(A) \in \mathcal{T}'\}$$

1. Show that $\mathcal{U}$ is a topology on Ω.
2. Show that $f : (\Omega', \mathcal{T}') \to (\Omega, \mathcal{T})$ is continuous, if and only if:

$$\forall A \in \mathcal{A} \ , \ f^{-1}(A) \in \mathcal{T}'$$

Exercise 10. Let $(\Omega', \mathcal{T}')$ be a topological space, and $(\Omega_i, \mathcal{T}_i)_{i \in I}$ be a family of topological spaces, indexed by a non-empty set I. Let Ω be the Cartesian product $\Omega = \Pi_{i \in I} \Omega_i$ and $\mathcal{T} = \odot_{i \in I} \mathcal{T}_i$ be the product topology on Ω. Let $(f_i)_{i \in I}$ be a family of maps $f_i : \Omega' \to \Omega_i$ indexed by I, and let $f : \Omega' \to \Omega$ be the map defined by $f(\omega) = (f_i(\omega))_{i \in I}$ for all $\omega \in \Omega'$. Let $p_i : \Omega \to \Omega_i$ be the canonical projection mapping.

1. Show that $p_i : (\Omega, \mathcal{T}) \to (\Omega_i, \mathcal{T}_i)$ is continuous for all $i \in I$.
2. Show that $f : (\Omega', \mathcal{T}') \to (\Omega, \mathcal{T})$ is continuous, if and only if each coordinate mapping $f_i : (\Omega', \mathcal{T}') \to (\Omega_i, \mathcal{T}_i)$ is continuous.

Exercise 11. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class C^1 such that $\phi(U) \subseteq V$.

1. For all $(l_1, l_2) \in \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$, we define:

$$\begin{aligned} N_1(l_1, l_2) &\stackrel{\triangle}{=} \|l_1\| + \|l_2\| \\ N_2(l_1, l_2) &\stackrel{\triangle}{=} \sqrt{\|l_1\|^2 + \|l_2\|^2} \\ N_\infty(l_1, l_2) &\stackrel{\triangle}{=} \max(\|l_1\|, \|l_2\|) \end{aligned}$$

   Show that N_1, N_2, N_∞ are all norms on $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$.

2. Show they induce the product topology on $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$.
3. We define the map $H : \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F) \to \mathcal{L}_{\mathbf{R}}(E, G)$ by:

$$\forall (l_1, l_2) \in \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F) \ , \ H(l_1, l_2) \stackrel{\triangle}{=} l_1 \circ l_2$$

   Show that $\|H(l_1, l_2)\| \leq \|l_1\|.\|l_2\|$, for all l_1, l_2.

4. Show that H is continuous.

5. We define $K : U \to \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$ by:

$$\forall a \in U \ , \ K(a) \stackrel{\triangle}{=} (d\psi(\phi(a)), d\phi(a))$$

Show that K is continuous.

6. Show that $\psi \circ \phi$ is differentiable on U.

7. Show that $d(\psi \circ \phi) = H \circ K$.

8. Conclude with the following:

Theorem 111 *Let E, F, G be three $\mathbf{R}$-normed spaces, U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class C^1 such that $\phi(U) \subseteq V$. Then, $\psi \circ \phi : U \to G$ is of class C^1.*

Exercise 12. Let E be an $\mathbf{R}$-normed space. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a,b] \to E$ and $g : [a,b] \to \mathbf{R}$ be two continuous maps which are differentiable at every point of $]a,b[$. We assume that:

$$\forall t \in]a,b[\ , \ \|f'(t)\| \leq g'(t)$$

1. Given $\epsilon > 0$, we define $\phi_\epsilon : [a,b] \to \mathbf{R}$ by:

$$\phi_\epsilon(t) \stackrel{\triangle}{=} \|f(t) - f(a)\| - g(t) + g(a) - \epsilon(t-a)$$

for all $t \in [a,b]$. Show that ϕ_ϵ is continuous.

2. Define $E_\epsilon = \{t \in [a,b] : \phi_\epsilon(t) \leq \epsilon\}$, and $c = \sup E_\epsilon$. Show that:

$$c \in [a,b] \text{ and } \phi_\epsilon(c) \leq \epsilon$$

3. Show the existence of $h > 0$, such that:

$$\forall t \in [a, a+h[\cap[a,b] \ , \ \phi_\epsilon(t) \leq \epsilon$$

4. Show that $c \in]a,b]$.

5. Suppose that $c \in]a,b[$. Show the existence of $t_0 \in]c,b]$ such that:

$$\left\| \frac{f(t_0) - f(c)}{t_0 - c} \right\| \leq \|f'(c)\| + \epsilon/2 \text{ and } g'(c) \leq \frac{g(t_0) - g(c)}{t_0 - c} + \epsilon/2$$

6. Show that $\|f(t_0) - f(c)\| \leq g(t_0) - g(c) + \epsilon(t_0 - c)$.

7. Show that $\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c-a) + \epsilon$.

8. Show that $\|f(t_0) - f(a)\| \leq g(t_0) - g(a) + \epsilon(t_0 - a) + \epsilon$.

9. Show that $c \in]a,b[$ leads to a contradiction.

10. Show that $\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b-a) + \epsilon$.
11. Conclude with the following:

Theorem 112 *Let E be an $\mathbf{R}$-normed space. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a,b] \to E$ and $g : [a,b] \to \mathbf{R}$ be two continuous maps which are differentiable at every point of $]a,b[$, and such that:*

$$\forall t \in]a,b[\ ,\ \|f'(t)\| \leq g'(t)$$

Then:

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

Definition 131 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. For all $i = 1, \ldots, n$, we say that ϕ has an ith* **partial derivative** *at $a \in U$, if and only if the limit:*

$$\frac{\partial \phi}{\partial x_i}(a) \stackrel{\triangle}{=} \lim_{h \neq 0, h \to 0} \frac{\phi(a + he_i) - \phi(a)}{h}$$

exists in E, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{R}^n$.

Exercise 13. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space.

1. Suppose ϕ is differentiable at $a \in U$. Show that for all $i \in \mathbf{N}_n$:

$$\lim_{h \neq 0, h \to 0} \frac{1}{\|he_i\|} \|\phi(a + he_i) - \phi(a) - d\phi(a)(he_i)\| = 0$$

2. Show that for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}(a)$ exists, and:

$$\frac{\partial \phi}{\partial x_i}(a) = d\phi(a)(e_i)$$

3. Conclude with the following:

Theorem 113 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. Then, if ϕ is differentiable at $a \in U$, for all $i = 1, \ldots, n$, $\frac{\partial \phi}{\partial x_i}(a)$ exists and we have:*

$$\forall h \stackrel{\triangle}{=} (h_1, \ldots, h_n) \in \mathbf{R}^n\ ,\ d\phi(a)(h) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i$$

Exercise 14. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space.

1. Show that if ϕ is differentiable at $a, b \in U$, then for all $i \in \mathbf{N}_n$:

$$\left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \| d\phi(b) - d\phi(a) \|$$

2. Conclude that if ϕ is of class C^1 on U, then $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U, for all $i \in \mathbf{N}_n$.

Exercise 15. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. We assume that $\frac{\partial \phi}{\partial x_i}$ exists on U, and is continuous at $a \in U$, for all $i \in \mathbf{N}_n$. We define $l : \mathbf{R}^n \to E$:

$$\forall h \stackrel{\triangle}{=} (h_1, \ldots, h_n) \in \mathbf{R}^n \ , \ l(h) \stackrel{\triangle}{=} \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i$$

1. Show that $l \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$.

2. Given $\epsilon > 0$, show the existence of $\eta > 0$ such that for all $h \in \mathbf{R}^n$ with $\|h\| < \eta$, we have $a + h \in U$, and:

$$\forall i = 1, \ldots, n \ , \ \left\| \frac{\partial \phi}{\partial x_i}(a+h) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \epsilon$$

3. Let $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$ be such that $\|h\| < \eta$. $(e_1, \ldots, e_n)$ being the canonical basis of $\mathbf{R}^n$, we define $k_0 = a$ and for $i \in \mathbf{N}_n$:

$$k_i \stackrel{\triangle}{=} a + \sum_{j=1}^{i} h_j e_j$$

Show that $k_0, \ldots, k_n \in U$, and that we have:

$$\phi(a+h) - \phi(a) - l(h) = \sum_{i=1}^{n} \left(\phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right)$$

4. Let $i \in \mathbf{N}_n$. Assume that $h_i > 0$. We define $f_i : [0, h_i] \to E$ by:

$$\forall t \in [0, h_i] \ , \ f_i(t) \stackrel{\triangle}{=} \phi(k_{i-1} + t e_i) - \phi(k_{i-1}) - t \frac{\partial \phi}{\partial x_i}(a)$$

Show f_i is well-defined, $f_i'(t)$ exists for all $t \in [0, h_i]$, and:

$$\forall t \in [0, h_i] \ , \ f_i'(t) = \frac{\partial \phi}{\partial x_i}(k_{i-1} + t e_i) - \frac{\partial \phi}{\partial x_i}(a)$$

5. Show f_i is continuous on $[0, h_i]$, differentiable on $]0, h_i[$, with:

$$\forall t \in]0, h_i[\ , \ \|f_i'(t)\| \leq \epsilon$$

6. Show that:

$$\left\| \phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \epsilon |h_i|$$

7. Show that the previous inequality still holds if $h_i \leq 0$.

8. Conclude that for all $h \in \mathbf{R}^n$ with $\|h\| < \eta$, we have:

$$\|\phi(a+h) - \phi(a) - l(h)\| \leq \epsilon\sqrt{n}\|h\|$$

9. Prove the following:

Theorem 114 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. If, for all $i \in \mathbf{N}_n$ $\frac{\partial \phi}{\partial x_i}$ exists on U and is continuous at $a \in U$, then ϕ is differentiable at $a \in U$.*

Exercise 16. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. We assume that for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U.

1. Show that ϕ is differentiable on U.

2. Show that for all $a, b \in U$ and $h \in \mathbf{R}^n$:

$$\|(d\phi(b) - d\phi(a))(h)\| \leq \left(\sum_{i=1}^{n} \left\|\frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a)\right\|^2\right)^{1/2} \|h\|$$

3. Show that for all $a, b \in U$:

$$\|d\phi(b) - d\phi(a)\| \leq \left(\sum_{i=1}^{n} \left\|\frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a)\right\|^2\right)^{1/2}$$

4. Show that $d\phi : U \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ is continuous.

5. Prove the following:

Theorem 115 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. Then, ϕ is of class C^1 on U, if and only if for all $i = 1, \ldots, n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U.*

Exercise 17. Let E, F be two $\mathbf{R}$-normed spaces and $l \in \mathcal{L}_{\mathbf{R}}(E, F)$. Let U be open in E and $l_{|U}$ be the restriction of l to U. Show that $l_{|U}$ is of class C^1 on U, and that we have:

$$\forall x \in U \ , \ d(l_{|U})(x) = l$$

Exercise 18. Let $E_1, \ldots, E_n$, $n \geq 1$, be n $\mathbf{K}$-normed spaces. Let $E = E_1 \times \ldots \times E_n$. Let $p \in [1, +\infty[$, and for all $x = (x_1, \ldots, x_n) \in E$:

$$\|x\|_p \stackrel{\triangle}{=} \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}$$

$$\|x\|_\infty \stackrel{\triangle}{=} \max_{i=1,\ldots,n} \|x_i\|$$

1. Using theorem (43), show that $\|.\|_p$ and $\|.\|_\infty$ are norms on E.
2. Show $\|.\|_p$ and $\|.\|_\infty$ induce the product topology on E.
3. Conclude that E is also an $\mathbf{K}$-normed space, and that the norm topology on E is exactly the product topology on E.

Exercise 19. Let E and F be two $\mathbf{R}$-normed spaces. Let U be open in E and $\phi, \psi : U \to F$ be two maps. We assume that both ϕ and ψ are differentiable at $a \in U$. Given $\alpha \in \mathbf{R}$, show that $\phi + \alpha\psi$ is differentiable at $a \in U$ and:

$$d(\phi + \alpha\psi)(a) = d\phi(a) + \alpha d\psi(a)$$

Exercise 20. Let E and F be $\mathbf{K}$-normed spaces. Let U be open in E and $\phi : U \to F$ be a map. Let N_E and N_F be two norms on E and F, inducing the same topologies as the norm topologies of E and F respectively. For all $l \in \mathcal{L}_{\mathbf{K}}(E, F)$, we define:

$$N(l) = \sup\{N_F(l(x)) : x \in E, N_E(x) = 1\}$$

1. Explain why the set $\mathcal{L}_{\mathbf{K}}(E, F)$ is unambiguously defined.
2. Show that the identity $id_E : (E, \|\cdot\|) \to (E, N_E)$ is continuous
3. Show the existence of $m_E, M_E > 0$ such that:
$$\forall x \in E \ , \ m_E\|x\| \leq N_E(x) \leq M_E\|x\|$$
4. Show the existence of $m, M > 0$ such that:
$$\forall l \in \mathcal{L}_{\mathbf{K}}(E, F) \ , \ m\|l\| \leq N(l) \leq M\|l\|$$
5. Show that $\|\cdot\|$ and N induce the same topology on $\mathcal{L}_{\mathbf{K}}(E, F)$.
6. Show that if $\mathbf{K} = \mathbf{R}$ and ϕ is differentiable at $a \in U$, then ϕ is also differentiable at a with respect to the norms N_E and N_F, and the differential $d\phi(a)$ is unchanged
7. Show that if $\mathbf{K} = \mathbf{R}$ and ϕ is of class C^1 on U, then ϕ is also of class C^1 on U with respect to the norms N_E and N_F.

Exercise 21. Let E and $F_1, \ldots, F_p$, $p \geq 1$, be $p + 1$ $\mathbf{R}$-normed spaces. Let U be open in E and $F = F_1 \times \ldots \times F_p$. Let $\phi : U \to F$ be a map. For all $i \in \mathbf{N}_p$, we denote $p_i : F \to F_i$ the canonical projection and we define $\phi_i = p_i \circ \phi$. We also consider $u_i : F_i \to F$, defined as:

$$\forall x_i \in F_i \ , \ u_i(x_i) \stackrel{\triangle}{=} (0, \ldots, \overbrace{x_i}^{i}, \ldots, 0)$$

1. Given $i \in \mathbf{N}_p$, show that $p_i \in \mathcal{L}_{\mathbf{R}}(F, F_i)$.
2. Given $i \in \mathbf{N}_p$, show that $u_i \in \mathcal{L}_{\mathbf{R}}(F_i, F)$ and $\phi = \sum_{i=1}^{p} u_i \circ \phi_i$.

3. Show that if ϕ is differentiable at $a \in U$, then for all $i \in \mathbf{N}_p$, $\phi_i : U \to F_i$ is differentiable at $a \in U$ and $d\phi_i(a) = p_i \circ d\phi(a)$.

4. Show that if ϕ_i is differentiable at $a \in U$ for all $i \in \mathbf{N}_p$, then ϕ is differentiable at $a \in U$ and:
$$d\phi(a) = \sum_{i=1}^{p} u_i \circ d\phi_i(a)$$

5. Suppose that ϕ is differentiable at $a, b \in U$. Let F be given the norm $\|\cdot\|_2$ of exercise (18). Show that for all $i \in \mathbf{N}_p$:
$$\|d\phi_i(b) - d\phi_i(a)\| \leq \|d\phi(b) - d\phi(a)\|$$

6. Show that:
$$\|d\phi(b) - d\phi(a)\| \leq \left(\sum_{i=1}^{p} \|d\phi_i(b) - d\phi_i(a)\|^2\right)^{1/2}$$

7. Show that ϕ is of class $C^1 \Leftrightarrow \phi_i$ is of class C^1 for all $i \in \mathbf{N}_p$.

8. Conclude with theorem (116)

Theorem 116 *Let $E, F_1, \ldots, F_p$, $(p \geq 1)$, be $p+1$ $\mathbf{R}$-normed spaces and U be open in E. Let F be the $\mathbf{R}$-normed space $F = F_1 \times \ldots \times F_p$ and $\phi = (\phi_1, \ldots, \phi_p) : U \to F$ be a map. Then, ϕ is differentiable at $a \in U$, if and only if $d\phi_i(a)$ exists for all $i \in \mathbf{N}_p$, in which case:*
$$\forall h \in E \ , \ d\phi(a)(h) = (d\phi_1(a)(h), \ldots, d\phi_p(a)(h))$$
Also, ϕ is of class C^1 on $U \Leftrightarrow \phi_i$ is of class C^1 on U, for all $i \in \mathbf{N}_p$.

Theorem 117 *Let $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbf{R}^n$ be a map, where $n \geq 1$ and U is open in $\mathbf{R}^n$. We assume that ϕ is differentiable at $a \in U$. Then, for all $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}(a)$ exists, and we have:*
$$d\phi(a) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(a) & \ldots & \frac{\partial \phi_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(a) & \ldots & \frac{\partial \phi_n}{\partial x_n}(a) \end{pmatrix}$$
Moreover, ϕ is of class C^1 on U, if and only if for all $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}$ exists and is continuous on U.

Exercise 22. Prove theorem (117)

Definition 132 *Let $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbf{R}^n$ be a map, where $n \geq 1$ and U is open in $\mathbf{R}^n$. We assume that ϕ is differentiable at $a \in U$. We call* **Jacobian** *of ϕ at a, denoted $J(\phi)(a)$, the determinant of the differential $d\phi(a)$ at a, i.e.*

$$J(\phi)(a) = \det \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(a) & \ldots & \frac{\partial \phi_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(a) & \ldots & \frac{\partial \phi_n}{\partial x_n}(a) \end{pmatrix}$$

Definition 133 *Let $n \geq 1$ and Ω, Ω' be open in $\mathbf{R}^n$. A bijection $\phi : \Omega \to \Omega'$ is called a* **C^1-diffeomorphism** *between Ω and Ω', if and only if $\phi : \Omega \to \mathbf{R}^n$ and $\phi^{-1} : \Omega' \to \mathbf{R}^n$ are both of class C^1.*

Exercise 23. Let Ω and Ω' be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism, $\psi = \phi^{-1}$, and I_n be the identity mapping of $\mathbf{R}^n$.

1. Explain why $J(\psi) : \Omega' \to \mathbf{R}$ and $J(\phi) : \Omega \to \mathbf{R}$ are continuous.
2. Show that $d\phi(\psi(x)) \circ d\psi(x) = I_n$, for all $x \in \Omega'$.
3. Show that $d\psi(\phi(x)) \circ d\phi(x) = I_n$, for all $x \in \Omega$.
4. Show that $J(\psi)(x) \neq 0$ for all $x \in \Omega'$.
5. Show that $J(\phi)(x) \neq 0$ for all $x \in \Omega$.
6. Show that $J(\psi) = 1/(J(\phi) \circ \psi)$ and $J(\phi) = 1/(J(\psi) \circ \phi)$.

Definition 134 *Let $n \geq 1$ and $\Omega \in \mathcal{B}(\mathbf{R}^n)$, be a Borel set in $\mathbf{R}^n$. We define the* **Lebesgue measure** *on Ω, denoted $dx_{|\Omega}$, as the restriction to $\mathcal{B}(\Omega)$ of the Lebesgue measure on $\mathbf{R}^n$, i.e the measure on $(\Omega, \mathcal{B}(\Omega))$ defined by:*

$$\forall B \in \mathcal{B}(\Omega)\ ,\ dx_{|\Omega}(B) \stackrel{\triangle}{=} dx(B)$$

Exercise 24. Show that $dx_{|\Omega}$ is a well-defined measure on $(\Omega, \mathcal{B}(\Omega))$.

Exercise 25. Let $n \geq 1$ and Ω, Ω' be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We assume that $d\psi(a) = I_n$, (identity mapping on $\mathbf{R}^n$), and given $\epsilon > 0$, we denote:

$$B(a, \epsilon) \stackrel{\triangle}{=} \{x \in \mathbf{R}^n : \ \|a - x\| < \epsilon\}$$

where $\|.\|$ is the usual norm in $\mathbf{R}^n$.

1. Why are $dx_{|\Omega'}$, $\phi(dx_{|\Omega})$ well-defined measures on $(\Omega', \mathcal{B}(\Omega'))$.
2. Show that for $\epsilon > 0$ sufficiently small, $B(a, \epsilon) \in \mathcal{B}(\Omega')$.

3. Show that it makes sense to investigate whether the limit:
$$\lim_{\epsilon\downarrow\downarrow 0}\frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))}$$
does exists in $\mathbf{R}$.

4. Given $r>0$, show the existence of $\epsilon_1>0$ such that for all $h\in\mathbf{R}^n$ with $\|h\|\leq\epsilon_1$, we have $a+h\in\Omega'$, and:
$$\|\psi(a+h)-\psi(a)-h\|\leq r\|h\|$$

5. Show for all $h\in\mathbf{R}^n$ with $\|h\|\leq\epsilon_1$, we have $a+h\in\Omega'$, and:
$$\|\psi(a+h)-\psi(a)\|\leq(1+r)\|h\|$$

6. Show that for all $\epsilon\in]0,\epsilon_1[$, we have $B(a,\epsilon)\subseteq\Omega'$, and:
$$\psi(B(a,\epsilon))\subseteq B(\psi(a),\epsilon(1+r))$$

7. Show that $d\phi(\psi(a))=I_n$.

8. Show the existence of $\epsilon_2>0$ such that for all $k\in\mathbf{R}^n$ with $\|k\|\leq\epsilon_2$, we have $\psi(a)+k\in\Omega$, and:
$$\|\phi(\psi(a)+k)-a-k\|\leq r\|k\|$$

9. Show for all $k\in\mathbf{R}^n$ with $\|k\|\leq\epsilon_2$, we have $\psi(a)+k\in\Omega$, and:
$$\|\phi(\psi(a)+k)-a\|\leq(1+r)\|k\|$$

10. Show for all $\epsilon\in]0,\epsilon_2(1+r)[$, we have $B(\psi(a),\frac{\epsilon}{1+r})\subseteq\Omega$, and:
$$B(\psi(a),\frac{\epsilon}{1+r})\subseteq\{\phi\in B(a,\epsilon)\}$$

11. Show that if $B(a,\epsilon)\subseteq\Omega'$, then $\psi(B(a,\epsilon))=\{\phi\in B(a,\epsilon)\}$.

12. Show if $0<\epsilon<\epsilon_0=\epsilon_1\wedge\epsilon_2(1+r)$, then $B(a,\epsilon)\subseteq\Omega'$, and:
$$B(\psi(a),\frac{\epsilon}{1+r})\subseteq\{\phi\in B(a,\epsilon)\}\subseteq B(\psi(a),\epsilon(1+r))$$

13. Show that for all $\epsilon\in]0,\epsilon_0[$:
$$\begin{array}{ll}(i) & dx(B(\psi(a),\frac{\epsilon}{1+r}))=(1+r)^{-n}dx_{|\Omega'}(B(a,\epsilon))\\ (ii) & dx(B(\psi(a),\epsilon(1+r)))=(1+r)^n dx_{|\Omega'}(B(a,\epsilon))\\ (iii) & dx(\{\phi\in B(a,\epsilon)\})=\phi(dx_{|\Omega})(B(a,\epsilon))\end{array}$$

14. Show that for all $\epsilon\in]0,\epsilon_0[$, $B(a,\epsilon)\subseteq\Omega'$, and:
$$(1+r)^{-n}\leq\frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))}\leq(1+r)^n$$

15. Conclude that:
$$\lim_{\epsilon\downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1$$

Exercise 26. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We put $A = d\psi(a)$.

1. Show that $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear bijection.

2. Define $\Omega'' = A^{-1}(\Omega)$. Show that this definition does not depend on whether $A^{-1}(\Omega)$ is viewed as inverse , or direct image.

3. Show that Ω'' is an open subset of $\mathbf{R}^n$.

4. We define $\tilde{\phi} : \Omega'' \to \Omega'$ by $\tilde{\phi}(x) = \phi \circ A(x)$. Show that $\tilde{\phi}$ is a C^1-diffeomorphism with $\tilde{\psi} = \tilde{\phi}^{-1} = A^{-1} \circ \psi$.

5. Show that $d\tilde{\psi}(a) = I_n$.

6. Show that:
$$\lim_{\epsilon\downarrow\downarrow 0} \frac{\tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1$$

7. Let $\epsilon > 0$ with $B(a,\epsilon) \subseteq \Omega'$. Justify each of the following steps:
$$\begin{aligned}
\tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon)) &= dx_{|\Omega''}(\{\tilde{\phi} \in B(a,\epsilon)\}) \\
&= dx(\{\tilde{\phi} \in B(a,\epsilon)\}) \\
&= dx(\{x \in \Omega'' :\ \phi \circ A(x) \in B(a,\epsilon)\}) \\
&= dx(\{x \in \Omega'' : A(x) \in \phi^{-1}(B(a,\epsilon))\}) \\
&= dx(\{x \in \mathbf{R}^n : A(x) \in \phi^{-1}(B(a,\epsilon))\}) \\
&= A(dx)(\{\phi \in B(a,\epsilon)\}) \\
&= |\det A|^{-1} dx(\{\phi \in B(a,\epsilon)\}) \\
&= |\det A|^{-1} dx_{|\Omega}(\{\phi \in B(a,\epsilon)\}) \\
&= |\det A|^{-1} \phi(dx_{|\Omega})(B(a,\epsilon))
\end{aligned}$$

8. Show that:
$$\lim_{\epsilon\downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |\det A|$$

9. Conclude with the following:

Theorem 118 *Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Then, for all $a \in \Omega'$, we have:*
$$\lim_{\epsilon\downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |J(\psi)(a)|$$
where $J(\psi)(a)$ is the Jacobian of ψ at a, $B(a,\epsilon)$ is the open ball in $\mathbf{R}^n$, and $dx_{|\Omega}$, $dx_{|\Omega'}$ are the Lebesgue measures on Ω and Ω' respectively.

EXERCISE 27. Let $n \geq 1$ and Ω, Ω' be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$.

1. Let $K \subseteq \Omega'$ be a non-empty compact subset of Ω' such that $dx_{|\Omega'}(K) = 0$. Given $\epsilon > 0$, show the existence of V open in Ω', such that $K \subseteq V \subseteq \Omega'$, and $dx_{|\Omega'}(V) \leq \epsilon$.
2. Explain why V is also open in $\mathbf{R}^n$.
3. Show that $M \stackrel{\triangle}{=} \sup_{x \in K} \|d\psi(x)\| \in \mathbf{R}^+$.
4. For all $x \in K$, show there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subseteq V$, and for all $h \in \mathbf{R}^n$ with $\|h\| \leq 3\epsilon_x$, we have $x + h \in \Omega'$, and:
$$\|\psi(x+h) - \psi(x)\| \leq (M+1)\|h\|$$
5. Show that for all $x \in K$, $B(x, 3\epsilon_x) \subseteq \Omega'$, and:
$$\psi(B(x, 3\epsilon_x)) \subseteq B(\psi(x), 3(M+1)\epsilon_x)$$
6. Show that $\psi(B(x, 3\epsilon_x)) = \{\phi \in B(x, 3\epsilon_x)\}$, for all $x \in K$.
7. Show the existence of $\{x_1, \ldots, x_p\} \subseteq K$, $(p \geq 1)$, such that:
$$K \subseteq B(x_1, \epsilon_{x_1}) \cup \ldots \cup B(x_p, \epsilon_{x_p})$$
8. Show the existence of $S \subseteq \{1, \ldots, p\}$ such that the $B(x_i, \epsilon_{x_i})$'s are pairwise disjoint for $i \in S$, and:
$$K \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_{x_i})$$
9. Show that $\{\phi \in K\} \subseteq \cup_{i \in S} B(\psi(x_i), 3(M+1)\epsilon_{x_i})$.
10. Show that $\phi(dx_{|\Omega})(K) \leq \sum_{i \in S} 3^n (M+1)^n dx(B(x_i, \epsilon_{x_i}))$.
11. Show that $\phi(dx_{|\Omega})(K) \leq 3^n (M+1)^n dx(V)$.
12. Show that $\phi(dx_{|\Omega})(K) \leq 3^n (M+1)^n \epsilon$.
13. Conclude that $\phi(dx_{|\Omega})(K) = 0$.
14. Show that $\phi(dx_{|\Omega})$ is a locally finite measure on $(\Omega', \mathcal{B}(\Omega'))$.
15. Show that for all $B \in \mathcal{B}(\Omega')$:
$$\phi(dx_{|\Omega})(B) = \sup\{\phi(dx_{|\Omega})(K) : \ K \subseteq B \ , \ K \text{ compact } \}$$
16. Show that for all $B \in \mathcal{B}(\Omega')$:
$$dx_{|\Omega'}(B) = 0 \ \Rightarrow \ \phi(dx_{|\Omega})(B) = 0$$
17. Conclude with the following:

Theorem 119 *Let $n \geq 1$, Ω, Ω' be open in $\mathbf{R}^n$, and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Then, the image measure $\phi(dx_{|\Omega})$, by ϕ of the Lebesgue measure on Ω, is absolutely continuous with respect to $dx_{|\Omega'}$, the Lebesgue measure on Ω', i.e.:*

$$\phi(dx_{|\Omega}) << dx_{|\Omega'}$$

EXERCISE 28. Let $n \geq 1$ and Ω, Ω' be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$.

1. Explain why there exists a sequence $(V_p)_{p\geq 1}$ of open sets in Ω', such that $V_p \uparrow \Omega'$ and for all $p \geq 1$, the closure of V_p in Ω', i.e. $\bar{V}_p^{\Omega'}$, is compact.
2. Show that each V_p is also open in $\mathbf{R}^n$, and that $\bar{V}_p^{\Omega'} = \bar{V}_p$.
3. Show that $\phi(dx_{|\Omega})(V_p) < +\infty$, for all $p \geq 1$.
4. Show that $dx_{|\Omega'}$ and $\phi(dx_{|\Omega})$ are two σ-finite measures on Ω'.
5. Show there is $h : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ measurable, with:
$$\forall B \in \mathcal{B}(\Omega') \ , \ \phi(dx_{|\Omega})(B) = \int_B h dx_{|\Omega'}$$
6. For all $p \geq 1$, we define $h_p = h1_{V_p}$, and we put:
$$\forall x \in \mathbf{R}^n \ , \ \tilde{h}_p(x) \stackrel{\triangle}{=} \begin{cases} h_p(x) & \text{if} \quad x \in \Omega' \\ 0 & \text{if} \quad x \notin \Omega' \end{cases}$$
Show that:
$$\int_{\mathbf{R}^n} \tilde{h}_p dx = \int_{\Omega'} h_p dx_{|\Omega'} = \phi(dx_{|\Omega})(V_p) < +\infty$$
and conclude that $\tilde{h}_p \in L^1_{\mathbf{R}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$.
7. Show the existence of some $N \in \mathcal{B}(\mathbf{R}^n)$, such that $dx(N) = 0$ and for all $x \in N^c$ and $p \geq 1$, we have:
$$\tilde{h}_p(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} \tilde{h}_p dx$$
8. Put $N' = N \cap \Omega'$. Show that $N' \in \mathcal{B}(\Omega')$ and $dx_{|\Omega'}(N') = 0$.
9. Let $x \in \Omega'$ and $p \geq 1$ be such that $x \in V_p$. Show that if $\epsilon > 0$ is such that $B(x,\epsilon) \subseteq V_p$, then $dx(B(x,\epsilon)) = dx_{|\Omega'}(B(x,\epsilon))$, and:
$$\int_{B(x,\epsilon)} \tilde{h}_p dx = \int_{\mathbf{R}^n} 1_{B(x,\epsilon)} \tilde{h}_p dx = \int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'}$$
10. Show that:
$$\int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'} = \int_{\Omega'} 1_{B(x,\epsilon)} h dx_{|\Omega'} = \phi(dx_{|\Omega})(B(x,\epsilon))$$

11. Show that for all $x \in \Omega' \setminus N'$, we have:

$$h(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(x,\epsilon))}{dx_{|\Omega'}(B(x,\epsilon))}$$

12. Show that $h = |J(\psi)|\ dx_{|\Omega'}$-a.s. and conclude with the following:

Theorem 120 *Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Then, the image measure by ϕ of the Lebesgue measure on Ω, is equal to the measure on $(\Omega', \mathcal{B}(\Omega'))$ with density $|J(\psi)|$ with respect to the Lebesgue measure on Ω', i.e.:*

$$\phi(dx_{|\Omega}) = \int |J(\psi)| dx_{|\Omega'}$$

Exercise 29. Prove the following:

Theorem 121 (Jacobian Formula 1) *Let $n \geq 1$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism where $\Omega,\ \Omega'$ are open in $\mathbf{R}^n$. Let $\psi = \phi^{-1}$. Then, for all $f : (\Omega', \mathcal{B}(\Omega')) \to [0, +\infty]$ non-negative and measurable:*

$$\int_\Omega f \circ \phi \, dx_{|\Omega} = \int_{\Omega'} f |J(\psi)| dx_{|\Omega'}$$

and:

$$\int_\Omega (f \circ \phi)|J(\phi)| dx_{|\Omega} = \int_{\Omega'} f dx_{|\Omega'}$$

Exercise 30. Prove the following:

Theorem 122 (Jacobian Formula 2) *Let $n \geq 1$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism where $\Omega,\ \Omega'$ are open in $\mathbf{R}^n$. Let $\psi = \phi^{-1}$. Then, for all measurable map $f : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have the equivalence:*

$$f \circ \phi \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \ \Leftrightarrow \ f|J(\psi)| \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

in which case:

$$\int_\Omega f \circ \phi \, dx_{|\Omega} = \int_{\Omega'} f |J(\psi)| dx_{|\Omega'}$$

and, furthermore:

$$(f \circ \phi)|J(\phi)| \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \ \Leftrightarrow \ f \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

in which case:

$$\int_\Omega (f \circ \phi)|J(\phi)| dx_{|\Omega} = \int_{\Omega'} f dx_{|\Omega'}$$

Exercise 31. Let $f : \mathbf{R}^2 \to [0, +\infty]$, with $f(x, y) = \exp(-(x^2 + y^2)/2)$.

1. Show that:
$$\int_{\mathbf{R}^2} f(x,y)dxdy = \left(\int_{-\infty}^{+\infty} e^{-u^2/2}du\right)^2$$

2. Define:
$$\Delta_1 \stackrel{\triangle}{=} \{(x,y)\in\mathbf{R}^2 :\ x>0\ ,\ y>0\}$$
$$\Delta_2 \stackrel{\triangle}{=} \{(x,y)\in\mathbf{R}^2 :\ x<0\ ,\ y>0\}$$
and let Δ_3 and Δ_4 be the other two open quarters of $\mathbf{R}^2$. Show:
$$\int_{\mathbf{R}^2} f(x,y)dxdy = \int_{\Delta_1\cup\ldots\cup\Delta_4} f(x,y)dxdy$$

3. Let $Q:\mathbf{R}^2\to\mathbf{R}^2$ be defined by $Q(x,y)=(-x,y)$. Show that:
$$\int_{\Delta_1} f(x,y)dxdy = \int_{\Delta_2} f\circ Q^{-1}(x,y)dxdy$$

4. Show that:
$$\int_{\mathbf{R}^2} f(x,y)dxdy = 4\int_{\Delta_1} f(x,y)dxdy$$

5. Let $D_1 =]0,+\infty[\times]0,\pi/2[\subseteq\mathbf{R}^2$, and define $\phi : D_1\to\Delta_1$ by:
$$\forall(r,\theta)\in D_1\ ,\ \phi(r,\theta)\stackrel{\triangle}{=}(r\cos\theta, r\sin\theta)$$
Show that ϕ is a bijection and that $\psi=\phi^{-1}$ is given by:
$$\forall(x,y)\in\Delta_1\ ,\ \psi(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x))$$

6. Show that ϕ is a C^1-diffeomorphism, with:
$$\forall(r,\theta)\in D_1\ ,\ d\phi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$
and:
$$\forall(x,y)\in\Delta_1\ ,\ d\psi(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

7. Show that $J(\phi)(r,\theta) = r$, for all $(r,\theta)\in D_1$.

8. Show that $J(\psi)(x,y) = 1/(\sqrt{x^2+y^2})$, for all $(x,y)\in\Delta_1$.

9. Show that:
$$\int_{\Delta_1} f(x,y)dxdy = \frac{\pi}{2}$$

10. Prove the following:

Theorem 123 *We have:*
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-u^2/2}du = 1$$

Solutions to Exercises

Exercise 1.

1. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$. From definition (81), we have $\langle x,x\rangle \geq 0$ for all $x \in \mathcal{H}$. So $\|\cdot\| = \sqrt{\langle x,x\rangle}$ is a well-defined map $\|\cdot\| : \mathcal{H} \to \mathbf{R}^+$. From (v) of definition (81), $\langle x,x\rangle = 0$ is equivalent to $x = 0$. It follows that $\|x\| = 0$ is equivalent to $x = 0$. Let $x \in \mathcal{H}$ and $\alpha \in \mathbf{K}$. We have:

$$\begin{aligned}\|\alpha x\| &= \sqrt{\langle \alpha x, \alpha x\rangle} \\ &= \sqrt{\alpha\langle x, \alpha x\rangle} \\ &= \sqrt{\alpha\bar{\alpha}\langle x, x\rangle} \\ &= \sqrt{|\alpha|^2\langle x, x\rangle} \\ &= |\alpha|\sqrt{\langle x, x\rangle} = |\alpha|\cdot\|x\|\end{aligned}$$

Finally, given $x, y \in \mathcal{H}$, the fact that:

$$\|x+y\| \leq \|x\| + \|y\|$$

has been proved in exercise (17) of Tutorial 10. From definition (95), we conclude that $\|\cdot\|$ is a norm on $\mathcal{H}$.

2. $\mathcal{H}$ is a $\mathbf{K}$-vector space and $\|\cdot\|$ is a norm on $\mathcal{H}$. From definition (125), we conclude that $(\mathcal{H}, \|\cdot\|)$ is a $\mathbf{K}$-normed space.

Exercise 1

Exercise 2.

1. Let $(E, \|\cdot\|)$ be a $\mathbf{K}$-normed space. Let $d(x,y) = \|x-y\|$. Then $d : E\times E \to \mathbf{R}^+$ is a well-defined map. Furthermore, since $\|x\| = 0$ is equivalent to $x = 0$, $d(x,y) = 0$ is equivalent to $x = y$. Since $\|\alpha x\| = |\alpha|\cdot\|x\|$ for all $x \in E$ and $\alpha \in \mathbf{K}$, taking $\alpha = -1$ it is clear that $d(x,y) = d(y,x)$ for all $x, y \in E$. Finally, given $x, y, z \in E$ we have:

$$\begin{aligned}d(x,y) &= \|x-y\| \\ &= \|x-z+z-y\| \\ &\leq \|x-z\| + \|z-y\| \\ &= d(x,z) + d(z,y)\end{aligned}$$

We conclude from definition (28) that d is a metric on E.

2. Let $x, y \in E$. We have:

$$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$$

and consequently $\|x\| - \|y\| \leq \|x-y\|$. Similarly:

$$\begin{aligned}\|y\| - \|x\| &\leq \|y-x\| \\ &= \|x-y\|\end{aligned}$$

and we conclude that:

$$\begin{aligned} |\,\|x\| - \|y\|\,| &= \max(\|x\| - \|y\|, \|y\| - \|x\|) \\ &\leq \|x - y\| \end{aligned}$$

Exercise 2

Exercise 3. Let E, F be two $\mathbf{K}$-normed spaces and $l : E \to F$ be a linear map. We claim that the following are equivalent:

$$\begin{array}{rl} (i) & l \text{ is continuous (w.r. to the norm topologies)} \\ (ii) & l \text{ is continuous at } x = 0. \\ (iii) & \exists K \in \mathbf{R}^+ \ , \ \forall x \in E \ , \ \|l(x)\| \leq K\|x\| \\ (iv) & \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\} < +\infty \end{array}$$

Suppose l is continuous. In particular, it is continuous at $x = 0$. In case you have any doubt, although we have not defined it in these tutorials, recall that a map $l : E \to F$, where E and F are topological spaces, is said to be continuous at $x \in E$, if and only if for all V open subsets of F with $l(x) \in V$, there exists U open subset of E with $x \in U \subseteq l^{-1}(V)$. Now if $l : E \to F$ is continuous, for all V open subsets of F, $l^{-1}(V)$ is an open subset of E. If furthermore $l(x) \in V$, then $x \in l^{-1}(V)$ and taking $U = l^{-1}(V)$, we have found U open subset of E with $x \in U \subseteq l^{-1}(V)$. So l is continuous at x. We have proved that $(i) \Rightarrow (ii)$. Suppose that l is continuous at $x = 0$. Let $\epsilon > 0$ and $B(0, \epsilon)$ denote the open ball in F. Since l is linear, $l(0) = 0$ and $B(0, \epsilon)$ is therefore an open subset of F containing $l(0)$. Having assumed that l is continuous at $x = 0$, there exists U open subset of E such that $0 \in U \subseteq l^{-1}(B(0, \epsilon))$. The topology on E being induced by the metric $d(x, y) = \|x-y\|$, there exists $\eta > 0$ such that $B(0, \eta) \subseteq U$, where $B(0, \eta)$ denotes the open ball in E. From $B(0, \eta) \subseteq U \subseteq l^{-1}(B(0, \epsilon))$ we see that for all $x \in E$:

$$\|x\| < \eta \ \Rightarrow \ \|l(x)\| < \epsilon$$

Suppose $x \neq 0$. Then $\|x\| \neq 0$ and $y = \eta x/(2\|x\|)$ is a well-defined element of E with $\|y\| = \eta/2 < \eta$. Hence, we have:

$$\begin{aligned} \frac{\eta}{2\|x\|}\|l(x)\| &= \left\| \frac{\eta}{2\|x\|} l(x) \right\| \\ &= \left\| l\left(\frac{\eta x}{2\|x\|} \right) \right\| = \|l(y)\| < \epsilon \end{aligned}$$

and consequently, setting $K = 2\epsilon/\eta \in \mathbf{R}^+$ we obtain $\|l(x)\| < K\|x\|$. So in particular, we have proved that $\|l(x)\| \leq K\|x\|$ for all $x \neq 0$. This inequality being obviously still valid if $x = 0$, we have found $K \in \mathbf{R}^+$ such that:

$$\forall x \in E \ , \ \|l(x)\| \leq K\|x\| \tag{1}$$

This shows that $(ii) \Rightarrow (iii)$. Suppose now that there exists $K \in \mathbf{R}^+$ such that (1) holds, and define:

$$\alpha \stackrel{\triangle}{=} \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\}$$

Given $x \in E$ such that $\|x\| = 1$, we have $\|l(x)\| \leq K\|x\| = K$. So K is an upper-bound of all $\|l(x)\|$'s as x runs through the set of all $x \in E$ with $\|x\| = 1$. Since α is the smallest of such upper-bounds, we obtain $\alpha \leq K$ and in particular $\alpha < +\infty$. This shows that $(iii) \Rightarrow (iv)$. Finally, suppose that $\alpha < +\infty$. Let $x, y \in E$ be such that $x \neq y$. Then $\|x - y\| \neq 0$ and $z = (x - y)/\|x - y\|$ is a well-defined element of E with $\|z\| = 1$. It follows that:

$$\begin{aligned} \frac{\|l(x) - l(y)\|}{\|x - y\|} &= \left\| l\left(\frac{x - y}{\|x - y\|}\right)\right\| \\ &= \|l(z)\| \leq \alpha \end{aligned}$$

and consequently $\|l(x) - l(y)\| \leq \alpha\|x - y\|$. This is obviously still valid if $x = y$, and it is therefore true for all $x, y \in E$. Since $\alpha < +\infty$, this shows that l is continuous, and we have proved that $(iv) \Rightarrow (i)$. This completes our proof that $(i), (ii), (iii)$ and (iv) are equivalent.

Exercise 3

Exercise 4. To show that $\mathcal{L}_{\mathbf{K}}(E, F)$ is a $\mathbf{K}$-vector space, we only need to show that it is a $\mathbf{K}$-vector subspace of the set of all maps $f : E \to F$. In other words, given $u, v \in \mathcal{L}_{\mathbf{K}}(E, F)$ and $\alpha \in \mathbf{K}$, we need to show that $u + \alpha v \in \mathcal{L}_{\mathbf{K}}(E, F)$. This in turn amounts to showing that $u + \alpha v$ is a linear map, and that it is continuous. Since u and v are continuous, from exercise (3) there exists $K_1, K_2 \in \mathbf{R}^+$ such that $\|u(x)\| \leq K_1\|x\|$ and $\|v(x)\| \leq K_2\|x\|$ for all $x \in E$. Hence:

$$\begin{aligned} \|(u + \alpha v)(x)\| &= \|u(x) + \alpha v(x)\| \\ &\leq \|u(x)\| + |\alpha| \cdot \|v(x)\| \\ &\leq (K_1 + |\alpha| K_2)\|x\| \end{aligned}$$

and consequently from exercise (3), $u + \alpha v$ is continuous (provided it is linear, which we are about to prove). Moreover, given $x, y \in E$ and $\beta \in \mathbf{K}$, we have:

$$\begin{aligned} (u + \alpha v)(x + \beta y) &= u(x + \beta y) + \alpha v(x + \beta y) \\ &= u(x) + \beta u(y) + \alpha v(x) + \alpha\beta v(y) \\ &= u(x) + \alpha v(x) + \beta(u(y) + \alpha v(y)) \\ &= (u + \alpha v)(x) + \beta(u + \alpha v)(y) \end{aligned}$$

This shows that $u + \alpha v$ is linear, and we have proved that $\mathcal{L}_{\mathbf{K}}(E, F)$ is indeed a $\mathbf{K}$-vector space.

Exercise 4

Exercise 5.

1. Let E, F be $\mathbf{K}$-normed spaces. Given $l \in \mathcal{L}_{\mathbf{K}}(E, F)$, let:

$$\|l\| \stackrel{\triangle}{=} \sup\{\|l(x)\| : x \in E\ ,\ \|x\| = 1\}$$

Note that from exercise (3), we have $\|l\| < +\infty$. Define:

$$\alpha \stackrel{\triangle}{=} \sup\{\|l(x)\| : x \in E\ ,\ \|x\| \leq 1\}$$

We claim that $\alpha = \|l\|$. Let $x \in E$ be such that $\|x\| = 1$. Then in particular $\|x\| \leq 1$, and consequently $\|l(x)\| \leq \alpha$. It follows that α is an upper-bound of all $\|l(x)\|$'s as x runs through that set of all $x \in E$ with $\|x\| = 1$. Since $\|l\|$ is the smallest of such upper-bounds, we obtain $\|l\| \leq \alpha$. To show the reverse inequality, consider $x \in E$ with $\|x\| \leq 1$, and assume that $x \neq 0$. Then $\|x\| \neq 0$ and $y = x/\|x\|$ is a well-defined element of E with $\|y\| = 1$. Hence, we have:

$$\frac{\|l(x)\|}{\|x\|} = \left\| l\left(\frac{x}{\|x\|}\right)\right\| = \|l(y)\| \leq \|l\|$$

and consequently $\|l(x)\| \leq \|l\| \cdot \|x\|$. Having assumed $\|x\| \leq 1$, we obtain $\|l(x)\| \leq \|l\|$. Since $l(0) = 0$, such inequality still holds for $x = 0$, and consequently we have proved that $\|l(x)\| \leq \|l\|$ for all $x \in E$ with $\|x\| \leq 1$. This shows that $\|l\|$ is an upper-bound of all $\|l(x)\|$'s as x runs through the set of all $x \in E$ with $\|x\| \leq 1$. Since α is the smallest of such upper-bounds, we obtain $\alpha \leq \|l\|$. We have proved that $\alpha = \|l\|$, i.e.:

$$\|l\| = \sup\{\|l(x)\| : x \in E\ ,\ \|x\| \leq 1\}$$

2. Define:

$$\alpha \stackrel{\triangle}{=} \sup\left\{\frac{\|l(x)\|}{\|x\|} : x \in E\ ,\ x \neq 0\right\}$$

We claim that $\|l\| = \alpha$. Let $x \in E$, $x \neq 0$. Then $y = x/\|x\|$ is such that $\|y\| = 1$, and consequently:

$$\frac{\|l(x)\|}{\|x\|} = \left\| l\left(\frac{x}{\|x\|}\right)\right\| = \|l(y)\| \leq \|l\|$$

This being true for all $x \in E$, $x \neq 0$, we obtain $\alpha \leq \|l\|$. To show the reverse inequality, consider $x \in E$ with $\|x\| = 1$. In particular $x \neq 0$ and consequently:

$$\|l(x)\| = \frac{\|l(x)\|}{\|x\|} \leq \alpha$$

This being true for all $x \in E$ with $\|x\| = 1$, we obtain $\|l\| \leq \alpha$. We have proved that $\alpha = \|l\|$, or equivalently:

$$\|l\| = \sup\left\{\frac{\|l(x)\|}{\|x\|} : x \in E\ ,\ x \neq 0\right\}$$

3. Let $x \in E$. Suppose $x \neq 0$. From 2. we obtain:

$$\frac{\|l(x)\|}{\|x\|} \leq \|l\|$$

and consequently $\|l(x)\| \leq \|l\| \cdot \|x\|$. Since $l(0) = 0$, we have proved that $\|l(x)\| \leq \|l\| \cdot \|x\|$ for all $x \in E$.

4. Since l is continuous, from exercise (3) we have $\|l\| < +\infty$. So $\|l\|$ is indeed an element of $\mathbf{R}^+$, which furthermore from 3. satisfies $\|l(x)\| \leq \|l\| \cdot \|x\|$ for all $x \in E$. Suppose K is another element of $\mathbf{R}^+$, such that:
$$\forall x \in E \ , \ \|l(x)\| \leq K\|x\|$$
Then for all $x \in E$, $x \neq 0$, we have $\|l(x)\|/\|x\| \leq K$. So K is an upper-bound of all $\|l(x)\|/\|x\|$, as x runs through the set of all $x \in E$, $x \neq 0$. Having proved in 2. that $\|l\|$ is the smallest of such upper-bounds, we obtain $\|l\| \leq K$. So $\|l\|$ is indeed the smallest $K \in \mathbf{R}^+$ with $\|l(x)\| \leq K\|x\|$ for all $x \in E$.

5. Since $\|l\| < +\infty$ for all $l \in \mathcal{L}_{\mathbf{K}}(E,F)$, the map $\|\cdot\|$ is indeed a map $\|\cdot\| : \mathcal{L}_{\mathbf{K}}(E,F) \to \mathbf{R}^+$. We claim that it is in fact a norm on $\mathcal{L}_{\mathbf{K}}(E,F)$. Suppose $\|l\| = 0$. Then from 3. for all $x \in E$:
$$\|l(x)\| \leq \|l\| \cdot \|x\| = 0$$
and consequently $l(x) = 0$ for all $x \in E$. This shows that $l = 0$ and we have proved that $\|l\| = 0 \Rightarrow l = 0$. Conversely, if $l = 0$:
$$\begin{aligned} \|l\| &= \sup\{\|l(x)\| : x \in E \ , \ \|x\| = 1\} \\ &= \sup\{0\} = 0 \end{aligned}$$
which shows that $\|l\| = 0$ is in fact equivalent to $l = 0$. Let $\alpha \in \mathbf{K}$. For all $x \in E$, using 3. we have:
$$\begin{aligned} \|(\alpha l)(x)\| &= \|\alpha l(x)\| \\ &= |\alpha| \cdot \|l(x)\| \\ &\leq |\alpha| \cdot \|l\| \cdot \|x\| \end{aligned}$$
and it follows from 4. that $\|\alpha l\| \leq |\alpha| \cdot \|l\|$. Suppose $\alpha \neq 0$. Then applying this inequality to α^{-1} and αl we obtain:
$$\begin{aligned} \|l\| &= \|\alpha^{-1}(\alpha l)\| \\ &\leq |\alpha^{-1}| \cdot \|\alpha l\| = |\alpha|^{-1}\|\alpha l\| \end{aligned}$$
and consequently $|\alpha| \cdot \|l\| \leq \|\alpha l\|$. This shows that $\|\alpha l\| = |\alpha| \cdot \|l\|$ for all $l \in \mathcal{L}_{\mathbf{K}}(E,F)$ and $\alpha \neq 0$. This equality being still true for $\alpha = 0$, we have proved that $\|\alpha l\| = |\alpha| \cdot \|l\|$ for all $l \in \mathcal{L}_{\mathbf{K}}(E,F)$ and $\alpha \in \mathbf{K}$. Let $l, l' \in \mathcal{L}_{\mathbf{K}}(E,F)$. Then for all $x \in E$:
$$\begin{aligned} \|(l+l')(x)\| &= \|l(x) + l'(x)\| \\ &\leq \|l(x)\| + \|l'(x)\| \\ &\leq \|l\| \cdot \|x\| + \|l'\| \cdot \|x\| \\ &= (\|l\| + \|l'\|)\|x\| \end{aligned}$$
and it follows from 4. that $\|l + l'\| \leq \|l\| + \|l'\|$. From definition (95), we conclude that $\|.\|$ is indeed a norm on $\mathcal{L}_{\mathbf{K}}(E,F)$.

6. Since $\mathcal{L}_{\mathbf{K}}(E,F)$ is a $\mathbf{K}$-vector space and $\|\cdot\|$ is a norm on $\mathcal{L}_{\mathbf{K}}(E,F)$, we conclude that $(\mathcal{L}_{\mathbf{K}}(E,F),\|\cdot\|)$ is a $\mathbf{K}$-normed space by virtue of definition (125).

Exercise 5

Exercise 6.

1. Let E,F be two $\mathbf{R}$-normed spaces and U be open in E. Let $\phi : U \to F$ be a map, and $a \in U$. We assume that $l_1, l_2 \in \mathcal{L}_{\mathbf{R}}(E,F)$ satisfy the requirements of definition (128). Let $\epsilon > 0$ be given. Since l_1 satisfies the requirement of definition (128), there exists $\delta_1 > 0$ such that for all $h \in E$:
$$\|h\| \leq \delta_1 \;\Rightarrow\; a+h \in U \text{ and } \|\phi(a+h)-\phi(a)-l_1(h)\| \leq \frac{\epsilon}{2}\|h\|$$
Similarly, there exists $\delta_2 > 0$ such that for all $h \in E$:
$$\|h\| \leq \delta_2 \;\Rightarrow\; a+h \in U \text{ and } \|\phi(a+h)-\phi(a)-l_2(h)\| \leq \frac{\epsilon}{2}\|h\|$$
Let $\delta = \min(\delta_1,\delta_2)$. Then $\delta > 0$, and for all $h \in E$ the condition $\|h\| \leq \delta$ implies that $a+h \in U$ and furthermore:
$$\begin{aligned} \|l_1(h)-l_2(h)\| &\leq \|\phi(a+h)-\phi(a)-l_2(h)\| \\ &+ \|\phi(a+h)-\phi(a)-l_1(h)\| \\ &\leq \frac{\epsilon}{2}\|h\| + \frac{\epsilon}{2}\|h\| \\ &= \epsilon\|h\| \end{aligned}$$
Hence, given $\epsilon > 0$, we have found $\delta > 0$ such that for all $h \in E$:
$$\|h\| \leq \delta \;\Rightarrow\; \|l_1(h)-l_2(h)\| \leq \epsilon\|h\|$$

2. Let $\epsilon > 0$ and $\delta > 0$ be such that for all $h \in E$:
$$\|h\| \leq \delta \;\Rightarrow\; \|l_1(h)-l_2(h)\| \leq \epsilon\|h\|$$
Let $x \in E$ with $\|x\| = 1$. Then $h = \delta x$ is an element of E with $\|h\| = \delta$. In particular $\|h\| \leq \delta$, and consequently we have:
$$\begin{aligned} \delta\|(l_1-l_2)(x)\| &= \delta\|l_1(x)-l_2(x)\| \\ &= \|l_1(\delta x)-l_2(\delta x)\| \\ &= \|l_1(h)-l_2(h)\| \\ &\leq \epsilon\|h\| = \epsilon\delta \end{aligned}$$
Since $\delta > 0$, it follows that $\|(l_1-l_2)(x)\| \leq \epsilon$ and we see that ϵ is an upper-bound of all $\|(l_1-l_2)(x)\|$'s as x runs through the set of all $x \in E$ with $\|x\| = 1$. Since $\|l_1-l_2\|$ is the smallest of such upper-bounds, we obtain $\|l_1-l_2\| \leq \epsilon$. This being true for all $\epsilon > 0$, we conclude that $\|l_1-l_2\| = 0$, i.e. $l_1 = l_2$.

Exercise 6

Exercise 7.

1. Let E, F be two $\mathbf{R}$-normed spaces and U be open in E. Let $\phi : U \to F$ be a map and $a \in U$. Suppose that ϕ is differentiable at a. Take $\epsilon = 1$. Since $d\phi(a)$ denotes the differential of ϕ at a, i.e. the unique element of $\mathcal{L}_{\mathbf{R}}(E, F)$ satisfying the requirements of (128), there exists $\delta > 0$ such that for all $h \in E$:

$$\|h\| \leq \delta \ \Rightarrow \ a + h \in U \text{ and } \|\phi(a+h) - \phi(a) - d\phi(a)(h)\| \leq \|h\|$$

   In particular, for all $h \in E$ the condition $\|h\| \leq \delta$ implies that $a + h \in U$ and furthermore:

$$\begin{aligned}
\|\phi(a+h) - \phi(a)\| &= \|\phi(a+h) - \phi(a)\| - \|d\phi(a)(h)\| \\
&+ \|d\phi(a)(h)\| \\
&\leq |\,\|\phi(a+h) - \phi(a)\| - \|d\phi(a)(h)\|\,| \\
&+ \|d\phi(a)(h)\| \\
&\leq \|\phi(a+h) - \phi(a) - d\phi(a)(h)\| \\
&+ \|d\phi(a)(h)\| \\
&\leq \|h\| + \|d\phi(a)\| \cdot \|h\| \\
&= K\|h\|
\end{aligned}$$

   where we have put $K = (1 + \|d\phi(a)\|) \in \mathbf{R}^+$. Hence, we have found $\delta > 0$ such that for all $h \in E$:

$$\|h\| \leq \delta \ \Rightarrow \ a + h \in U \text{ and } \|\phi(a+h) - \phi(a)\| \leq K\|h\|$$

   This shows that ϕ is continuous at a. We have proved that if ϕ is differentiable at a, then ϕ is continuous at a.

2. Suppose $\phi : U \to F$ is of class C^1. From definition (130), the differential map $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is well-defined, i.e. $d\phi(a)$ exists for all $a \in U$. Furthermore, $d\phi$ is said to be a continuous map. For this to be meaningful, U and $\mathcal{L}_{\mathbf{R}}(E, F)$ need to be topological spaces. E being an $\mathbf{R}$-normed space, it is naturally endowed with the norm topology, as defined in (126). Since U is a subset of E, the obvious topology on U is the topology induced by the topology on E, as defined in (23). Now from exercise (5), $\mathcal{L}_{\mathbf{R}}(E, F)$ is an $\mathbf{R}$-normed space. It is therefore a topological space, when endowed with the norm topology, as defined in (126).

3. Suppose $\phi : U \to F$ is of class C^1. Then in particular, for all $a \in U$ the differential $d\phi(a)$ exists. From 1. it follows that ϕ is continuous at a, for all $a \in U$. We conclude that ϕ is continuous.

4. We assume that $E = \mathbf{R}$. Note that $\mathbf{R}$ is a vector space over itself, and that $|\cdot|$ is a norm on $\mathbf{R}$. So $(\mathbf{R}, |\cdot|)$ is an $\mathbf{R}$-normed space. Let $a \in U$.

We assume that the limit:

$$\phi'(a) \stackrel{\triangle}{=} \lim_{t\neq 0, t\to 0} \frac{\phi(a+t)-\phi(a)}{t}$$

exists in F. We claim that ϕ is differentiable at a, and furthermore that the differential $d\phi(a)$ of ϕ at a is given by:

$$\forall t \in \mathbf{R}\ ,\ d\phi(a)(t) = t\cdot\phi'(a)$$

Let $l \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}, F)$ be defined by $l(t) = t\cdot\phi'(a)$. Note that $l(t)$ is nothing but the product of $\phi'(a) \in F$ with the scalar $t \in \mathbf{R}$. So l is well-defined, and it is clearly a linear map. Moreover, for all $t \in \mathbf{R}$, we have:

$$\|l(t)\| = \|t\cdot\phi'(a)\| = |t|\cdot\|\phi'(a)\|$$

and in particular $\|l(t)\| \leq \|\phi'(a)\|\cdot|t|$. So l is continuous, and it is indeed an element of $\mathcal{L}_{\mathbf{R}}(\mathbf{R}, F)$. To show that ϕ is differentiable at a with $d\phi(a) = l$, we only need to show that l satisfies the requirements of definition (128). Let $\epsilon > 0$ be given. Having assumed that the limit $\phi'(a)$ exists, there is $\delta > 0$ such that for all $t \in \mathbf{R}$, $t \neq 0$, the condition $|t| \leq \delta$ implies $a + t \in U$ and:

$$\left\|\frac{\phi(a+t)-\phi(a)}{t} - \phi'(a)\right\| \leq \epsilon$$

Hence, we have:

$$\begin{aligned}\|\phi(a+t)-\phi(a)-l(t)\| &= \|\phi(a+t)-\phi(a)-t\cdot\phi'(a)\| \\ &= |t|\cdot\left\|\frac{\phi(a+t)-\phi(a)}{t} - \phi'(a)\right\| \\ &\leq \epsilon|t|\end{aligned}$$

This last inequality being still valid for $t = 0$, we have:

$$|t| \leq \delta \ \Rightarrow\ a+t \in U \text{ and } \|\phi(a+t)-\phi(a)-l(t)\| \leq \epsilon|t|$$

So l satisfies the requirements of definition (128) and we have proved that ϕ is differentiable at a with $d\phi(a) = l$. This shows that the existence of $\phi'(a)$ implies that of $d\phi(a)$. Conversely, suppose that $d\phi(a)$ exists, i.e. that ϕ is differentiable at a. We claim that $\phi'(a)$ exists, and furthermore that $\phi'(a) = d\phi(a)(1)$. Let $\epsilon > 0$. There exists $\delta > 0$ such that for all $t \in \mathbf{R}$:

$$|t| \leq \delta \ \Rightarrow\ a+t \in U \text{ and } \|\phi(a+t)-\phi(a)-d\phi(a)(t)\| \leq \epsilon|t|$$

In particular, if $t \in \mathbf{R}$, $t \neq 0$, the condition $|t| \leq \delta$ implies that $a + t \in U$, and furthermore, denoting $l = d\phi(a)$:

$$\begin{aligned}\left\|\frac{\phi(a+t)-\phi(a)}{t} - l(1)\right\| &= \frac{1}{|t|}\|\phi(a+t)-\phi(a)-tl(1)\| \\ &= \frac{1}{|t|}\|\phi(a+t)-\phi(a)-l(t)\| \\ &\leq \frac{1}{|t|}\epsilon|t| = \epsilon\end{aligned}$$

This shows that the limit $\phi'(a)$ exists and is equal to $d\phi(a)(1)$. We conclude that in the case when $E = \mathbf{R}$, $\phi : U \to F$ is differentiable at a, if and only if the derivative $\phi'(a)$ exists, in which case $d\phi(a) \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}, F)$ is given by $d\phi(a)(t) = t \cdot \phi'(a)$ for all $t \in \mathbf{R}$. In particular, we have $d\phi(a)(1) = \phi'(a)$.

Exercise 7

Exercise 8.

1. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. We assume that ϕ is differentiable at $a \in U$, and we put $l_1 = d\phi(a)$. We assume that ψ is differentiable at $\phi(a) \in V$, and we put $l_2 = d\psi(\phi(a))$. Since $\phi(U) \subseteq V$, for all $x \in U$ we have $\phi(x) \in V$. So $\psi(\phi(x))$ is a well-defined element of G. It follows that $\psi \circ \phi : U \to G$ is a well-defined map.

2. Let $\epsilon > 0$. Since $l_1 \in \mathcal{L}_{\mathbf{R}}(E, F)$, $\|l_1\|$ is a well-defined element of $\mathbf{R}^+$. Since $l_2 \in \mathcal{L}_{\mathbf{R}}(F, G)$, $\|l_2\|$ is a well-defined element of $\mathbf{R}^+$. Take $\eta = \min(1, \epsilon(1 + \|l_1\| + \|l_2\|)^{-1})$. Then $\eta > 0$, and:
$$\begin{aligned} \eta(\eta + \|l_1\| + \|l_2\|) &\leq \eta(1 + \|l_1\| + \|l_2\|) \\ &\leq \epsilon \end{aligned}$$

3. Since ψ is differentiable at $\phi(a) \in V$ and $l_2 = d\psi(\phi(a))$, l_2 satisfies the requirements of definition (128). There is $\delta_2 > 0$ such that for all $h_2 \in F$ with $\|h_2\| \leq \delta_2$, $\phi(a) + h_2 \in V$ and:
$$\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2(h_2)\| \leq \eta \|h_2\|$$

4. Let $h_2 \in F$ with $\|h_2\| \leq \delta_2$. Let $h \in E$. Using 3. we obtain:
$$\begin{aligned} \|\psi(\phi(a) + h_2) &- \psi \circ \phi(a) - l_2 \circ l_1(h)\| \\ &\leq \|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2(h_2)\| \\ &+ \|l_2(h_2) - l_2 \circ l_1(h)\| \\ &\leq \eta\|h_2\| + \|l_2(h_2 - l_1(h))\| \\ &\leq \eta\|h_2\| + \|l_2\| \cdot \|h_2 - l_1(h)\| \end{aligned}$$

5. Since ϕ is differentiable at $a \in U$ and $l_1 = d\phi(a)$, l_1 satisfies the requirements of definition (128). There exists $\delta_1 > 0$ such that for all $h \in E$ with $\|h\| \leq \delta_1$, we have $a + h \in U$ and:
$$\|\phi(a + h) - \phi(a) - l_1(h)\| \leq \eta \|h\| \tag{2}$$
Moreover, from 1. of exercise (7), ϕ is continuous at a. Since $\delta_2 > 0$, there exists $\delta_1' > 0$ such that for all $h \in E$ with $\|h\| \leq \delta_1'$, we have $a + h \in U$ and:
$$\|\phi(a + h) - \phi(a)\| \leq \delta_2 \tag{3}$$

Taking $\delta = \min(\delta_1, \delta_1')$, we have found $\delta > 0$ such that for all $h \in E$ with $\|h\| \leq \delta$, we have $a + h \in U$ and furthermore both inequalities (2) and (3) hold.

6. Let $h \in E$ with $\|h\| \leq \delta$, Then $a + h \in U$ and furthermore both inequalities (2) and (3) hold. Let $h_2 = \phi(a + h) - \phi(a)$. Then (3) can be written as $\|h_2\| \leq \delta_2$, and applying 4.:

$$\begin{aligned} \|\psi \circ \phi(a + h) &- \psi \circ \phi(a) - l_2 \circ l_1(h)\| \\ &= \|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| \\ &\leq \eta\|h_2\| + \|l_2\| \cdot \|h_2 - l_1(h)\| \\ &= \eta\|h_2\| + \|l_2\| \cdot \|\phi(a + h) - \phi(a) - l_1(h)\| \\ \text{using (2)} \rightarrow &\leq \eta\|h_2\| + \|l_2\|\eta\|h\| \\ &= \eta\|\phi(a + h) - \phi(a)\| + \eta\|l_2\| \cdot \|h\| \\ &\leq \eta\|\phi(a + h) - \phi(a) - l_1(h)\| \\ &+ \eta\|l_1(h)\| + \eta\|l_2\| \cdot \|h\| \\ \text{using (2)} \rightarrow &\leq \eta^2\|h\| + \eta\|l_1\| \cdot \|h\| + \eta\|l_2\| \cdot \|h\| \\ &= \eta(\eta + \|l_1\| + \|l_2\|)\|h\| \\ \text{using 2.} \rightarrow &\leq \epsilon\|h\| \end{aligned}$$

7. Since $l_1 \in \mathcal{L}_{\mathbf{R}}(E, F)$, $l_1 : E \rightarrow F$ is linear and continuous. Since $l_2 \in \mathcal{L}_{\mathbf{R}}(F, G)$, $l_2 : F \rightarrow G$ is linear and continuous. So $l_2 \circ l_1 : E \rightarrow G$ is linear and continuous, and $l_2 \circ l_1 \in \mathcal{L}_{\mathbf{R}}(E, G)$.

8. From 6. and 7. we conclude that $l_2 \circ l_1 \in \mathcal{L}_{\mathbf{R}}(E, G)$ is such that given $\epsilon > 0$, we have found $\delta > 0$ such that for all $h \in E$ with $\|h\| \leq \delta$, we have $a + h \in U$ and:

$$\|\psi \circ \phi(a + h) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| \leq \epsilon\|h\|$$

From definition (128), it follows that $\psi \circ \phi : U \rightarrow G$ is differentiable at $a \in U$, and furthermore from definition (129):

$$\begin{aligned} d(\psi \circ \phi)(a) &= l_2 \circ l_1 \\ &= d\psi(\phi(a)) \circ d\phi(a) \end{aligned}$$

This completes the proof of theorem (110).

Exercise 8

Exercise 9.

1. Let $(\Omega', \mathcal{T}')$ and $(\Omega, \mathcal{T})$ be two topological spaces, and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a set of subsets of Ω generating the topology $\mathcal{T}$, i.e. such that $\mathcal{T} = \mathcal{T}(\mathcal{A})$. Let $f : \Omega' \rightarrow \Omega$ be a map, and define:

$$\mathcal{U} \stackrel{\triangle}{=} \{A \subseteq \Omega \ : \ f^{-1}(A) \in \mathcal{T}'\}$$

We claim that $\mathcal{U}$ is a topology on Ω. Since $f^{-1}(\emptyset) = \emptyset \in \mathcal{T}'$ and $f^{-1}(\Omega) = \Omega' \in \mathcal{T}'$, both $\emptyset$ and Ω are elements of $\mathcal{U}$. Let $(A_i)_{i\in I}$ be a family of elements of $\mathcal{U}$. Then:

$$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i) \in \mathcal{T}'$$

So $\cup_{i\in I} A_i \in \mathcal{U}$, and we have proved that $\mathcal{U}$ is closed under arbitrary unions. Let $A, B \in \mathcal{U}$. Then:

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \in \mathcal{T}'$$

So $A \cap B \in \mathcal{U}$, and we have proved that $\mathcal{U}$ is closed under finite intersections. From definition (13), we conclude that $\mathcal{U}$ is a topology on Ω.

2. Suppose $f : (\Omega', \mathcal{T}') \to (\Omega, \mathcal{T})$ is continuous. Then from definition (27), for all $A \in \mathcal{T}$ we have $f^{-1}(A) \in \mathcal{T}'$. In particular, since $\mathcal{A} \subseteq \mathcal{T}(\mathcal{A}) = \mathcal{T}$, for all $A \in \mathcal{A}$ we have $f^{-1}(A) \in \mathcal{T}'$. Conversely, suppose $f^{-1}(A) \in \mathcal{T}'$ for all $A \in \mathcal{A}$. Then $\mathcal{A} \subseteq \mathcal{U}$, where $\mathcal{U}$ is the topology on Ω defined in 1. However from exercise (11) of Tutorial 6, the topology $\mathcal{T}(\mathcal{A})$ generated by $\mathcal{A}$ is the smallest topology on Ω containing $\mathcal{A}$, in the inclusion sense. Hence, it follows from $\mathcal{A} \subseteq \mathcal{U}$ and the fact that $\mathcal{U}$ is a topology, that $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{U}$. However by assumption, we have $\mathcal{T}(\mathcal{A}) = \mathcal{T}$. So $\mathcal{T} \subseteq \mathcal{U}$, and we conclude that $f^{-1}(A) \in \mathcal{T}'$ for all $A \in \mathcal{T}$. This shows that f is continuous. We have proved that f is continuous if and only if $f^{-1}(A) \in \mathcal{T}'$ for all $A \in \mathcal{A}$.

Exercise 9

Exercise 10.

1. Let $p_i : \Omega \to \Omega_i$ be the canonical projection mapping. Given $i \in I$ and $A_i \in \mathcal{T}_i$ we have:

$$p_i^{-1}(A_i) = A_i \times \prod_{j\in I\setminus\{i\}} \Omega_j$$

It follows from definition (52), that $p_i^{-1}(A_i)$ is an open rectangle, i.e. a rectangle of $(\mathcal{T}_j)_{j\in I}$, and in particular it is an element of the product topology $\mathcal{T}$. This shows that p_i is continuous.

2. Suppose each $f_i : (\Omega', \mathcal{T}') \to (\Omega_i, \mathcal{T}_i)$ is a continuous map. From definition (56), the product topology $\mathcal{T}$ on Ω is the topology generated by the open rectangles, i.e. the rectangles of $(\mathcal{T}_i)_{i\in I}$. In other words, $\mathcal{T} = \mathcal{T}(\mathcal{A})$ where $\mathcal{A} = \amalg_{i\in I}\mathcal{T}_i$. From exercise (9), to show that f is continuous, it is sufficient to show that $f^{-1}(A) \in \mathcal{T}'$ for all $A \in \mathcal{A}$. So let $A \in \mathcal{A}$ be an open rectangle. From definition (52), A can be written as $A = \Pi_{i\in I} A_i$, where each A_i is an element of $\mathcal{T}_i \cup \{\Omega_i\} = \mathcal{T}_i$, and the set $J = \{i \in I : A_i \neq \Omega_i\}$ is finite. Hence, we have:

$$\begin{aligned} f^{-1}(A) &= \{\omega \in \Omega' : f(\omega) \in A\} \\ &= \{\omega \in \Omega' : (f_i(\omega))_{i\in I} \in \Pi_{i\in I} A_i\} \end{aligned}$$

$$\begin{aligned} &= \{\omega \in \Omega' : f_i(\omega) \in A_i, \forall i \in I\} \\ &= \{\omega \in \Omega' : f_i(\omega) \in A_i, \forall i \in J\} \\ &= \bigcap_{i\in J} f_i^{-1}(A_i) \end{aligned}$$

Having assumed that f_i is continuous for all $i \in I$, it follows from $A_i \in \mathcal{T}_i$ that $f_i^{-1}(A_i) \in \mathcal{T}'$, and consequently since J is finite, $f^{-1}(A) = \cap_{i\in J} f_i^{-1}(A_i)$ is an element of $\mathcal{T}'$. Hence, we have proved that $f^{-1}(A) \in \mathcal{T}'$ for all $A \in \mathcal{A}$, and we conclude that f is continuous. Conversely, suppose $f : (\Omega', \mathcal{T}') \to (\Omega, \mathcal{T})$ is continuous. Since $p_i : (\Omega, \mathcal{T}) \to (\Omega_i, \mathcal{T}_i)$ is continuous, each $f_i = p_i \circ f$ is a continuous map.

Exercise 10

Exercise 11.

1. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class C^1 such that $\phi(U) \subseteq V$. Given $(l_1, l_2) \in \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$, we define:

$$\begin{aligned} N_1(l_1, l_2) &\stackrel{\triangle}{=} \|l_1\| + \|l_2\| \\ N_2(l_1, l_2) &\stackrel{\triangle}{=} \sqrt{\|l_1\|^2 + \|l_2\|^2} \\ N_\infty(l_1, l_2) &\stackrel{\triangle}{=} \max(\|l_1\|, \|l_2\|) \end{aligned}$$

Then each $N_i : \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F) \to \mathbf{R}^+$ is a well-defined map, $i \in \{1, 2, \infty\}$, and we claim that it is in fact a norm on $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$. Note that we are implicitly saying that $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$ is an $\mathbf{R}$-vector space, a fact that has not been justified in these Tutorials. For those not familiar with the product structure of vector spaces, recall that given two elements (l_1, l_2) and (l_1', l_2') of $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$, and $\alpha \in \mathbf{R}$, a *vector addition* $\oplus$ is defined as:

$$(l_1, l_2) \oplus (l_1', l_2') \stackrel{\triangle}{=} (l_1 + l_1', l_2 + l_2')$$

and a *scalar multiplication* $\otimes$ is defined as:

$$\alpha \otimes (l_1, l_2) = (\alpha l_1, \alpha l_2)$$

It is cumbersome but not difficult to show that $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$ together with the operators $\oplus$ and $\otimes$, satisfy the requirements of (89) defining an $\mathbf{R}$-vector space, where the *zero element* of $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$ is understood to be $(0, 0)$. It is customary to denote $\oplus$ and $\otimes$ simply by $+$ and $\cdot$, and we shall do so from now on. Now, given $(x, y) \in \mathbf{R}^2$, we define $\|(x, y)\|_1 = |x| + |y|$, $\|(x, y)\|_2 = \sqrt{|x|^2 + |y|^2}$ as well as $\|(x, y)\|_\infty = \max(|x|, |y|)$. Then it is clear that $N_i(l_1, l_2) = \|(\|l_1\|, \|l_2\|)\|_i$ for all $i \in \{1, 2, \infty\}$. In order to prove that N_i is a norm, we shall first prove that $\|\cdot\|_i$ is a norm on $\mathbf{R}^2$, a fact that many of us are already familiar with. For those who require a proof, here is the following: note that $\|\cdot\|_2$ is nothing but the norm defined in (81), associated with the usual inner-product of $\mathbf{R}^2$. From exercise (1), $\|\cdot\|_2$ is therefore a norm on $\mathbf{R}^2$. So we may assume that $i \in \{1, \infty\}$. It is clear that $\|(x,y)\|_i = 0$ is equivalent to $(x,y) = (0,0)$ and furthermore that $\|\alpha(x,y)\|_i = |\alpha| \cdot \|(x,y)\|_i$ for all $\alpha \in \mathbf{R}$. Hence, we only need to prove the triangle inequality for $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Given (x,y) and (x',y') in $\mathbf{R}^2$, we have:

$$\begin{aligned}\|(x,y)+(x',y')\|_1 &= \|(x+x',y+y')\|_1\\ &= |x+x'|+|y+y'|\\ &\leq |x|+|x'|+|y|+|y'|\\ &= \|(x,y)\|_1+\|(x',y')\|_1\end{aligned}$$

Moreover, we have:

$$\begin{aligned}|x+x'| &\leq |x|+|x'|\\ &\leq \max(|x|,|y|)+\max(|x'|,|y'|)\\ &= \|(x,y)\|_\infty+\|(x',y')\|_\infty\end{aligned}$$

and similarly $|y+y'| \leq \|(x,y)\|_\infty + \|(x',y')\|_\infty$. Hence:

$$\begin{aligned}\|(x,y)+(x',y')\|_\infty &= \|(x+x',y+y')\|_\infty\\ &= \max(|x+x'|,|y+y'|)\\ &\leq \|(x,y)\|_\infty+\|(x',y')\|_\infty\end{aligned}$$

So we have proved that $\|\cdot\|_i$ is a norm on $\mathbf{R}^2$ for all $i \in \{1,2,\infty\}$. Note that all this will be generalized in a later tutorial, when we formally study normed vector spaces, and in particular the norm $\|\cdot\|_p$ on $\mathbf{R}^n$ or $\mathbf{C}^n$, where $p \in [1,+\infty]$. Having proved that $\|\cdot\|_i$ is a norm on $\mathbf{R}^2$, we shall now prove that N_i is a norm on $\mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$. Since $N_i(l_1,l_2) = \|(\|l_1\|,\|l_2\|)\|_i$, the condition $N_i(l_1,l_2) = 0$ is equivalent to $\|(\|l_1\|,\|l_2\|)\|_i = 0$, which is equivalent to $(\|l_1\|,\|l_2\|) = (0,0)$, which is in turn equivalent to $(l_1,l_2) = (0,0)$. Moreover, if $\alpha \in \mathbf{R}$, we have:

$$\begin{aligned}N_i[\alpha(l_1,l_2)] &= N_i[(\alpha l_1,\alpha l_2)]\\ &= \|\,(\|\alpha l_1\|,\|\alpha l_2\|)\,\|_i\\ &= \|\,(|\alpha|\cdot\|l_1\|,|\alpha|\cdot\|l_2\|)\,\|_i\\ &= \|\,|\alpha|(\|l_1\|,\|l_2\|)\,\|_i\\ &= |\alpha|\cdot\|\,(\|l_1\|,\|l_2\|)\,\|_i\\ &= |\alpha|N_i(l_1,l_2)\end{aligned}$$

Finally, if $(l_1,l_2),(l_1',l_2') \in \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$:

$$\begin{aligned}N_i[(l_1,l_2)+(l_1',l_2')] &= N_i[(l_1+l_1',l_2+l_2')]\\ &= \|\,(\|l_1+l_1'\|,\|l_2+l_2'\|)\,\|_i\\ &\leq \|\,(\|l_1\|+\|l_1'\|,\|l_2\|+\|l_2'\|)\,\|_i\\ &= \|\,(\|l_1\|,\|l_2\|)+(\|l_1'\|,\|l_2'\|)\,\|_i\end{aligned}$$

$$\begin{array}{rl} \leq & \|\,(\|l_1\|,\|l_2\|)\,\|_i + \|\,(\|l'_1\|,\|l'_2\|)\,\|_i \\ = & N_i(l_1,l_2) + N_i(l'_1,l'_2) \end{array}$$

We have proved that N_i is a norm on $\mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$.

2. Let $X = \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$ and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_\infty$ be the topologies on X induced by the norms N_1, N_2 and N_∞ respectively. Let $\mathcal{T}$ denote the product topology on X. We shall prove the equality $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_\infty = \mathcal{T}$. For all $(l_1,l_2) \in X$, we have:

$$\begin{array}{rcl} [N_2(l_1,l_2)]^2 & = & \|l_1\|^2 + \|l_2\|^2 \\ & \leq & \|l_1\|^2 + \|l_2\|^2 + 2\|l_1\|\cdot\|l_2\| \\ & = & (\|l_1\| + \|l_2\|)^2 \\ & = & [N_1(l_1,l_2)]^2 \\ & \leq & [2\max(\|l_1\|,\|l_2\|)]^2 \\ & = & 4[N_\infty(l_1,l_2)]^2 \\ & = & 4\max(\|l_1\|^2,\|l_2\|^2) \\ & \leq & 4(\|l_1\|^2 + \|l_2\|^2) \\ & = & 4[N_2(l_1,l_2)]^2 \end{array}$$

from which we obtain $N_2 \leq N_1 \leq 2N_\infty \leq 2N_2$. Consider the identity mapping $j : X \to X$, defined by $j(l_1,l_2) = (l_1,l_2)$ for all $(l_1,l_2) \in X$. Then j is a linear mapping and the inequality $N_2 \leq N_1$ can be written as:

$$\forall (l_1,l_2) \in X \ , \ N_2[j(l_1,l_2)] \leq N_1(l_1,l_2)$$

From exercise (3), it follows that $j : (X,N_1) \to (X,N_2)$ is a continuous map. Hence, for all U open in (X,N_2), i.e. for all $U \in \mathcal{T}_2$, we have $j^{-1}(U)$ open in (X,N_1), i.e. $U \in \mathcal{T}_1$. This shows that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Similarly, the inequality $N_1 \leq 2N_\infty$ implies that $\mathcal{T}_1 \subseteq \mathcal{T}_\infty$, and $N_\infty \leq N_2$ that $\mathcal{T}_\infty \subseteq \mathcal{T}_2$. Hence, we have proved that $\mathcal{T}_2 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_\infty \subseteq \mathcal{T}_2$, or equivalently $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_\infty$. It remains to show that $\mathcal{T} = \mathcal{T}_\infty$. From definition (56), the product topology on X is the topology generated by the open rectangles of X, i.e. the sets of the form $A \times B$ where A is open in $\mathcal{L}_{\mathbf{R}}(F,G)$ and B is open in $\mathcal{L}_{\mathbf{R}}(E,F)$. To show that $\mathcal{T} \subseteq \mathcal{T}_\infty$, it is sufficient to prove that any such $A \times B$ is an element of $\mathcal{T}_\infty$. Indeed, $\mathcal{T}$ being the smallest topology on X containing all open rectangles, if $\mathcal{T}_\infty$ is shown to contain all open rectangles, then $\mathcal{T} \subseteq \mathcal{T}_\infty$. We therefore consider $A \times B$ open rectangle in X, and we shall prove that $A \times B \in \mathcal{T}_\infty$. If $A \times B = \emptyset$, then $A \times B \in \mathcal{T}_\infty$ is clear. Otherwise, there exists $(l_1,l_2) \in A \times B$. Since A is open in $\mathcal{L}_{\mathbf{R}}(F,G)$ and $l_1 \in A$, there exists $\epsilon_1 > 0$ such that $B(l_1,\epsilon_1) \subseteq A$, where $B(l_1,\epsilon_1)$ denotes the open ball in $\mathcal{L}_{\mathbf{R}}(F,G)$. Similarly, since B is open in $\mathcal{L}_{\mathbf{R}}(E,F)$ and $l_2 \in B$, there exists $\epsilon_2 > 0$ such that $B(l_2,\epsilon_2) \subseteq B$, where $B(l_2,\epsilon_2)$ denotes the open ball in $\mathcal{L}_{\mathbf{R}}(E,F)$. Note that we are using identical notations $B(\cdot,\cdot)$ to refer to open balls in $\mathcal{L}_{\mathbf{R}}(F,G)$ and $\mathcal{L}_{\mathbf{R}}(E,F)$, but this is unlikely to confuse anyone. Let $\epsilon = \min(\epsilon_1,\epsilon_2)$. Then $\epsilon > 0$, and furthermore for all $(l'_1, l'_2) \in X$ we have:

$$\begin{aligned} N_\infty[(l'_1, l'_2) - (l_1, l_2)] < \epsilon &\Leftrightarrow N_\infty[(l'_1 - l_1, l'_2 - l_2)] < \epsilon \\ &\Leftrightarrow \max(\|l'_1 - l_1\|, \|l'_2 - l_2\|) < \epsilon \\ &\Rightarrow \|l'_1 - l_1\| < \epsilon_1 \ , \ \|l'_2 - l_2\| < \epsilon_2 \\ &\Leftrightarrow l'_1 \in B(l_1, \epsilon_1) \ , \ l'_2 \in B(l_2, \epsilon_2) \\ &\Rightarrow l'_1 \in A \ , \ l'_2 \in B \\ &\Leftrightarrow (l'_1, l'_2) \in A \times B \end{aligned}$$

Hence, given $(l_1, l_2) \in A \times B$, we have found $\epsilon > 0$ such that $B_\infty[(l_1, l_2), \epsilon] \subseteq A \times B$, where $B_\infty[(l_1, l_2), \epsilon]$ denotes the open ball in X with respect to the norm N_∞. This shows that $A \times B$ is open with respect to the topology induced by N_∞, i.e. that $A \times B \in \mathcal{T}_\infty$. We have proved that $\mathcal{T} \subseteq \mathcal{T}_\infty$. To show the reverse inclusion, consider $U \in \mathcal{T}_\infty$. Given $(l_1, l_2) \in U$, there exists $\epsilon > 0$ such that $B_\infty[(l_1, l_2), \epsilon] \subseteq U$. For all $(l'_1, l'_2) \in X$:

$$\begin{aligned} (l'_1, l'_2) \in B(l_1, \epsilon) \times B(l_2, \epsilon) &\Leftrightarrow \|l'_1 - l_1\| < \epsilon \ , \ \|l'_2 - l_2\| < \epsilon \\ &\Leftrightarrow \max(\|l'_1 - l_1\|, \|l'_2 - l_2\|) < \epsilon \\ &\Leftrightarrow N_\infty[(l'_1 - l_1, l'_2 - l_2)] < \epsilon \\ &\Leftrightarrow N_\infty[(l'_1, l'_2) - (l_1, l_2)] < \epsilon \\ &\Leftrightarrow (l'_1, l'_2) \in B_\infty[(l_1, l_2), \epsilon] \end{aligned}$$

and consequently $B(l_1, \epsilon) \times B(l_2, \epsilon) = B_\infty[(l_1, l_2), \epsilon]$. However, $B(l_1, \epsilon)$ being an open ball in $\mathcal{L}_{\mathbf{R}}(F, G)$, it is an open subset of $\mathcal{L}_{\mathbf{R}}(F, G)$. Similarly, $B(l_2, \epsilon)$ is an open subset of $\mathcal{L}_{\mathbf{R}}(E, F)$. It follows that $B(l_1, \epsilon) \times B(l_2, \epsilon)$ is an open rectangle in X, and in particular is an element of the product topology $\mathcal{T}$. We have proved that $B_\infty[(l_1, l_2), \epsilon] = B(l_1, \epsilon) \times B(l_2, \epsilon)$ is an element of $\mathcal{T}$. Hence, given $(l_1, l_2) \in U$, we have found some $U_{(l_1, l_2)} = B_\infty[(l_1, l_2), \epsilon] \in \mathcal{T}$ such that $(l_1, l_2) \in U_{(l_1, l_2)} \subseteq U$. Hence:

$$U = \bigcup_{(l_1, l_2) \in U} U_{(l_1, l_2)} \in \mathcal{T}$$

and we have proved that $\mathcal{T}_\infty \subseteq \mathcal{T}$. This completes our proof of $\mathcal{T}_\infty = \mathcal{T}$, and finally $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_\infty = \mathcal{T}$.

3. Let $X = \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$ and $H : X \to \mathcal{L}_{\mathbf{R}}(E, G)$ be the map defined by $H(l_1, l_2) = l_1 \circ l_2$, for all $(l_1, l_2) \in X$. Note that if $l_1 \in \mathcal{L}_{\mathbf{R}}(F, G)$ and $l_2 \in \mathcal{L}_{\mathbf{R}}(E, F)$, then $l_1 \circ l_2 : E \to G$ is a well-defined map, which furthermore is linear and continuous. So H is a well-defined map which has indeed values in $\mathcal{L}_{\mathbf{R}}(E, G)$. Given $(l_1, l_2) \in X$, for all $x \in E$ we have:

$$\begin{aligned} \|H(l_1, l_2)(x)\| &= \|(l_1 \circ l_2)(x)\| \\ &= \|l_1(l_2(x))\| \\ &\leq \|l_1\| \cdot \|l_2(x)\| \\ &\leq \|l_1\| \cdot \|l_2\| \cdot \|x\| \end{aligned}$$

Hence, using 4. of exercise (5), $\|H(l_1,l_2)\| \leq \|l_1\| \cdot \|l_2\|$.

4. For those familiar with the notion, H is a bilinear map such that $\|H(l_1,l_2)\| \leq \|l_1\| \cdot \|l_2\|$ for all $(l_1,l_2) \in X$, where $X = \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$. It follows that H is continuous. As we have not had a tutorial on multilinear maps, here is a direct proof: Let (l_1,l_2) and (l'_1,l'_2) be elements of X. Then:

$$\begin{array}{rcl} \|H(l'_1,l'_2) - H(l_1,l_2)\| & = & \|l'_1 \circ l'_2 - l_1 \circ l_2\| \\ & \leq & \|l'_1 \circ l'_2 - l_1 \circ l'_2\| + \|l_1 \circ l'_2 - l_1 \circ l_2\| \\ l_1 \text{ is linear} \rightarrow & = & \|(l'_1 - l_1) \circ l'_2\| + \|l_1 \circ (l'_2 - l_2)\| \\ & = & \|H(l'_1 - l_1, l'_2)\| + \|H(l_1, l'_2 - l_2)\| \\ & \leq & \|l'_1 - l_1\| \cdot \|l'_2\| + \|l_1\| \cdot \|l'_2 - l_2\| \\ & \leq & (\|l'_2\| + \|l_1\|) \max(\|l'_1 - l_1\|, \|l'_2 - l_2\|) \\ & = & (\|l'_2\| + \|l_1\|) N_\infty(l'_1 - l_1, l'_2 - l_2) \\ & = & (\|l'_2\| + \|l_1\|) N_\infty[(l'_1,l'_2) - (l_1,l_2)] \end{array}$$

So we have proved that:

$$\|H(l'_1,l'_2) - H(l_1,l_2)\| \leq (\|l'_2\| + \|l_1\|) N_\infty[(l'_1,l'_2) - (l_1,l_2)] \qquad (4)$$

Suppose now that $N_\infty[(l'_1,l'_2) - (l_1,l_2)] \leq 1$. Then:

$$\begin{array}{rcl} \|l'_2\| & \leq & \|l'_2 - l_2\| + \|l_2\| \\ & \leq & \max(\|l'_1 - l_1\|, \|l'_2 - l_2\|) + \|l_2\| \\ & = & N_\infty[(l'_1,l'_2) - (l_1,l_2)] + \|l_2\| \\ & \leq & 1 + \|l_2\| \end{array}$$

and consequently, using (4) we obtain:

$$\|H(l'_1,l'_2) - H(l_1,l_2)\| \leq (1 + \|l_1\| + \|l_2\|) N_\infty[(l'_1,l'_2) - (l_1,l_2)]$$

Hence, assuming $(l_1,l_2) \in X$ given and $\epsilon > 0$, defining $\eta > 0$ as $\eta = \min[1, (1 + \|l_1\| + \|l_2\|)^{-1}\epsilon]$, it is clear that:

$$N_\infty[(l'_1,l'_2) - (l_1,l_2)] \leq \eta \;\Rightarrow\; \|H(l'_1,l'_2) - H(l_1,l_2)\| \leq \epsilon$$

Having proved in 2. that the product topology on X is induced by the norm N_∞, it follows that H is continuous at (l_1,l_2). This being true for all $(l_1,l_2) \in X$, H is continuous.

5. Let $K : U \to \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$ be the map defined by $K(a) = (d\psi(\phi(a)), d\phi(a))$ for all $a \in U$. Note that given $a \in U$, having assumed that ϕ is of class C^1 on U, in particular the differential $d\phi(a)$ is a well-defined element of $\mathcal{L}_{\mathbf{R}}(E,F)$. Furthermore, having assumed that ψ is of class C^1 on V and $\phi(U) \subseteq V$, in particular $\phi(a) \in V$ and the differential $d\psi(\phi(a))$ is a well-defined element of $\mathcal{L}_{\mathbf{R}}(F,G)$. It follows that $K(a)$ is a well-defined element of $X = \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$. So K is a well-defined map, which has indeed values in X. From exercise (10), in order to show that K is continuous, it is sufficient to show that each coordinate

mapping $a \to d\psi(\phi(a))$ and $a \to d\phi(a)$ is continuous. However, since ϕ is of class C^1, the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E,F)$ is a continuous map. Similarly, since ψ is of class C^1, the differential $d\psi : V \to \mathcal{L}_{\mathbf{R}}(F,G)$ is a continuous map. Since $\phi : U \to F$ is differentiable on U, it follows from exercise (7) that it is continuous. Since $\phi(U) \subseteq V$, we conclude that $d\psi \circ \phi : U \to \mathcal{L}_{\mathbf{R}}(F,G)$ is a continuous map. Having proved that the two coordinate mappings $d\phi$ and $d\psi \circ \phi$ are continuous, we have proved that K is a continuous map.

6. Let $a \in U$. Then ϕ is differentiable at a and ψ is differentiable at $\phi(a) \in V$. From theorem (110), it follows that $\psi \circ \phi$ is differentiable at a. This being true for all $a \in U$, $\psi \circ \phi$ is differentiable on U.

7. From theorem (110), for all $a \in U$ we have:

$$\begin{aligned} d(\psi \circ \phi)(a) &= d\psi(\phi(a)) \circ d\phi(a) \\ &= H(d\psi(\phi(a)), d\phi(a)) \\ &= H(K(a)) \\ &= H \circ K(a) \end{aligned}$$

This being true for all $a \in U$, $d(\psi \circ \phi) = H \circ K$.

8. Given three $\mathbf{R}$-normed spaces E, F and G, given U open in E and V open in F, given $\phi : U \to F$ and $\psi : V \to G$ of class C^1 with $\phi(U) \subseteq V$, we have shown in 6. that $\psi \circ \phi$ is differentiable on U. Furthermore, we have shown in 7. that $d(\psi \circ \phi)$ can be expressed as $d(\psi \circ \phi) = H \circ K$, where $K : U \to \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F)$ has been shown in 5. to be continuous, and $H : \mathcal{L}_{\mathbf{R}}(F,G) \times \mathcal{L}_{\mathbf{R}}(E,F) \to \mathcal{L}_{\mathbf{R}}(E,G)$ has been shown in 4. to be continuous. It follows that $d(\psi \circ \phi) : U \to \mathcal{L}_{\mathbf{R}}(E,G)$ is a continuous map. From definition (130), we conclude that $\psi \circ \phi : U \to G$ is of class C^1. This completes the proof of theorem (111).

Exercise 11

Exercise 12.

1. Let E be an $\mathbf{R}$-normed space. Let $a, b \in \mathbf{R}$, $a < b$. We assume that $f : [a,b] \to E$ and $g : [a,b] \to \mathbf{R}$ are two continuous maps which are differentiable at every point of $]a,b[$, with:

$$\forall t \in]a,b[\ ,\ \|f'(t)\| \leq g'(t)$$

Let $\epsilon > 0$. We define $\phi_\epsilon : [a,b] \to \mathbf{R}$ by:

$$\phi_\epsilon(t) \stackrel{\triangle}{=} \|f(t) - f(a)\| - g(t) + g(a) - \epsilon(t - a)$$

for all $t \in [a,b]$. For all $x, y \in E$, we have:

$$|\ \|x\| - \|y\|\ | \leq \|x - y\|$$

It follows that the map $\|\cdot\| : E \to \mathbf{R}^+$ is a continuous map. Having assumed that $f : [a, b] \to E$ is continuous, from:

$$\|f(t) - f(a) - f(t') + f(a)\| = \|f(t) - f(t')\|$$

it is clear that $t \to f(t) - f(a)$ is also continuous. Hence, we see that $t \to \|f(t) - f(a)\|$ is continuous and finally, since g is itself continuous, we conclude that ϕ_ϵ is a continuous map.

2. Let $E_\epsilon = \{t \in [a, b] : \phi_\epsilon(t) \leq \epsilon\}$ and $c = \sup E_\epsilon$. Since $\phi_\epsilon(a) = 0$, in particular $\phi_\epsilon(a) \leq \epsilon$ and consequently $a \in E_\epsilon$. This shows that $a \leq c$. Furthermore, for all $t \in E_\epsilon$, we have $t \leq b$. So b is an upper-bound of E_ϵ. Since c is the smallest of such upper-bounds, we obtain $c \leq b$. We have proved that $c \in [a, b]$. In particular $\phi_\epsilon(c)$ is well-defined. Suppose $\phi_\epsilon(c) > \epsilon$. Then $c \in \phi_\epsilon^{-1}(]\epsilon, +\infty[)$. Having proved that ϕ_ϵ is continuous, the fact that $]\epsilon, +\infty[$ is an open subset of $\mathbf{R}$ implies that $\phi_\epsilon^{-1}(]\epsilon, +\infty[)$ is an open subset of $[a, b]$. From $c \in \phi_\epsilon^{-1}(]\epsilon, +\infty[)$, we deduce the existence of $\eta > 0$, such that:

$$]c - \eta, c + \eta[\cap[a, b] \subseteq \phi_\epsilon^{-1}(]\epsilon, +\infty[) \quad (5)$$

Now let $t \in E_\epsilon$. Then $t \in [a, b]$, $t \leq c$ and furthermore $\phi_\epsilon(t) \leq \epsilon$. It follows from (5) that t cannot be an element of $]c - \eta, c]$, and consequently $t \leq c - \eta$. This shows that $c - \eta$ is an upper-bound of E_ϵ, contradicting the fact that c is the smallest of such upper-bounds. Indeed, note that $c \in [a, b]$ implies that $c < +\infty$ and consequently $c - \eta < c$. Our initial assumption is therefore absurd, and we have proved that $\phi_\epsilon(c) \leq \epsilon$. When dealing with this question, it may have been tempting to some to use the following argument: since $E_\epsilon = \{t \in [a, b] : \phi_\epsilon \leq \epsilon\}$ and ϕ_ϵ is continuous, E_ϵ is a closed subset of $[a, b]$, which furthermore is non-empty since $a \in E_\epsilon$. It follows that $c = \sup E_\epsilon \in E_\epsilon$. This argument is valid, but one has to be careful about the following point: if E_ϵ is a closed subset of $\mathbf{R}$, it may not be true that $\sup E_\epsilon \in E_\epsilon$ (take $E_\epsilon = \mathbf{R}$). The fact that E_ϵ is a closed subset of $[a, b]$ (which is itself closed in $\bar{\mathbf{R}}$) is of crucial importance here. A rigorous argument goes as follows: The topology of $[a, b]$ is induced by that of $\mathbf{R}$, but also more importantly by that of $\bar{\mathbf{R}}$. The fact that E_ϵ is closed in $[a, b]$ implies the existence of some F closed in $\bar{\mathbf{R}}$, such that $E_\epsilon = F \cap [a, b]$. However, the interval $[a, b]$ is also closed in $\bar{\mathbf{R}}$ (it is compact and $\bar{\mathbf{R}}$ is metrizable). So E_ϵ is in fact also a closed subset of $\bar{\mathbf{R}}$. Being non-empty, we conclude from exercise (9) (part 5.) of Tutorial 8 that $c = \sup E_\epsilon \in E_\epsilon$.

3. Since ϕ_ϵ is continuous and $\phi_\epsilon(a) = 0$, there exists $h > 0$ with:

$$\forall t \in [a, a + h[\cap[a, b] \ , \ \phi_\epsilon(t) \leq |\phi_\epsilon(t)| \leq \epsilon$$

4. Since $a < b$, we have $]a, a + h[\cap[a, b] \neq \emptyset$. Let t be an arbitrary element of $]a, a + h[\cap[a, b]$. Then $t \in [a, b]$ and from 3. we have $\phi_\epsilon(t) \leq \epsilon$. So $t \in E_\epsilon$ and consequently $t \leq c$. Since $t \in]a, a + h[$, we conclude in particular that $a < c$. So $c \in]a, b]$.

5. Suppose $c \in]a,b[$. By assumption, both derivatives $f'(c) \in E$ and $g'(c) \in \mathbf{R}$ are well-defined. From the existence of $f'(c)$ we deduce that of $\delta_1 > 0$ such that for all $t \neq c$:

$$t \in]c-\delta_1, c+\delta_1[\cap[a,b] \ \Rightarrow \ \left\| \frac{f(t)-f(c)}{t-c} - f'(c) \right\| \leq \frac{\epsilon}{2} \tag{6}$$

From the existence of $g'(c)$ we deduce that of $\delta_2 > 0$ such that for all $t \neq c$:

$$t \in]c-\delta_2, c+\delta_2[\cap[a,b] \ \Rightarrow \ \left| \frac{g(t)-g(c)}{t-c} - g'(c) \right| \leq \frac{\epsilon}{2} \tag{7}$$

Let $\delta = \min(\delta_1,\delta_2) > 0$. Having assumed that $c < b$, the set $]c,b]\cap]c,c+\delta[$ is not empty. Let t_0 be an arbitrary element of $]c,b]\cap]c,c+\delta[$. From (6) we obtain:

$$\begin{aligned} \left\| \frac{f(t_0)-f(c)}{t_0-c} \right\| &\leq \|f'(c)\| + \left\| \frac{f(t_0)-f(c)}{t_0-c} - f'(c) \right\| \\ &\leq \|f'(c)\| + \frac{\epsilon}{2} \end{aligned}$$

From (7) we obtain:

$$\begin{aligned} g'(c) &= \frac{g(t_0)-g(c)}{t_0-c} + g'(c) - \frac{g(t_0)-g(c)}{t_0-c} \\ &\leq \frac{g(t_0)-g(c)}{t_0-c} + \left| \frac{g(t_0)-g(c)}{t_0-c} - g'(c) \right| \\ &\leq \frac{g(t_0)-g(c)}{t_0-c} + \frac{\epsilon}{2} \end{aligned}$$

6. Since $\|f'(c)\| \leq g'(c)$, it follows from 5. that:

$$\begin{aligned} \|f(t_0)-f(c)\| &= |t_0-c| \cdot \left\| \frac{f(t_0)-f(c)}{t_0-c} \right\| \\ &\leq |t_0-c| \cdot (\|f'(c)\| + \epsilon/2) \\ &\leq |t_0-c| \cdot (g'(c) + \epsilon/2) \\ &\leq |t_0-c| \cdot \left(\frac{g(t_0)-g(c)}{t_0-c} + \frac{\epsilon}{2} + \frac{\epsilon}{2} \right) \\ &= (t_0-c) \cdot \left(\frac{g(t_0)-g(c)}{t_0-c} + \epsilon \right) \\ &= g(t_0) - g(c) + \epsilon(t_0-c) \end{aligned}$$

7. Having proved in 2. that $\phi_\epsilon(c) \leq \epsilon$, we have:

$$\|f(c)-f(a)\| \leq g(c) - g(a) + \epsilon(c-a) + \epsilon$$

8. From 6. and 7. we obtain:

$$\begin{aligned} \|f(t_0)-f(a)\| &\leq \|f(t_0)-f(c)\| + \|f(c)-f(a)\| \\ &\leq g(t_0) - g(c) + \epsilon(t_0-c) \end{aligned}$$

$$\begin{aligned} &+ \quad g(c) - g(a) + \epsilon(c-a) + \epsilon \\ &= \quad g(t_0) - g(a) + \epsilon(t_0 - a) + \epsilon \end{aligned}$$

9. It follows from 8. that $\phi_\epsilon(t_0) \leq \epsilon$. This shows that $t_0 \in E_\epsilon$ and consequently $t_0 \leq c$. This contradicts that fact that $t_0 \in]c, b]$. Hence, our initial assumption that $c \in]a, b[$ is absurd.

10. We have proved in 4. that $c \in]a, b]$. However, $c \in]a, b[$ leads to a contradiction. It follows that $c = b$. Since $\phi_\epsilon(c) \leq \epsilon$, we conclude that $\phi_\epsilon(b) \leq \epsilon$. Hence:

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b-a) + \epsilon$$

11. Given an $\mathbf{R}$-normed space E, given $a, b \in \mathbf{R}$, $a < b$, given two continuous maps $f : [a, b] \to E$ and $g : [a, b] \to \mathbf{R}$ which are differentiable at every point of $]a, b[$, and such that:

$$\forall t \in]a, b[\ ,\ \|f'(t)\| \leq g'(t)$$

we have proved in 10. that given $\epsilon > 0$:

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b-a) + \epsilon$$

This being true for all $\epsilon > 0$, we conclude that:

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

This completes the proof of theorem (112)

Exercise 12

Exercise 13.

1. Let U be open in $\mathbf{R}^n$ and $\phi : U \to E$ be a map where E is an $\mathbf{R}$-normed space. We assume that ϕ is differentiable at $a \in U$. The differential $d\phi(a) \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ satisfies the requirements of definition (128). Given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbf{R}^n$, the condition $\|x\| \leq \delta$ implies that $a + x \in U$ and:

$$\|\phi(a+x) - \phi(a) - d\phi(a)(x)\| \leq \epsilon\|x\|$$

If $(e_1, \ldots, e_n)$ denotes the canonical basis of $\mathbf{R}^n$, then for all $h \in \mathbf{R}$ with $|h| \leq \delta$, given an arbitrary $i \in \mathbf{N}_n$, the vector $x = he_i$ is such that $\|x\| = |h| \leq \delta$. So $a + he_i \in U$ and:

$$\|\phi(a + he_i) - \phi(a) - d\phi(a)(he_i)\| \leq \epsilon\|he_i\|$$

This being true for all $h \in \mathbf{R}$ with $|h| \leq \delta$, we have proved that:

$$\lim_{h\neq 0, h\to 0} \frac{1}{\|he_i\|}\|\phi(a + he_i) - \phi(a) - d\phi(a)(he_i)\| = 0$$

2. Let $i \in \mathbf{N}_n$. Putting $l = d\phi(a)$, we have:

$$\left\| \frac{\phi(a+he_i)-\phi(a)}{h} - l(e_i) \right\| = \frac{1}{|h|}\|\phi(a+he_i)-\phi(a)-hl(e_i)\|$$
$$= \frac{1}{\|he_i\|}\|\phi(a+he_i)-\phi(a)-l(he_i)\|$$

and it follows from 1. that:

$$\lim_{h\neq 0, h\to 0} \left\| \frac{\phi(a+he_i)-\phi(a)}{h} - d\phi(a)(e_i) \right\| = 0$$

We conclude from definition (131) that the partial derivative $\frac{\partial \phi}{\partial x_i}(a)$ exists and is equal to $d\phi(a)(e_i)$.

3. Given an open subset U of $\mathbf{R}^n$, given a map $\phi : U \to E$ where E is an $\mathbf{R}$-normed space, we have proved that if ϕ is differentiable at $a \in U$, then $\frac{\partial \phi}{\partial x_i}(a)$ exists for all $i \in \mathbf{N}_n$, and furthermore:

$$\frac{\partial \phi}{\partial x_i}(a) = d\phi(a)(e_i)$$

Let $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$. We have:

$$\begin{aligned} d\phi(a)(h) &= d\phi(a)\left(\sum_{i=1}^n h_i e_i\right) \\ &= \sum_{i=1}^n h_i d\phi(a)(e_i) \\ &= \sum_{i=1}^n h_i \frac{\partial \phi}{\partial x_i}(a) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(a) h_i \end{aligned}$$

This completes the proof of theorem (113).

Exercise 13

Exercise 14.

1. Let U be open in $\mathbf{R}^n$ and $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. Suppose ϕ is differentiable at $a, b \in U$. Let $i \in \mathbf{N}_n$. From exercise (3), we have:

$$\|d\phi(b) - d\phi(a)\| = \sup \|(d\phi(b) - d\phi(a))(x)\|$$

where the supremum is taken over all $x \in \mathbf{R}^n$ with $\|x\| = 1$. Taking $x = e_i$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{R}^n$, since $\|e_i\| = 1$ we obtain in particular:

$$\begin{aligned} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\| &= \|d\phi(b)(e_i) - d\phi(a)(e_i)\| \\ &= \|(d\phi(b) - d\phi(a))(e_i)\| \\ &\leq \|d\phi(b) - d\phi(a)\| \end{aligned}$$

2. We now assume that ϕ is of class C^1 on U. In particular, $d\phi(a)$ exists for all $a \in U$. From theorem (113), it follows that the partial derivative $\frac{\partial \phi}{\partial x_i}(a)$ exists for all $a \in U$ and $i \in \mathbf{N}_n$. Furthermore, the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ is continuous. It follows from 1. that $\frac{\partial \phi}{\partial x_i} : U \to E$ is also a continuous map. We have proved that if ϕ is of class C^1 on U, then $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U, for all $i \in \mathbf{N}_n$.

Exercise 14

Exercise 15.

1. Let U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. We assume that $\frac{\partial \phi}{\partial x_i}$ exists on U, and is continuous at $a \in U$, for all $i \in \mathbf{N}_n$. We define $l : \mathbf{R}^n \to E$ by:

$$l(h) \stackrel{\triangle}{=} \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i$$

for all $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$. Having assumed that $\frac{\partial \phi}{\partial x_i}$ exists on U for all $i \in \mathbf{N}_n$, in particular each $\frac{\partial \phi}{\partial x_i}(a)$ is a well-defined element of E. Given $h \in \mathbf{R}^n$, each product $\frac{\partial \phi}{\partial x_i}(a) \cdot h_i$ of the scalar $h_i \in \mathbf{R}$ and vector $\frac{\partial \phi}{\partial x_i}(a)$ is therefore itself well-defined. It follows that $l(h)$ is a well-defined element of E. So $l : \mathbf{R}^n \to E$ is a well-defined map, which furthermore is clearly linear. Given $h \in \mathbf{R}^n$, using the Cauchy-Schwarz inequality (50), we obtain:

$$\begin{aligned}
\|l(h)\| &= \left\| \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i \right\| \\
&\leq \sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(a) h_i \right\| \\
&= \sum_{i=1}^{n} |h_i| \cdot \left\| \frac{\partial \phi}{\partial x_i}(a) \right\| \\
&\leq \left(\sum_{i=1}^{n} |h_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{1/2} \\
&= M \cdot \|h\|
\end{aligned}$$

where we have put $M = (\sum_{i=1}^{n} \|\frac{\partial \phi}{\partial x_i}(a)\|^2)^{1/2}$. Having found $M \in \mathbf{R}^+$ such that $\|l(h)\| \leq M\|h\|$ for all $h \in \mathbf{R}^n$, we conclude from exercise (3) that l is continuous. So we have proved that $l \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$. Of course the fact that l is continuous is a consequence of a far more general result: any linear linear map $l : F \to E$ defined on a finite dimensional normed space F, is in fact continuous. We shall prove this result in a later tutorial.

2. Let $\epsilon > 0$. Having assumed that each partial derivative $\frac{\partial \phi}{\partial x_i}$ is continuous at $a \in U$, for all $i \in \mathbf{N}_n$ there exists $\eta_i > 0$ such that for all $h \in \mathbf{R}^n$, the

condition $\|h\| < \eta_i$ implies that $a + h \in U$ and furthermore:

$$\left\|\frac{\partial \phi}{\partial x_i}(a+h) - \frac{\partial \phi}{\partial x_i}(a)\right\| \leq \epsilon$$

Taking $\eta = \min(\eta_1, \ldots, \eta_n) > 0$, the condition $\|h\| < \eta$ implies that $a + h \in U$ and furthermore:

$$\forall i \in \mathbf{N}_n\ ,\ \left\|\frac{\partial \phi}{\partial x_i}(a+h) - \frac{\partial \phi}{\partial x_i}(a)\right\| \leq \epsilon$$

3. Let $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$ with $\|h\| < \eta$. Let $(e_1, \ldots, e_n)$ denote the canonical basis of $\mathbf{R}^n$. Let $k_0 = a$ and for all $i \in \mathbf{N}_n$:

$$k_i = a + \sum_{j=1}^{i} h_j e_j$$

From 2. the condition $\|h'\| < \eta$ implies that $a + h' \in U$, for all $h' \in \mathbf{R}^n$. However, it is clear that $k_0 \in U$ and for all $i \in \mathbf{N}_n$:

$$\begin{aligned} \|k_i - a\| &= \left\|\sum_{j=1}^{i} h_j e_j\right\| \\ &= \left(\sum_{j=1}^{i} h_j^2\right)^{1/2} \\ &\leq \left(\sum_{j=1}^{n} h_j^2\right)^{1/2} = \|h\| < \eta \end{aligned}$$

So $k_i = a + (k_i - a)$ is an element of U. Moreover:

$$\begin{aligned} \phi(a+h) &- \phi(a) - l(h) = \phi(k_n) - \phi(k_0) - l(h) \\ &= \sum_{i=1}^{n} (\phi(k_i) - \phi(k_{i-1})) - l(h) \\ &= \sum_{i=1}^{n} \left(\phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a)\right) \end{aligned}$$

4. Let $i \in \mathbf{N}_n$. Suppose $h_i > 0$ and define $f_i : [0, h_i] \to E$ by:

$$f_i(t) = \phi(k_{i-1} + t e_i) - \phi(k_{i-1}) - t \frac{\partial \phi}{\partial x_i}(a)$$

for all $t \in [0, h_i]$. Given $t \in [0, h_i]$, the product $t \cdot \frac{\partial \phi}{\partial x_i}(a)$ is a well-defined element of E, and $\phi(k_{i-1})$ is also well-defined since $k_{i-1} \in U$.

Furthermore, following a similar proof to that of 3.:

$$\begin{aligned}\|k_{i-1}+te_i-a\| &= \left(\sum_{j=1}^{i-1}h_j^2+t^2\right)^{1/2}\\ &\leq \left(\sum_{j=1}^{i}h_j^2\right)^{1/2} < \eta\end{aligned}$$

and consequently $k_{i-1}+te_i \in U$. It follows that $\phi(k_{i-1}+te_i)$ is also a well-defined element of E. We conclude that $f_i(t)$ is a well-defined element of E of all $t\in[0,h_i]$, and we have proved that $f_i:[0,h_i]\to E$ is well-defined. Let $t\in[0,h_i]$ and $u\neq 0$ such that $t+u\in[0,h_i]$. Define $k^*=k_{i-1}+te_i\in U$. We have:

$$\begin{aligned}\frac{f_i(t+u)-f_i(t)}{u} &= \frac{1}{u}[\phi(k_{i-1}+(t+u)e_i)-\phi(k_{i-1})\\ &- (t+u)\frac{\partial\phi}{\partial x_i}(a)]\\ &- \frac{1}{u}[\phi(k_{i-1}+te_i)-\phi(k_{i-1})-t\frac{\partial\phi}{\partial x_i}(a)]\\ &= \frac{1}{u}[\phi(k^*+ue_i)-\phi(k^*)]-\frac{\partial\phi}{\partial x_i}(a)\end{aligned}$$

Having assumed that the partial derivative $\frac{\partial\phi}{\partial x_i}$ exists at every point of U, in particular it exists at $k^*\in U$, and consequently from definition (131), we obtain:

$$\lim_{u\neq 0,u\to 0}\frac{f_i(t+u)-f_i(t)}{u}=\frac{\partial\phi}{\partial x_i}(k^*)-\frac{\partial\phi}{\partial x_i}(a)$$

So the derivative $f_i'(t)$ exists for all $t\in[0,h_i]$ and furthermore:

$$f_i'(t)=\frac{\partial\phi}{\partial x_i}(k_{i-1}+te_i)-\frac{\partial\phi}{\partial x_i}(a)$$

5. The fact that f_i is continuous on $[0,h_i]$ can be seen in various ways. One the one hand, having proved that $f_i'(t)$ exists for all $t\in[0,h_i]$, f_i is necessarily continuous on $[0,h_i]$. On the other hand, the map $t\to k_{i-1}+te_i$ is clearly continuous with values in U, while $\phi:U\to E$ being differentiable, is also continuous by virtue of exercise (7). It follows that $t\to\phi(k_{i-1}+te_i)$ is a continuous map, and it is clear from there that f_i is continuous on $[0,h_i]$. Having proved that $f_i'(t)$ exists for all $t\in[0,h_i]$, in particular $f_i'(t)$ exists for all $t\in]0,h_i[$. So f_i is differentiable on $]0,h_i[$. Note that our use of the word *differentiable* means nothing more here than the existence of the derivative $f_i'(t)$. Fortunately, from 4. of exercise (7), this is equivalent to the word *differentiable* in the sense of definition (128). Since we have proved that for all $t\in]0,h_i[$, we have $\|k_{i-1}+te_i-a\|<\eta$, using 2. we

obtain:
$$\|f_i'(t)\| = \left\| \frac{\partial \phi}{\partial x_i}(k_{i-1} + te_i) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \epsilon$$

6. Since $f_i : [0, h_i] \to E$ is continuous on $[0, h_i]$ and differentiable on $]0, h_i[$ with $\|f_i'(t)\| \leq \epsilon$ for all $t \in]0, h_i[$, applying theorem (112) we obtain:
$$\begin{aligned}\left\| \phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right\| &= \|f_i(h_i)\| \\ &= \|f_i(h_i) - f_i(0)\| \\ &\leq \epsilon(h_i - 0) = \epsilon |h_i|\end{aligned}$$

7. Suppose now that $h_i \leq 0$. The inequality obtained in 6. is clearly true if $h_i = 0$. So we may assume that $h_i < 0$. Similarly to 4. we define $f_i : [h_i, 0] \to E$ by:
$$f_i(t) = \phi(k_{i-1} + te_i) - \phi(k_{i-1}) - t \frac{\partial \phi}{\partial x_i}(a)$$
Then f_i is well-defined, continuous on $[h_i, 0]$ and differentiable on $]h_i, 0[$, with the property that:
$$f_i'(t) = \frac{\partial \phi}{\partial x_i}(k_{i-1} + te_i) - \frac{\partial \phi}{\partial x_i}(a)$$
for all $t \in]h_i, 0[$. In particular, we still have $\|f_i'(t)\| \leq \epsilon$ for all $t \in]h_i, 0[$, and applying theorem (112) once more, we obtain:
$$\begin{aligned}\left\| \phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right\| &= \|f_i(h_i)\| \\ &= \|f_i(0) - f_i(h_i)\| \\ &\leq \epsilon(0 - h_i) = \epsilon |h_i|\end{aligned}$$
Hence, the inequality obtained in 6. is still valid for $h_i \leq 0$.

8. Using 3. and 6. we obtain:
$$\begin{aligned}\|\phi(a + h) &- \phi(a) - l(h)\| \\ &= \left\| \sum_{i=1}^{n} \left(\phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right) \right\| \\ &\leq \sum_{i=1}^{n} \left\| \phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right\| \\ &\leq \sum_{i=1}^{n} \epsilon |h_i| \\ &\leq \left(\sum_{i=1}^{n} \epsilon^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{n} |h_i|^2 \right)^{1/2} \\ &= \epsilon \sqrt{n} \|h\|\end{aligned}$$

This has been proved for any $h \in \mathbf{R}^n$ with $\|h\| < \eta$.

9. Given U open in $\mathbf{R}^n$, given a map $\phi : U \to E$ where E is an $\mathbf{R}$-normed space, having assumed that $\frac{\partial \phi}{\partial x_i}$ exists at every point of U and is continuous at $a \in U$ for all $i \in \mathbf{N}_n$, given $\epsilon > 0$, we have found $\eta > 0$ such that for all $h \in \mathbf{R}^n$, the condition $\|h\| < \eta$ implies that $a + h \in U$ together with:

$$\|\phi(a+h) - \phi(a) - l(h)\| \leq \epsilon\sqrt{n}\|h\|$$

Applying this result to $\epsilon/\sqrt{n}$ instead of ϵ, taking $\delta = \eta/2 > 0$, the condition $\|h\| \leq \delta$ implies that $a + h \in U$ together with:

$$\|\phi(a+h) - \phi(a) - l(h)\| \leq \epsilon\|h\|$$

It follows that $l \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ satisfies the requirements of definition (128), and we have proved that ϕ is differentiable at $a \in U$. This completes the proof of theorem (114).

Exercise 15

Exercise 16.

1. Let U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map where E is an $\mathbf{R}$-normed space. We assume that for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U. Then in particular, given $a \in U$, for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}$ exists at every point of U and is continuous at $a \in U$. From theorem (114), it follows that ϕ is differentiable at a. This being true for all $a \in U$, we have proved that ϕ is differentiable on U.

2. Let $a, b \in U$ and $h \in \mathbf{R}^n$. Since ϕ is differentiable at a and b, using theorem (113) and the Cauchy-Schwarz inequality (50):

$$\begin{aligned}
\|(d\phi(b) - d\phi(a))(h)\| &= \|d\phi(b)(h) - d\phi(a)(h)\| \\
&= \left\| \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(b) h_i - \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i \right\| \\
&\leq \sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\| \cdot |h_i| \\
&\leq \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |h_i|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{1/2} \cdot \|h\|
\end{aligned}$$

3. Let $a, b \in U$. It follows from 2. together with 4. of exercise (5):

$$\|d\phi(b) - d\phi(a)\| \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{1/2} \tag{8}$$

4. Let $a \in U$ and $\epsilon > 0$ be given. Having assumed that $\frac{\partial \phi}{\partial x_i}$ is continuous on U for all $i \in \mathbf{N}_n$, in particular $\frac{\partial \phi}{\partial x_i}$ is continuous at a for all $i \in \mathbf{N}_n$. Hence, given $i \in \mathbf{N}_n$, there exists $\eta_i > 0$ such that for all $b \in U$, we have:
$$\|a-b\| \leq \eta_i \ \Rightarrow \ \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \frac{\epsilon}{\sqrt{n}}$$
Taking $\eta = \min(\eta_1, \dots, \eta_n) > 0$, for all $b \in U$, using (8):
$$\|a-b\| \leq \eta \ \Rightarrow \ \|d\phi(b) - d\phi(a)\| \leq \epsilon$$
This shows that $d\phi : U \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ is continuous at a. This being true for all $a \in U$, we have proved that $d\phi$ is continuous.

5. Given U open in $\mathbf{R}^n$, given a map $\phi : U \to E$ where E is an $\mathbf{R}$-normed space, having assumed that $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U for all $i \in \mathbf{N}_n$, we have proved that ϕ is differentiable on U and furthermore that $d\phi : U \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ is a continuous map. From definition (130), it follows that ϕ is of class C^1 on U. Conversely, if we assume that ϕ is of class C^1 on U, then from 2. of exercise (14), $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U for all $i \in \mathbf{N}_n$. This completes the proof of theorem (115).

Exercise 16

Exercise 17. Let E, F be two $\mathbf{R}$-normed spaces and $l \in \mathcal{L}_{\mathbf{R}}(E, F)$. Let U be an open subset of E. Let $l_{|U}$ denote the restriction of l to U, i.e. the map $l_{|U} : U \to F$ defined by $(l_{|U})(x) = l(x)$ for all $x \in U$. Let $a \in U$. Since U is open in E, there exists $\delta > 0$ such that the condition $\|h\| < \delta$ implies $a + h \in U$ for all $h \in E$. So there exists $\delta > 0$ such that the condition $\|h\| \leq \delta$ implies $a + h \in U$, and:
$$\|(l_{|U})(a+h) - (l_{|U})(a) - l(h)\| = \|l(a+h) - l(a) - l(h)\| = 0$$
It follows that l satisfies the requirements of definition (128) in relation to $l_{|U}$. We conclude that $l_{|U}$ is differentiable at a, and furthermore that $d(l_{|U})(a) = l \in \mathcal{L}_{\mathbf{R}}(E, F)$. This being true for all $a \in U$, $l_{|U}$ is differentiable on U, and since $d(l_{|U}) : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is the constant map $d(l_{|U})(x) = l$, $d(l_{|U})$ is continuous. So $l_{|U}$ is of class C^1.

Exercise 17

Exercise 18.

1. Let $E_1, \dots, E_n$, $n \geq 1$, be n $\mathbf{K}$-normed spaces. Let $E = E_1 \times \dots \times E_n$. Let $p \in [1, +\infty[$, and for all $x = (x_1, \dots, x_n) \in E$:
$$\|x\|_p \ \stackrel{\triangle}{=} \ \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$
$$\|x\|_\infty \ \stackrel{\triangle}{=} \ \max_{i=1,\dots,n} \|x_i\|$$

We claim that $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are norms on E. It is clear that $\|x\|_p = 0$ and $\|x\|_\infty = 0$ are both equivalent to $x_i = 0$ for all $i \in \mathbf{N}_n$, which is itself equivalent to $x = 0$. Note that although the same notation is used, the 0's of $\|x\|_p = 0$, $x_i = 0$ and $x = 0$, do not refer to the same things. The first one is the element of $\mathbf{R}$, the second is the identity element of E_i and the last one refers to $(0,\ldots,0)$, the identity element of E, where the entries of $(0,\ldots,0)$ are themselves *different zeroes*, each particular one being the identity element of the corresponding E_i... We have not yet defined an *Abelian group* in these tutorials, but we shall still venture the following comment: in the context where an Abelian group is clearly understood ($\mathbf{R}$ is an Abelian group, a vector space is an Abelian group), it is customary to denote its identity element by 0. Now for all $x \in E$ and $\alpha \in \mathbf{K}$ we have $\|\alpha x\|_\infty = |\alpha| \cdot \|x\|_\infty$, and furthermore:

$$\begin{aligned}
\|\alpha x\|_p &= \|\, \alpha \cdot (x_1,\ldots,x_n)\, \|_p \\
&= \|\, (\alpha x_1,\ldots,\alpha x_n)\, \|_p \\
&= \left(\sum_{i=1}^n \|\alpha x_i\|^p \right)^{1/p} \\
&= \left(\sum_{i=1}^n (|\alpha| \cdot \|x_i\|)^p \right)^{1/p} \\
&= \left(|\alpha|^p \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \\
&= |\alpha| \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} = |\alpha| \cdot \|x\|_p
\end{aligned}$$

It remains to prove the triangle inequalities for $\|\cdot\|_\infty$ and $\|\cdot\|_p$. Let $x \in E$ and $y \in E$. For all $i \in \mathbf{N}_n$, we have:

$$\begin{aligned}
\|x_i + y_i\| &\leq \|x_i\| + \|y_i\| \\
&\leq \max_i \|x_i\| + \max_i \|y_i\| \\
&= \|x\|_\infty + \|y\|_\infty
\end{aligned}$$

This being true for all $i \in \mathbf{N}_n$, we obtain:

$$\|x + y\|_\infty = \max_{i=1,\ldots,n} \|x_i + y_i\| \leq \|x\|_\infty + \|y\|_\infty$$

In order to prove the triangle inequality for $\|\cdot\|_p$, one may think of two possible strategies: On the one hand, it is likely that mimicking the proof of theorem (43) will lead to a valid and simplified proof of the triangle inequality, the crucial point being the convexity of $x \to x^p$, $x > 0$, for $p \in [1, +\infty[$. On the other hand, it is possible to re-interpret the triangle inequality in a way which makes it a particular case of theorem (43). This is the approach we shall follow: Let $x = (x_1,\ldots,x_n) \in E$ and $y =$

$(y_1, \ldots, y_n) \in E$. Define $\Omega = \mathbf{N}_n$ and let $\mathcal{F} = \mathcal{P}(\Omega)$ be the power set of Ω. Then $\mathcal{F}$ is obviously a σ-algebra on Ω. We define $\mu : \mathcal{F} \to [0, +\infty]$ by:

$$\forall A \in \mathcal{F}\ ,\ \mu(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} 1_A(i)$$

Then $\mu(\emptyset) = 0$, and if $A = \uplus_{k\geq 1} A_k$ is a union of pairwise disjoint elements of $\mathcal{F}$, we have $1_A = \sum_{k\geq 1} 1_{A_k}$ and consequently:

$$\begin{aligned} \mu(A) &= \sum_{i=1}^{n} 1_A(i) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{+\infty} 1_{A_k} \right)(i) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{+\infty} 1_{A_k}(i) \right) \\ \text{All terms} \geq 0 \ \rightarrow \quad &= \sum_{k=1}^{+\infty} \left(\sum_{i=1}^{n} 1_{A_k}(i) \right) \\ &= \sum_{k=1}^{+\infty} \mu(A_k) \end{aligned}$$

So μ is a measure on $(\Omega, \mathcal{F})$. We define $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ by setting $f(i) = \|x_i\|$ and $g(i) = \|y_i\|$ for all $i \in \Omega$. Then f and g are non-negative, and clearly measurable since $\mathcal{F}$ is the whole of the power set $\mathcal{P}(\Omega)$. Applying theorem (43), we obtain:

$$\begin{aligned} \|x+y\|_p &= \|\,(x_1, \ldots, x_n) + (y_1, \ldots, y_n)\,\|_p \\ &= \|\,(x_1 + y_1, \ldots, x_n + y_n)\,\|_p \\ &= \left(\sum_{i=1}^{n} \|x_i + y_i\|^p \right)^{1/p} \\ &\leq \left(\sum_{i=1}^{n} (\|x_i\| + \|y_i\|)^p \right)^{1/p} \\ \mu(\{i\}) = 1 \ \rightarrow \quad &= \left(\sum_{i=1}^{n} \int_{\{i\}} (f+g)^p d\mu \right)^{1/p} \\ &= \left(\int (f+g)^p d\mu \right)^{1/p} \\ \text{Theorem (43)} \rightarrow \quad &\leq \left(\int f^p d\mu \right)^{1/p} + \left(\int g^p d\mu \right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} + \left(\sum_{i=1}^n \|y_i\|^p\right)^{1/p} \\
&= \|x\|_p + \|y\|_p
\end{aligned}
$$

This completes our proof of the triangle inequality for $\|\cdot\|_p$, and we have proved that $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are norms on E.

2. Let $\mathcal{T}_p$ and $\mathcal{T}_\infty$ denote the topologies induced on E by $\|\cdot\|_p$ and $\|\cdot\|_\infty$ respectively. Let $\mathcal{T}$ denote the product topology on E. For all $x \in E$, we have:

$$
\begin{aligned}
\|x\|_p &= \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \\
&\leq \left(\sum_{i=1}^n (\|x\|_\infty)^p\right)^{1/p} \\
&= n^{1/p} \cdot \|x\|_\infty \\
&= n^{1/p} \cdot (\max_i \|x_i\|^p)^{1/p} \\
&\leq n^{1/p} \cdot \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} = n^{1/p} \cdot \|x\|_p
\end{aligned}
$$

Having proved that $\|\cdot\|_p \leq n^{1/p}\|\cdot\|_\infty \leq n^{1/p}\|\cdot\|_p$, it follows from exercise (3) that the identity mapping $j : (E, \|\cdot\|_p) \to (E, \|\cdot\|_\infty)$ is a homeomorphism, i.e. that j and j^{-1} are continuous. This shows that $\mathcal{T}_p = \mathcal{T}_\infty$. In order to prove that $\mathcal{T} \subseteq \mathcal{T}_\infty$, it is sufficient to prove that $\mathcal{T}_\infty$ contains every open rectangle in E. Hence, we consider $A = A_1 \times \ldots \times A_n$, where each A_i is an open subset of E_i. Suppose $x = (x_1, \ldots, x_n)$ is an element of A. Then for all $i \in \mathbf{N}_n$, x_i is an element of A_i which is open in E_i. There exists $\epsilon_i > 0$ such that $B(x_i, \epsilon_i) \subseteq A_i$, where $B(x_i, \epsilon_i)$ denotes the open ball in E_i. Let $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n) > 0$ and let $B_\infty(x, \epsilon)$ denote the open ball in E, relative to the norm $\|\cdot\|_\infty$. For all $y = (y_1, \ldots, y_n) \in E$, we have:

$$
\begin{aligned}
y \in B_\infty(x, \epsilon) &\Leftrightarrow \|y - x\|_\infty < \epsilon \\
&\Leftrightarrow \max_i \|y_i - x_i\| < \epsilon \\
&\Rightarrow \|y_i - x_i\| < \epsilon_i, \forall i \in \mathbf{N}_n \\
&\Leftrightarrow y_i \in B(x_i, \epsilon_i), \forall i \in \mathbf{N}_n \\
&\Rightarrow y_i \in A_i, \forall i \in \mathbf{N}_n \\
&\Leftrightarrow y \in A
\end{aligned}
$$

This shows that $B_\infty(x, \epsilon) \subseteq A$, and we have proved that for all $x \in A$, there exists $\epsilon > 0$ such that $B_\infty(x, \epsilon) \subseteq A$. It follows that $A \in \mathcal{T}_\infty$ and we have proved that $\mathcal{T} \subseteq \mathcal{T}_\infty$. Note that there is no need to consider separately the case $A = \emptyset$ in the previous argument. To show that $\mathcal{T}_\infty \subseteq \mathcal{T}$, consider $A \in \mathcal{T}_\infty$. Given $x \in A$, there exists $\epsilon > 0$ such that $B_\infty(x,\epsilon) \subseteq A$. For all $y \in E$, we have:

$$\begin{aligned} y \in B_\infty(x,\epsilon) &\Leftrightarrow \|y-x\|_\infty < \epsilon \\ &\Leftrightarrow \max_i \|y_i - x_i\| < \epsilon \\ &\Leftrightarrow \|y_i - x_i\| < \epsilon, \forall i \in \mathbf{N}_n \\ &\Leftrightarrow y_i \in B(x_i,\epsilon), \forall i \in \mathbf{N}_n \\ &\Leftrightarrow y \in B(x_1,\epsilon) \times \ldots \times B(x_n,\epsilon) \end{aligned}$$

It follows that $B_\infty(x,\epsilon) = B(x_1,\epsilon)\times\ldots\times B(x_n,\epsilon)$ and $B_\infty(x,\epsilon)$ is therefore an open rectangle in E, and in particular an element of the product topology $\mathcal{T}$. Hence, for all $x \in A$, there exists some $A_x \in \mathcal{T}$ such that $x \in A_x \subseteq A$. From $A = \cup_{x\in A} A_x$ we conclude that $A \in \mathcal{T}$, and we have proved that $\mathcal{T}_\infty \subseteq \mathcal{T}$. This completes our proof of $\mathcal{T}_p = \mathcal{T}_\infty = \mathcal{T}$.

3. Although we have not explicitly justified this point, E is a **K**-vector space as defined in (89), where the scalar multiplication and vector addition are given by the formulas:

$$\begin{aligned} \alpha\cdot(x_1,\ldots,x_n) &\stackrel{\triangle}{=} (\alpha x_1,\ldots,\alpha x_n) \\ (x_1+\ldots,x_n)+(y_1,\ldots,y_n) &\stackrel{\triangle}{=} (x_1+y_1,\ldots,x_n+y_n) \end{aligned}$$

For all $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ elements of E, and $\alpha \in \mathbf{K}$. Since $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are norms on E, it follows from definition (125) that $(E,\|\cdot\|_p)$ and $(E,\|\cdot\|_\infty)$ are **K**-normed spaces. Having proved that $\mathcal{T}_p = \mathcal{T}_\infty = \mathcal{T}$, we conclude that the norm topologies on E relative to both $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are equal to the product topology on E.

Exercise 18

Exercise 19. Let E and F be two **R**-normed spaces. Let U be open in E and $\phi,\psi : U \to F$ be two maps. We assume that both ϕ and ψ are differentiable at $a \in U$. Let $\alpha \in \mathbf{R}$. Let $k = d\phi(a)$ and $l = d\psi(a)$. Since both k and l are elements of $\mathcal{L}_{\mathbf{R}}(E,F)$, from exercise (4) the map $m = k + \alpha l$ is an element of $\mathcal{L}_{\mathbf{R}}(E,F)$. To show that $\phi+\alpha\psi$ is differentiable at a with $d(\phi+\alpha\psi)(a) = m$, we have to show that m satisfies the requirements of definition (128), in relation to $\phi+\alpha\psi$. There is nothing to do if $\alpha = 0$, so we may assume that $\alpha \neq 0$. Since both k and l satisfy the requirements of definition (128), in relation to ϕ and ψ respectively, given $\epsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $h \in E$, $\|h\| \leq \delta_1$ implies that $a + h \in U$, with:

$$\|\phi(a+h) - \phi(a) - k(h)\| \leq \frac{\epsilon}{2}\|h\|$$

and $\|h\| \leq \delta_2$ implies that $a + h \in U$, with:

$$\|\psi(a+h) - \psi(a) - l(h)\| \leq \frac{\epsilon}{2|\alpha|}\|h\|$$

Note that to obtain δ_1 and δ_2, we obviously applied definition (128) to different values of 'ϵ'. Defining $\chi = \phi + \alpha\psi$, if $\delta = \min(\delta_1, \delta_2) > 0$, the condition $\|h\| \leq \delta$ implies that $a + h \in U$, with:

$$\begin{array}{rcl} \|\chi(a+h) - \chi(a) - m(h)\| & \leq & \|\phi(a+h) - \phi(a) - k(h)\| \\ & + & |\alpha| \cdot \|\psi(a+h) - \psi(a) - l(h)\| \\ & \leq & \frac{\epsilon}{2}\|h\| + |\alpha|\frac{\epsilon}{2|\alpha|}\|h\| \\ & = & \epsilon\|h\| \end{array}$$

This shows that m satisfies the requirements of definition (128), and we have proved that $\chi = \phi + \alpha\psi$ is differentiable with $d\chi(a) = m$, i.e.:

$$d(\phi + \alpha\psi)(a) = d\phi(a) + \alpha d\psi(a)$$

Exercise 19

Exercise 20.

1. Let E and F be two $\mathbf{K}$-normed spaces. Let N_E and N_F be two norms on E and F, inducing the same topologies as the norm topologies on E and F respectively. From definition (127), the set $\mathcal{L}_{\mathbf{K}}(E, F)$ is that of all linear maps $l : E \to F$ which are continuous. In the presence of alternative norms N_E and N_F on E and F respectively, the word *continuous* is potentially vague, as it may not be clear which topologies are being referred to. Fortunately, by assumption the norms $\|\cdot\|$ and N_E induce the same topology on E, whereas $\|\cdot\|$ and N_F induce the same topology on F. As far as *continuity* is concerned, it is therefore unnecessary to be more specific about which particular norm on E ($\|\cdot\|$ or N_E), and which particular norm on F ($\|\cdot\|$ or N_F) is being considered. Consequently, the set $\mathcal{L}_{\mathbf{K}}(E, F)$ is unambiguously defined, without the need to introduce more precise but cumbersome notations such as $\mathcal{L}_{\mathbf{K}}[(E, \|\cdot\|), (F, \|\cdot\|)]$ or $\mathcal{L}_{\mathbf{K}}[(E, N_E), (F, N_F)]$ etc.

2. Let $id_E : (E, \|\cdot\|) \to (E, N_E)$ be the identity mapping. Since $\|\cdot\|$ and N_E induce the same topology on E, if A is open with respect to the topology induced by N_E, then $A = id_E^{-1}(A)$ is also open with respect to the topology induced by $\|\cdot\|$. It follows that id_E is a continuous map.

3. Having proved that $id_E : (E, \|\cdot\|) \to (E, N_E)$ is a continuous map, being also linear, it follows from exercise (3) that there exists $M_E \in \mathbf{R}^+$ such that:
$$\forall x \in E \ , \ N_E[id_E(x)] \leq M_E\|x\|$$
If $M_E = 0$ (which is possible when E is reduced to the trivial case $E = \{0\}$), it is always possible to replace M_E by an arbitrary positive constant. Hence, there exists $M_E > 0$ such that $N_E \leq M_E\|\cdot\|$. However, since $\|\cdot\|$ and N_E induce the same topology on E, the map $id_E^{-1} : (E, N_E) \to (E, \|\cdot\|)$ is also continuous. Hence, we can find $M_E^* > 0$ such that:
$$\forall x \in \ E \ , \ \|id_E^{-1}(x)\| \leq M_E^* N_E(x)$$

Defining $m_E = 1/M_E^* > 0$, we obtain $m_E\|\cdot\| \leq N_E$. We have proved the existence of $m_E, M_E > 0$ such that:

$$\forall x \in E\ ,\ m_E\|x\| \leq N_E(x) \leq M_E\|x\|$$

4. Since $\|\cdot\|$ and N_F induce the same topology on F, applying 3. to the space F and the norms $\|\cdot\|$ and N_F, we obtain the existence of $m_F, M_F > 0$ such that:

$$\forall y \in F\ ,\ m_F\|y\| \leq N_F(y) \leq M_F\|y\|$$

Let $l \in \mathcal{L}_{\mathbf{K}}(E,F)$ and $x \in E$ with $N_E(x) = 1$. We have:

$$\begin{aligned} \|N_F(l(x))\| &\leq M_F\|l(x)\| \\ &\leq M_F\|l\|\cdot\|x\| \\ &\leq M_F\|l\|\cdot\frac{N_E(x)}{m_E} = \frac{M_F}{m_E}\|l\| \end{aligned}$$

Defining $M = M_F/m_E > 0$, we have proved that $M\|l\|$ is an upper-bound of all $\|N_F(l(x))\|$'s as x ranges through the set of all $x \in E$ with $N_E(x) = 1$. Since $N(l)$ is by definition the smallest of such upper-bounds, we obtain $N(l) \leq M\|l\|$. This being true for all $l \in \mathcal{L}_{\mathbf{K}}(E,F)$, we have found $M > 0$ such that $N \leq M\|\cdot\|$. In order to show the existence of $m > 0$ such that $m\|\cdot\| \leq N$, one may reach a quick conclusion by interchanging the roles of $\|\cdot\|$ and N_E on the one hand, and $\|\cdot\|$ and N_F on the other hand, to obtain $M^* > 0$ such that $\|\cdot\| \leq M^*N$, and conclude with $m = 1/M^*$. As this may seem confusing or unconvincing to some, we shall proceed without emphasis to this symmetry. Let $x \in E$ be such that $\|x\| = 1$. Using 3. of exercise (5) applied to the norms N_E on E, N_F on F, and associated N on $\mathcal{L}_{\mathbf{K}}(E,F)$:

$$\begin{aligned} \|l(x)\| &\leq \frac{1}{m_F}N_F(l(x)) \\ \text{3. of ex. (5)} \rightarrow \quad &\leq \frac{1}{m_F}N(l)\cdot N_E(x) \\ &\leq \frac{1}{m_F}N(l)M_E\|x\| = \frac{M_E}{m_F}N(l) \end{aligned}$$

Defining $m = m_F/M_E > 0$, we have proved that $m^{-1}N(l)$ is an upper-bound of all $\|l(x)\|$'s as x ranges through the set of all $x \in E$ with $\|x\| = 1$. Since $\|l\|$ is the smallest of such upper-bounds, we obtain $\|l\| \leq m^{-1}N(l)$, or equivalently $m\|l\| \leq N(l)$. This being true for all $l \in \mathcal{L}_{\mathbf{K}}(E,F)$, we have found $m > 0$ such that $m\|\cdot\| \leq N$. Hence, there is $m, M > 0$ such that:

$$\forall l \in \mathcal{L}_{\mathbf{K}}(E,F)\ ,\ m\|l\| \leq N(l) \leq M\|l\|$$

5. Having found $m, M > 0$ such that $m\|\cdot\| \leq N \leq M\|\cdot\|$, it is clear from exercise (3) that $j : (\mathcal{L}_{\mathbf{K}}(E,F), \|\cdot\|) \rightarrow (\mathcal{L}_{\mathbf{K}}(E,F), N)$, the identity mapping, is a homeomorphism, i.e. that both j and j^{-1} are continuous. It follows that $\|\cdot\|$ and N induce the same topology on $\mathcal{L}_{\mathbf{K}}(E,F)$. Indeed,

let $\mathcal{T}_{\|\cdot\|}$ and $\mathcal{T}_N$ be the topologies on $\mathcal{L}_{\mathbf{K}}(E,F)$ induced by $\|\cdot\|$ and N respectively. Let $A \in \mathcal{T}_N$. Since j is continuous, $A = j^{-1}(A)$ is an element of $\mathcal{T}_{\|\cdot\|}$. This shows that $\mathcal{T}_N \subseteq \mathcal{T}_{\|\cdot\|}$, and similarly $\mathcal{T}_{\|\cdot\|} \subseteq \mathcal{T}_N$.

6. Suppose that $\mathbf{K} = \mathbf{R}$ and $\phi : U \to F$ is differentiable at $a \in U$. Let $l = d\phi(a) \in \mathcal{L}_{\mathbf{R}}(E,F)$. Our assumption of ϕ being differentiable at a, means specifically that l satisfies the requirements of definition (128), in relation to the normed spaces $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$. Saying that ϕ is also differentiable at a with respect to the norms N_E and N_F, is just an informal way of saying that l should also satisfy the requirements of definition (128), in relation to the normed spaces (E, N_E) and (F, N_F). This is exactly what we need to prove. For this purpose, we consider $m_E, M_E > 0$ such that $m_E\|\cdot\| \leq N_E \leq M_E\|\cdot\|$, and $m_F, M_F > 0$ such that $m_F\|\cdot\| \leq N_F \leq M_F\|\cdot\|$. Let $\epsilon > 0$ be given. Applying definition (128) to $\epsilon' = \epsilon m_E/M_F$ in relation to $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$, there exists $\delta' > 0$ such that for all $h \in E$, the condition $\|h\| \leq \delta'$ implies that $a + h \in U$, and furthermore:
$$\|\phi(a+h) - \phi(a) - l(h)\| \leq \epsilon \frac{m_E}{M_F}\|h\|$$
Defining $\delta = m_E\delta' > 0$, for all $h \in E$ the condition $N_E(h) \leq \delta$ implies that $m_E\|h\| \leq m_E\delta'$ and consequently $\|h\| \leq \delta'$. Hence, the condition $N_E(h) \leq \delta$ implies that $a + h \in U$ and furthermore:
$$\begin{aligned} N_F(\phi(a+h) - \phi(a) - l(h)) &\leq M_F\|\phi(a+h) - \phi(a) - l(h)\| \\ &\leq M_F \epsilon \frac{m_E}{M_F}\|h\| \\ &\leq M_F \epsilon \frac{m_E}{M_F}\frac{N_E(h)}{m_E} \\ &= \epsilon N_E(h) \end{aligned}$$
This shows that l satisfies the requirements of definition (128) in relation to the normed spaces (E, N_E) and (F, N_F). We have proved that changing the norms on E and F with equivalent norms N_E and N_F, i.e. norms inducing the same topologies on E and F, does not affect the differentiability of $\phi : U \to F$ at $a \in U$, or the value of the differential $d\phi(a) \in \mathcal{L}_{\mathbf{R}}(E,F)$.

7. Suppose that $K = \mathbf{R}$ and $\phi : U \to F$ is of class C^1 on U. In particular, ϕ is differentiable on U. It follows from 6. that ϕ is also differentiable on U with respect to the norms N_E and N_F. Let $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E,F)$ be the differential of ϕ. From 6., $d\phi$ is also the differential of ϕ with respect to the norms N_E and N_F. Having assumed that ϕ is of class C^1 on U, the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E,F)$ is continuous. More precisely, $d\phi$ is a continuous map, with respect to the norm topology on $\mathcal{L}_{\mathbf{R}}(E,F)$ and the topology on U induced by the norm topology on E. If we replace the norms on E and F by N_E and N_F respectively, by assumption the norm topology on E is unchanged, and so is the topology on U. From 5. the topology on $\mathcal{L}_{\mathbf{R}}(E,F)$ is also unchanged. It follows that $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E,F)$ is also

continuous with respect to the topologies on U and $\mathcal{L}_{\mathbf{R}}(E,F)$ induced by the norms N_E and N_F. This shows that ϕ is of class C^1 on U, with respect to the norms N_E and N_F.

Exercise 20

Exercise 21.

1. Let $F = F_1 \times \ldots \times F_p$ be the product of p, $p \geq 1$, $\mathbf{R}$-normed spaces. Given $i \in \mathbf{N}_p$, let $p_i : F \to F_i$ be the canonical projection defined by $p_i(x_1, \ldots, x_p) = x_i$ for all $x = (x_1, \ldots, x_p) \in F$. Given $x = (x_1, \ldots, x_p) \in F$ and $y = (y_1, \ldots, y_p) \in F$, given $\alpha \in \mathbf{R}$, we have:

$$\begin{aligned}
p_i(x + \alpha y) &= p_i[(x_1, \ldots, x_p) + \alpha \cdot (y_1, \ldots, y_p)] \\
&= p_i[(x_1, \ldots, x_p) + (\alpha y_1, \ldots, \alpha y_p)] \\
&= p_i[(x_1 + \alpha y_1, \ldots, x_p + \alpha y_p)] \\
&= x_i + \alpha y_i \\
&= p_i(x) + \alpha p_i(y)
\end{aligned}$$

Hence, $p_i : F \to F_i$ is a linear map. From exercise (10), p_i is continuous with respect to the product topology on F. From exercise (18), the product topology on F coincides with the norm topology on F viewed as an $\mathbf{R}$-normed space. So p_i is also continuous with respect to the norm topology on F. This shows that $p_i \in \mathcal{L}_{\mathbf{R}}(F, F_i)$. Note that there is no need to be very specific about which norm on F is being referred to, by virtue of exercise (18) and (20). It is understood that any norm chosen on F, if not specifically of a type described in exercise (18), will at least induce the same topology, i.e. the product topology on F. To show that p_i is continuous, assuming for example that F is endowed with the norm $\|\cdot\|_q$ of exercise (18) with $q \in [1, +\infty[$, one can argue directly that for all $x \in F$:

$$\|p_i(x)\| = \|x_i\| \leq \left(\sum_{i=1}^{p} \|x_i\|^q \right)^{1/q} = \|x\|_q$$

It follows from exercise (3) that p_i is continuous.

2. Given $i \in \mathbf{N}_p$, let $u_i : F_i \to F$ be defined as:

$$\forall x_i \in F_i \ , \ u_i(x_i) \stackrel{\triangle}{=} (0, \ldots, \overbrace{x_i}^{i}, \ldots, 0)$$

For all $x_i, y_i \in F_i$ and $\alpha \in \mathbf{R}$, we have:

$$\begin{aligned}
u_i(x_i + \alpha y_i) &= (0, \ldots, x_i + \alpha y_i, \ldots, 0) \\
&= (0, \ldots, x_i, \ldots, 0) + \alpha \cdot (0, \ldots, y_i, \ldots, 0) \\
&= u_i(x_i) + \alpha \cdot u_i(y_i)
\end{aligned}$$

Hence, $u_i : F_i \to F$ is linear. Using the norm $\|\cdot\|_\infty$ on F as defined in exercise (18), we obtain:

$$\|u_i(x_i)\|_\infty = \max(0, \ldots, \|x_i\|, \ldots, 0) = \|x_i\|$$

and it follows from exercise (3) that $u_i : F_i \to F$ is continuous. We have proved that $u_i \in \mathcal{L}_{\mathbf{R}}(F_i, F)$. Now for all $x \in F$:

$$\begin{aligned}\left(\sum_{i=1}^{p} u_i \circ p_i\right)(x) &= \sum_{i=1}^{p}(u_i \circ p_i)(x) \\ &= \sum_{i=1}^{p} u_i(p_i(x)) \\ &= \sum_{i=1}^{p}(0, \ldots, x_i, \ldots, 0) \\ &= (x_1, \ldots, x_p) = x\end{aligned}$$

This being true for all $x \in F$, we obtain:

$$\sum_{i=1}^{p} u_i \circ p_i = id_F$$

where $id_F : F \to F$ denotes the identity mapping. It follows that if E is an $\mathbf{R}$-normed space, U is open in E and $\phi : U \to F$ is a map, then:

$$\begin{aligned}\phi &= id_F \circ \phi \\ &= \left(\sum_{i=1}^{p} u_i \circ p_i\right) \circ \phi \\ &= \sum_{i=1}^{p}(u_i \circ p_i) \circ \phi \\ &= \sum_{i=1}^{p} u_i \circ (p_i \circ \phi) \\ &= \sum_{i=1}^{p} u_i \circ \phi_i\end{aligned}$$

where $\phi_i : U \to F_i$ is defined as $\phi = p_i \circ \phi$.

3. Suppose $\phi : U \to F$ is differentiable at $a \in U$. Let $i \in \mathbf{N}_p$. Having proved in 1. that $p_i \in \mathcal{L}_{\mathbf{R}}(F, F_i)$, it follows from exercise (17) that $p_i : F \to F_i$ is differentiable on F, with $dp_i(x) = p_i$ for all $x \in F$. Applying theorem (110), we conclude that $p_i \circ \phi = \phi_i$ is differentiable at $a \in U$, with:

$$\begin{aligned}d\phi_i(a) &= d(p_i \circ \phi)(a) \\ &= dp_i(\phi(a)) \circ d\phi(a) = p_i \circ d\phi(a)\end{aligned}$$

4. Suppose that for all $i \in \mathbf{N}_p$, $\phi_i : U \to F_i$ is differentiable at $a \in U$. Having proved in 2. that $u_i \in \mathcal{L}_{\mathbf{R}}(F_i, F)$, it follows from exercise (17) that $u_i : F_i \to F$ is differentiable on F_i, with $du_i(x_i) = u_i$ for all $x_i \in F_i$.

Applying theorem (110), the map $u_i \circ \phi_i : U \to F$ is therefore differentiable at $a \in U$, with:

$$d(u_i \circ \phi_i)(a) = du_i(\phi_i(a)) \circ d\phi_i(a) = u_i \circ d\phi_i(a)$$

Having proved in 2. that $\phi = \sum_{i=1}^{p} u_i \circ \phi_i$, we conclude from exercise (19) that ϕ is differentiable at $a \in U$, with:

$$\begin{aligned} d\phi(a) &= d\left(\sum_{i=1}^{p} u_i \circ \phi_i\right)(a) \\ &= \sum_{i=1}^{p} d(u_i \circ \phi_i)(a) \\ &= \sum_{i=1}^{p} u_i \circ d\phi_i(a) \end{aligned}$$

5. Let $a, b \in U$. We assume that ϕ is differentiable at a and b. Then $d\phi(a)$ and $d\phi(b)$ are well-defined elements of $\mathcal{L}_{\mathbf{R}}(E, F)$. From 3. $d\phi_i(a)$ and $d\phi_i(b)$ are well-defined elements of $\mathcal{L}_{\mathbf{R}}(E, F_i)$ for all $i \in \mathbf{N}_p$. Given $i \in \mathbf{N}_p$, we claim that:

$$\|d\phi_i(b) - d\phi_i(a)\| \leq \|d\phi(b) - d\phi(a)\|$$

Note that $\|d\phi_i(b) - d\phi_i(a)\|$ is well-defined from exercise (5):

$$\|d\phi_i(b) - d\phi_i(a)\| \stackrel{\triangle}{=} \sup \|(d\phi_i(b) - d\phi_i(a))(x)\|$$

where the *sup* is taken over all $x \in E$ with $\|x\| = 1$. Also:

$$\|d\phi(b) - d\phi(a)\| \stackrel{\triangle}{=} \sup \|(d\phi(b) - d\phi(a))(x)\|$$

where the *sup* is taken over all $x \in E$ with $\|x\| = 1$. Note however that this expression is dependent upon a specific choice of norm on F, in order for $\|(d\phi(b) - d\phi(a))(x)\|$ to be meaningful. As a possible choice, we shall work with the norm $\|\cdot\|_2$ of exercise (18), so that specifically:

$$\|d\phi(b) - d\phi(a)\| \stackrel{\triangle}{=} \sup \|(d\phi(b) - d\phi(a))(x)\|_2$$

where the supremum is taken over all $x \in E$ with $\|x\| = 1$. Now for all $y = (y_1, \ldots, y_p) \in F$ and $i \in \mathbf{N}_p$, we have:

$$\|p_i(y)\| = \|y_i\| \leq \left(\sum_{j=1}^{p} \|y_j\|^2\right)^{1/2} = \|y\|_2$$

Having proved in 3. that $d\phi_i(a) = p_i \circ d\phi(a)$, we have similarly $d\phi_i(b) = p_i \circ d\phi(b)$ and consequently for all $x \in E$ with $\|x\| = 1$:

$$\begin{aligned} \|(d\phi_i(b) - d\phi_i(a))(x)\| &= \|\, p_i[(d\phi(b) - d\phi(a))(x)]\,\| \\ &\leq \|(d\phi(b) - d\phi(a))(x)\|_2 \\ &\leq \|d\phi(b) - d\phi(a)\| \end{aligned}$$

from which we conclude that:

$$\|d\phi_i(b) - d\phi_i(a)\| \leq \|d\phi(b) - d\phi(a)\|$$

6. For all $x \in E$ with $\|x\| = 1$, since $d\phi_i(a) = p_i \circ d\phi(a)$:

$$\begin{aligned}
\|(d\phi(b) - d\phi(a))(x)\| &\stackrel{\triangle}{=} \|(d\phi(b) - d\phi(a))(x)\|_2 \\
&= \left(\sum_{i=1}^{p} \|\, p_i[(d\phi(b) - d\phi(a))(x)]\,\|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{p} \|(d\phi_i(b) - d\phi_i(a))(x)\|^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{p} \|d\phi_i(b) - d\phi_i(a)\|^2\right)^{1/2}
\end{aligned}$$

from which we conclude that:

$$\|d\phi(b) - d\phi(a)\| \leq \left(\sum_{i=1}^{p} \|d\phi_i(b) - d\phi_i(a)\|^2\right)^{1/2}$$

7. Suppose $\phi : U \to F$ is of class C^1 on U. Let $i \in \mathbf{N}_p$. Since ϕ is differentiable on U, from 3. $\phi_i : U \to F_i$ is also differentiable on U . Since $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is a continuous map, it follows from 5. that $d\phi_i : U \to \mathcal{L}_{\mathbf{R}}(E, F_i)$ is also a continuous map. This shows that $\phi_i : U \to F_i$ is of class C^1 on U. We have proved that if ϕ is of class C^1, then $\phi_i = p_i \circ \phi$ is of class C^1 for all $i \in \mathbf{N}_p$. Conversely, suppose all ϕ_i's are of class C^1 on U. Then in particular, all ϕ_i's are differentiable on U. It follows from 4. that ϕ is also differentiable on U. Furthermore, each $d\phi_i : U \to \mathcal{L}_{\mathbf{R}}(E, F_i)$ is a continuous map. In particular, given $a \in U$, each $d\phi_i$ is continuous at a. Given $\epsilon > 0$, for all $i \in \mathbf{N}_p$ there exists $\eta_i > 0$ such that for all $b \in U$:

$$\|b - a\| \leq \eta_i \;\Rightarrow\; \|d\phi_i(b) - d\phi_i(a)\| \leq \frac{\epsilon}{\sqrt{p}}$$

Defining $\eta = \min(\eta_1, \ldots, \eta_p) > 0$, for all $b \in U$, using 6.:

$$\|b - a\| \leq \eta \;\Rightarrow\; \|d\phi(b) - d\phi(a)\| \leq \epsilon$$

This shows that $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is continuous at a. This being true for all $a \in U$, we have proved that $d\phi$ is a continuous map. So ϕ is of class C^1 on U. We conclude that ϕ is of class C^1 on U, if and only if ϕ_i is of class C^1 on U for all $i \in \mathbf{N}_p$. Note that this conclusion would still hold, if F were given any other norm N inducing the product topology on F, instead of $\|\cdot\|_2$. Indeed from exercise (18) the norm $\|\cdot\|_2$ does induce the product topology on F. So any other norm N inducing the product topology, induces the same topology as $\|\cdot\|_2$. It follows from exercise (20) that ϕ being of class C^1 on U relative to the norm $\|\cdot\|_2$, is equivalent to

ϕ being of class C^1 on U relative to the norm N. Given $i \in \mathbf{N}_p$, the map $\phi_i : U \to F_i$ is unaffected by a change of norm on F. It follows that the conclusion we have reached having assumed that F is endowed with the norm $\|\cdot\|_2$, is still valid when F is endowed with the norm N.

8. Given $p+1$ **R**-normed spaces E and $F_1, \ldots, F_p$, given U open in E and $F = F_1 \times \ldots \times F_p$, given a map $\phi = (\phi_1, \ldots, \phi_p) : U \to F$, for all $a \in U$ we have proved in 3. and 4. that ϕ is differentiable at a, if and only if ϕ_i is differentiable at a for all $i \in \mathbf{N}_p$. We have proved in 7. that ϕ is of class C^1 on U, if and only if ϕ_i is of class C^1 on U for all $i \in \mathbf{N}_p$. Now suppose $a \in U$ and ϕ is differentiable at a. For all $h \in E$, using 4. we obtain:

$$\begin{aligned} d\phi(a)(h) &= \left(\sum_{i=1}^{p} u_i \circ d\phi_i(a)\right)(h) \\ &= \sum_{i=1}^{p} (u_i \circ d\phi_i(a))(h) \\ &= \sum_{i=1}^{p} u_i[d\phi_i(a)(h)] \\ &= \sum_{i=1}^{p} (0, \ldots, d\phi_i(a)(h), \ldots, 0) \\ &= (d\phi_1(a)(h), \ldots, d\phi_p(a)(h)) \end{aligned}$$

This completes the proof of theorem (116).

Exercise 21

Exercise 22. Let $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbf{R}^n$ be a map, where U is an open subset of $\mathbf{R}^n$. We assume that ϕ is differentiable at $a \in U$. Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbf{R}^n$. Note that if we consider $(\mathbf{R}, |\cdot|)$ as a normed vector space over itself, then the usual inner-product of $\mathbf{R}^n$ induces the norm $\|\cdot\|_2$ of exercise (18), and in particular, it induces the product topology on $\mathbf{R}^n$. It follows that $\mathbf{R}^n$ is a particular case of finite product of **R**-normed spaces, as per theorem (116). Having assumed that ϕ is differentiable at $a \in U$, from theorem (116) each $\phi_i : U \to \mathbf{R}$ is differentiable at $a \in U$. Given $i \in \mathbf{N}_n$, applying theorem (113) to ϕ_i, it follows that for all $j \in \mathbf{N}_n$, the partial derivative $\frac{\partial \phi_i}{\partial x_j}(a)$ exists and furthermore for all $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$, we have:

$$d\phi_i(a)(h) = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j}(a) h_j$$

In particular, $d\phi_i(a)(e_j) = \frac{\partial \phi_i}{\partial x_j}(a)$ for all $j \in \mathbf{N}_n$. Hence, we obtain from theorem (116):

$$d\phi(a)(e_j) = (d\phi_1(a)(e_j), \ldots, d\phi_n(a)(e_j))$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} d\phi_i(a)(e_j)e_i \\
&= \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_j}(a)e_i = Me_j
\end{aligned}
$$

where $M \in \mathcal{M}_n(\mathbf{R})$ is the $n \times n$ matrix:

$$
M = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(a) & \dots & \frac{\partial \phi_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(a) & \dots & \frac{\partial \phi_n}{\partial x_n}(a) \end{pmatrix}
$$

Having proved that $d\phi(a)(e_j) = Me_j$ for all $j \in \mathbf{N}_n$, we conclude that $d\phi(a) = M$. Now from theorem (116), ϕ being of class C^1 on U is equivalent to ϕ_i being of class C^1 on U for all $i \in \mathbf{N}_n$. From theorem (115), this in turn is equivalent to $\frac{\partial \phi_i}{\partial x_j}$ existing and being continuous on U, for all $j \in \mathbf{N}_n$ and $i \in \mathbf{N}_n$. Hence, we have proved that ϕ is of class C^1 on U, if and only if for all $i, j \in \mathbf{N}_n$, the partial derivative $\frac{\partial \phi_i}{\partial x_j}$ exists and is continuous on U. This completes the proof of theorem (117).

Exercise 22

Exercise 23.

1. The set $\mathcal{M}_n(\mathbf{R})$ of $n \times n$ matrices with entries in $\mathbf{R}$, is the set of all maps $M : \mathbf{N}_n \times \mathbf{N}_n \to \mathbf{R}$, i.e. $\mathcal{M}_n(\mathbf{R}) = \mathbf{R}^{\mathbf{N}_n \times \mathbf{N}_n}$. There is an obvious topology on $\mathcal{M}_n(\mathbf{R})$, namely the one induced by the inner-product:

$$
\langle M, N \rangle \stackrel{\triangle}{=} \sum_{i,j=1}^{n} M_{i,j} N_{i,j}
$$

with associated norm:

$$
\|M\|_2 = \left(\sum_{i,j=1}^{n} M_{i,j}^2 \right)^{1/2}
$$

which induces the product topology on $\mathbf{R}^{\mathbf{N}_n \times \mathbf{N}_n}$, by virtue of exercise (18). In these tutorials, we have consistently identified elements of $\mathcal{M}_n(\mathbf{R})$ with the set of linear maps $l : \mathbf{R}^n \to \mathbf{R}^n$. This set coincides with $\mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$, as every such linear map is continuous. Indeed, if $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbf{R}^n$ and $l : \mathbf{R}^n \to \mathbf{R}^n$ is linear, for all $x = (x_1, \dots, x_n) \in \mathbf{R}^n$:

$$
\begin{aligned}
\|l(x)\| &= \left\| \sum_{i=1}^{n} x_i l(e_i) \right\| \\
&\leq \sum_{i=1}^{n} |x_i| \cdot \|l(e_i)\|
\end{aligned}
$$

$$\leq \left(\sum_{i=1}^{n} \|l(e_i)\|^2\right)^{1/2} \cdot \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = K\|x\|$$

where $K = (\sum_{i=1}^{n} \|l(e_i)\|^2)^{1/2} \in \mathbf{R}^+$. Now, the identification of $\mathcal{M}_n(\mathbf{R})$ with $\mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$ gives us another obvious topology on $\mathcal{M}_n(\mathbf{R})$, namely the one induced by the norm on $\mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$, specifically the norm $\|\cdot\|$ defined by:

$$\|M\| \stackrel{\triangle}{=} \sup\{\|Mx\| : x \in \mathbf{R}^n, \|x\| = 1\}$$

Because we haven't yet proved that all norms on a finite dimensional space induce the same topology, we shall now prove that $\|\cdot\|_2$ and $\|\cdot\|$ induce the same topology on $\mathcal{M}_n(\mathbf{R})$, namely the product topology on $\mathbf{R}^{\mathbf{N}_n \times \mathbf{N}_n}$. Let $M \in \mathcal{M}_n(\mathbf{R})$. We have:

$$\begin{aligned}
\|M\|_2 &= \left(\sum_{i,j=1}^{n} M_{i,j}^2\right)^{1/2} \\
&= \left(\sum_{j=1}^{n}\sum_{i=1}^{n} M_{i,j}^2\right)^{1/2} \\
&= \left(\sum_{j=1}^{n} \|Me_j\|^2\right)^{1/2} \\
\|e_j\| = 1 \;\rightarrow\; &\leq \left(\sum_{j=1}^{n} \|M\|^2\right)^{1/2} = \sqrt{n}\|M\|
\end{aligned}$$

Furthermore, if $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ with $\|x\| = 1$:

$$\begin{aligned}
\|Mx\| &= \left\|\sum_{j=1}^{n} x_j M e_j\right\| \\
&\leq \sum_{j=1}^{n} |x_j| \cdot \|Me_j\| \\
&\leq \left(\sum_{j=1}^{n} \|Me_j\|^2\right)^{1/2} \cdot \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} \\
&= \left(\sum_{j=1}^{n}\sum_{i=1}^{n} M_{i,j}^2\right)^{1/2} \cdot \|x\| \\
&= \|M\|_2
\end{aligned}$$

from which we obtain $\|M\| \leq \|M\|_2$. Hence, we have proved that $\|\cdot\| \leq \|\cdot\|_2 \leq \sqrt{n}\|\cdot\|$, which shows that the identity mapping $j : (\mathcal{M}_n(\mathbf{R}), \|\cdot\|) \rightarrow$

$(\mathcal{M}_n(\mathbf{R}), \|\cdot\|_2)$ is a homeomorphism. So $\|\cdot\|$ and $\|\cdot\|_2$ induce the same topology on $\mathcal{M}_n(\mathbf{R})$, namely the product topology on $\mathbf{R}^{\mathbf{N}_n \times \mathbf{N}_n}$. Having clarified which topology is to be assumed on $\mathcal{M}_n(\mathbf{R})$, it is now meaningful to state that the determinant $\det : \mathcal{M}_n(\mathbf{R}) \to \mathbf{R}$ is a continuous map. As we haven't had a tutorial on the determinant, we shall have to accept this fact. However, for those familiar with the formula:

$$\det M = \sum_\sigma \epsilon(\sigma) M_{1,\sigma(1)} \cdot \ldots \cdot M_{n,\sigma(n)}$$

where the sum is taken over all permutations $\sigma : \mathbf{N}_n \to \mathbf{N}_n$ (and $\epsilon(\sigma) \in \{-1, 1\}$ denotes the *sign* of a permutation σ), the fact that $\det : \mathcal{M}_n(\mathbf{R}) \to \mathbf{R}$ is a continuous map is a lot easier to believe. Indeed, det can be expressed as a linear combination (with coefficients in $\{-1, 1\}$) of products of the form $p_{i_1,j_1} \ldots p_{i_n,j_n}$, where $p_{i,j} : \mathbf{R}^{\mathbf{N}_n \times \mathbf{N}_n} \to \mathbf{R}$ is the (continuous) canonical projection. Having (hopefully) accepted the continuity of $\det : \mathcal{M}_n(\mathbf{R}) \to \mathbf{R}$, we are now in a position to prove that $J(\phi) : \Omega \to \mathbf{R}$ is itself continuous. From definition (132):

$$J(\phi)(a) = \det[d\phi(a)] = (\det \circ d\phi)(a)$$

This being true for all $a \in \Omega$, we obtain $J(\phi) = \det \circ d\phi$. However, since ϕ is assumed to be of class C^1 on Ω, the map $d\phi : \Omega \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$ (or equivalently $d\phi : \Omega \to \mathcal{M}_n(\mathbf{R})$) is a continuous map. It follows that $J(\phi) = \det \circ d\phi : \Omega \to \mathbf{R}$ is itself continuous. Likewise, since $\psi : \Omega' \to \mathbf{R}^n$ is of class C^1 on Ω', $J(\psi) : \Omega' \to \mathbf{R}$ is continuous.

2. Let $I_n : \mathbf{R}^n \to \mathbf{R}^n$ be the identity mapping. From $\psi = \phi^{-1}$ we obtain $\phi \circ \psi = (I_n)_{|\Omega'}$, where $(I_n)_{|\Omega'}$ is the restriction of I_n to Ω'. From exercise (17), $(I_n)_{|\Omega'}$ is differentiable and $d(I_n)_{|\Omega'}(x) = I_n$ for all $x \in \Omega'$. Hence, from theorem (110) and for all $x \in \Omega'$:

$$d\phi(\psi(x)) \circ d\psi(x) = d(\phi \circ \psi)(x) = d(I_n)_{|\Omega'}(x) = I_n$$

3. Similarly to 2., from $\psi \circ \phi = (I_n)_{|\Omega}$ we obtain for all $x \in \Omega$:

$$d\psi(\phi(x)) \circ d\phi(x) = d(\psi \circ \phi)(x) = d(I_n)_{|\Omega}(x) = I_n$$

4. Let $x \in \Omega'$. From 2. and definition (132) we obtain:

$$\begin{array}{rcl} 1 &=& \det I_n \\ &=& \det[d\phi(\psi(x)) \circ d\psi(x)] \\ \text{Granted} \to &=& \det[d\phi(\psi(x))] \det[d\psi(x)] \\ \text{Definition (132)} \to &=& J(\phi)(\psi(x)) J(\psi)(x) \end{array} \qquad (9)$$

It follows in particular that $J(\psi)(x) \neq 0$ for all $x \in \Omega'$.

5. Let $x \in \Omega$. From 3. we have similarly to 4.:

$$1 = \det I_n$$

$$\begin{aligned} &= \det[d\psi(\phi(x)) \circ d\phi(x)] \\ &= \det[d\psi(\phi(x))] \det[d\phi(x)] \\ &= J(\psi)(\phi(x))J(\phi)(x) \end{aligned} \quad (10)$$

and it follows that $J(\phi)(x) \neq 0$ for all $x \in \Omega$. Note that it is perfectly acceptable to deduce $J(\phi)(x) \neq 0$ directly from 3. by interchanging the roles of ϕ and ψ.

6. Let $x \in \Omega'$. Going back to (9), we have:

$$J(\psi)(x) = \frac{1}{J(\phi)(\psi(x))} = \frac{1}{(J(\phi) \circ \psi)(x)}$$

This being true for all $x \in \Omega'$, $J(\psi) = 1/(J(\phi) \circ \psi)$. Similarly, going back to (10) we obtain $J(\phi) = 1/(J(\psi) \circ \phi)$.

Exercise 23

Exercise 24. Let $\Omega \in \mathcal{B}(\mathbf{R}^n)$ be a Borel subset of $\mathbf{R}^n$ and $B \in \mathcal{B}(\Omega)$ be a Borel subset of Ω. Then $dx_{|\Omega}(B)$ is defined by $dx_{|\Omega}(B) = dx(B)$. For this to be meaningful, we need to ensure that $dx(B)$ is well-defined, i.e. that $B \in \mathcal{B}(\mathbf{R}^n)$. This amounts to proving the inclusion $\mathcal{B}(\Omega) \subseteq \mathcal{B}(\mathbf{R}^n)$, which can be seen from theorem (10):

$$\begin{aligned} \mathcal{B}(\Omega) &\triangleq \sigma(\mathcal{T}_\Omega) \\ &\triangleq \sigma((\mathcal{T}_{\mathbf{R}^n})_{|\Omega}) \\ \text{Theorem (10)} \rightarrow \quad &= \sigma(\mathcal{T}_{\mathbf{R}^n})_{|\Omega} \\ &\triangleq \mathcal{B}(\mathbf{R}^n)_{|\Omega} \\ &\triangleq \{B \cap \Omega : B \in \mathcal{B}(\mathbf{R}^n)\} \\ \Omega \in \mathcal{B}(\mathbf{R}^n) \ \rightarrow \quad &\subseteq \mathcal{B}(\mathbf{R}^n) \end{aligned}$$

So $dx_{|\Omega}$ is well-defined, and it is clearly a measure on $(\Omega, \mathcal{B}(\Omega))$.

Exercise 24

Exercise 25.

1. Let Ω, Ω' be open in $\mathbf{R}^n$ and $\phi : \Omega \rightarrow \Omega'$ be a C^1-diffeomorphism. Being open in $\mathbf{R}^n$, in particular Ω and Ω' are Borel subsets of $\mathbf{R}^n$. From exercise (24), it follows that $dx_{|\Omega'}$ is a well-defined measure on $(\Omega', \mathcal{B}(\Omega'))$, while $dx_{|\Omega}$ is a well-defined measure on $(\Omega, \mathcal{B}(\Omega))$. Furthermore, being differentiable, the map $\phi : \Omega \rightarrow \Omega'$ is continuous and therefore measurable. It follows from definition (123) that the image measure $\phi(dx_{|\Omega})$ is a well-defined measure on $(\Omega', \mathcal{B}(\Omega'))$. We have proved that $dx_{|\Omega'}$ and $\phi(dx_{|\Omega})$ are well-defined measures on $(\Omega', \mathcal{B}(\Omega'))$.

2. Let $a \in \Omega'$. Since Ω' is open in $\mathbf{R}^n$, there exists $\eta > 0$ such that $B(a, \eta) \subseteq \Omega'$, where $B(a, \eta)$ denotes the open ball in $\mathbf{R}^n$. Let $0 < \epsilon \leq \eta$. Then $B(a, \epsilon) \subseteq B(a, \eta) \subseteq \Omega'$, and consequently $B(a, \epsilon) = B(a, \epsilon) \cap \Omega'$. Since

$B(a,\epsilon)$ is open in $\mathbf{R}^n$, the equality $B(a,\epsilon) = B(a,\epsilon) \cap \Omega'$ shows that it is also open in Ω'. In particular, $B(a,\epsilon)$ is a Borel subset of Ω'. We have found $\eta > 0$ such that $B(a,\epsilon) \in \mathcal{B}(\Omega')$ for all $\epsilon > 0$ with $\epsilon \leq \eta$. This shows that $B(a,\epsilon) \in \mathcal{B}(\Omega')$ for $\epsilon > 0$ sufficiently small.

3. From 2. $B(a,\epsilon)$ is an element of $\mathcal{B}(\Omega')$ for $\epsilon > 0$ sufficiently small. From 1. $dx_{|\Omega'}$ and $\phi(dx_{|\Omega})$ are well-defined measures on $(\Omega', \mathcal{B}(\Omega'))$. It follows that the quantities $dx_{|\Omega'}(B(a,\epsilon))$ and $\phi(dx_{|\Omega})(B(a,\epsilon))$ are meaningful elements of $[0,+\infty]$ for $\epsilon > 0$ sufficiently small. In fact, from definition (134), we have:
$$dx_{|\Omega'}(B(a,\epsilon)) = dx(B(a,\epsilon)) \in]0,+\infty[$$
It follows that the ratio $\phi(dx_{|\Omega})(B(a,\epsilon))/dx_{|\Omega'}(B(a,\epsilon))$ is well-defined in $[0,+\infty]$ for $\epsilon > 0$ sufficiently small. Hence, it does make sense to investigate whether the limit:
$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))}$$
exists in $[0,+\infty]$, and whether this limit is an element of $\mathbf{R}$.

4. We assume that $d\psi(a) = I_n$. Let $r > 0$ be given. Since I_n satisfies the requirements of definition (128) in relation to ψ at $a \in \Omega'$, there exists $\epsilon_1 > 0$ such that for all $h \in \mathbf{R}^n$, the condition $\|h\| \leq \epsilon_1$ implies that $a + h \in \Omega'$, and:
$$\|\psi(a+h) - \psi(a) - h\| \leq r\|h\|$$

5. Let $h \in \mathbf{R}^n$ with $\|h\| \leq \epsilon_1$. Then $a + h \in \Omega'$, and:
$$\begin{aligned} \|\psi(a+h) - \psi(a)\| &\leq \|\psi(a+h) - \psi(a) - h\| + \|h\| \\ &\leq r\|h\| + \|h\| \\ &= (1+r)\|h\| \end{aligned}$$

6. Let $\epsilon \in]0,\epsilon_1[$ and $x \in B(a,\epsilon)$. Then $h = x - a$ satisfies the condition $\|h\| < \epsilon$, and in particular $\|h\| \leq \epsilon_1$. It follows that $a + h \in \Omega'$ and consequently $x \in \Omega'$. So $B(a,\epsilon) \subseteq \Omega'$. Furthermore, if $x \in B(a,\epsilon)$ and $h = x - a$, we obtain from 5.:
$$\begin{aligned} \|\psi(x) - \psi(a)\| &= \|\psi(a+h) - \psi(a)\| \\ &\leq (1+r)\|h\| \\ &< \epsilon(1+r) \end{aligned}$$
This shows that $\psi(x) \in B(\psi(a), \epsilon(1+r))$. This being true for all $x \in B(a,\epsilon)$, we have proved that:
$$\psi(B(a,\epsilon)) \subseteq B(\psi(a), \epsilon(1+r))$$

7. From 2. of exercise (23), we have $d\phi(\psi(a)) \circ d\psi(a) = I_n$. Since $d\psi(a) = I_n$, we obtain $d\phi(\psi(a)) = I_n$.

8. It follows from 7. that I_n satisfies the requirements of definition (128) in relation to ϕ at $\psi(a) \in \Omega$. Having fixed $r > 0$ in 4., there exists $\epsilon_2 > 0$ such that for all $k \in \mathbf{R}^n$, the condition $\|k\| \leq \epsilon_2$ implies that $\psi(a)+k \in \Omega$, and:

$$\begin{array}{rcl} \|\phi(\psi(a)+k) - a - k\| & = & \|\phi(\psi(a)+k) - \phi(\psi(a)) - I_n(k)\| \\ & \leq & r\|k\| \end{array}$$

9. Let $k \in \mathbf{R}^n$ with $\|k\| \leq \epsilon_2$. Then $\psi(a) + k \in \Omega$, and:

$$\begin{array}{rcl} \|\phi(\psi(a)+k) - a\| & \leq & \|\phi(\psi(a)+k) - a - k\| + \|k\| \\ & \leq & (1+r)\|k\| \end{array}$$

10. Let $\epsilon \in]0, \epsilon_2(1+r)[$. Let $y \in B(\psi(a), \epsilon(1+r)^{-1})$. Define $k = y - \psi(a)$. Then k satisfies the condition $\|k\| < \epsilon(1+r)^{-1}$ and in particular $\|k\| \leq \epsilon_2$. It follows from 9. that $\psi(a) + k \in \Omega$. So $y \in \Omega$, and we have proved that $B(\psi(a), \epsilon(1+r)^{-1}) \subseteq \Omega$. Furthermore, if $y \in B(\psi(a), \epsilon(1+r)^{-1})$ and $k = y - \psi(a)$:

$$\begin{array}{rcl} \|\phi(y) - a\| & = & \|\phi(\psi(a)+k) - a\| \\ \text{From 9. } \rightarrow & \leq & (1+r)\|k\| \\ & < & (1+r)\epsilon(1+r)^{-1} = \epsilon \end{array}$$

So $\phi(y) \in B(a,\epsilon)$, i.e. $y \in \{\phi \in B(a,\epsilon)\}$. We have proved that:

$$B(\psi(a), \frac{\epsilon}{1+r}) \subseteq \{\phi \in B(a,\epsilon)\}$$

11. Suppose $\epsilon > 0$ is such that $B(a,\epsilon) \subseteq \Omega'$. We claim that:

$$\psi(B(a,\epsilon)) = \{\phi \in B(a,\epsilon)\}$$

Let $y \in \psi(B(a,\epsilon))$. There is $x \in B(a,\epsilon)$ such that $y = \psi(x)$. It follows that $\phi(y) = \phi(\psi(x)) = x \in B(a,\epsilon)$. So $y \in \{\phi \in B(a,\epsilon)\}$. This shows the inclusion $\subseteq$. To show the reverse inclusion, suppose $y \in \Omega$ is such that $\phi(y) \in B(a,\epsilon)$. Define $x = \phi(y)$. Then $x \in B(a,\epsilon)$ and $\psi(x) = \psi(\phi(y)) = y$. So $y \in \psi(B(a,\epsilon))$. This shows the inclusion $\supseteq$.

12. Let $\epsilon_0 = \epsilon_1 \wedge \epsilon_2(1+r)$. Let $\epsilon \in]0, \epsilon_0[$. In particular, $\epsilon \in]0, \epsilon_1[$ and it follows from 6. that $B(a,\epsilon) \subseteq \Omega'$. Also, from 6. and 11.:

$$\{\phi \in B(a,\epsilon)\} = \psi(B(a,\epsilon)) \subseteq B(\psi(a), \epsilon(1+r))$$

Moreover, since $\epsilon \in]0, \epsilon_2(1+r)[$, from 10. we obtain:

$$B(\psi(a), \frac{\epsilon}{1+r}) \subseteq \{\phi \in B(a,\epsilon)\}$$

We have proved that $B(a,\epsilon) \subseteq \Omega'$, and:

$$B(\psi(a), \frac{\epsilon}{1+r}) \subseteq \{\phi \in B(a,\epsilon)\} \subseteq B(\psi(a), \epsilon(1+r))$$

13. Let $\epsilon \in]0, \epsilon_0[$. From 12. we have $B(a,\epsilon) \subseteq \Omega'$ and consequently:

$$B(a,\epsilon) = B(a,\epsilon) \cap \Omega' \in \mathcal{B}(\mathbf{R}^n)_{|\Omega'} = \mathcal{B}(\Omega')$$

where the last equality has been fully justified in exercise (24). So $B(a,\epsilon) \in \mathcal{B}(\Omega')$. Using exercise (12) of Tutorial 16:

$$\begin{aligned} dx(B(\psi(a), \frac{\epsilon}{1+r})) &= \frac{\epsilon^n}{(1+r)^n} dx(B(0,1)) \\ &= (1+r)^{-n} dx(B(a,\epsilon)) \\ &= (1+r)^{-n} dx_{|\Omega'}(B(a,\epsilon)) \end{aligned}$$

Moreover:

$$\begin{aligned} dx(B(\psi(a), \epsilon(1+r))) &= \epsilon^n (1+r)^n dx(B(0,1)) \\ &= (1+r)^n dx(B(a,\epsilon)) \\ &= (1+r)^n dx_{|\Omega'}(B(a,\epsilon)) \end{aligned}$$

Finally, since $B(a,\epsilon) \in \mathcal{B}(\Omega')$ and ϕ is measurable, we have $\{\phi \in B(a,\epsilon)\} \in \mathcal{B}(\Omega)$ and consequently from definition (123):

$$dx(\{\phi \in B(a,\epsilon)\}) = dx_{|\Omega}(\{\phi \in B(a,\epsilon)\}) = \phi(dx_{|\Omega})(B(a,\epsilon))$$

14. Let $\epsilon \in]0, \epsilon_0[$. From 12. we have $B(a,\epsilon) \subseteq \Omega'$, and:

$$dx(B(\psi(a), \frac{\epsilon}{1+r})) \leq dx(\{\phi \in B(a,\epsilon)\}) \leq dx(B(\psi(a), \epsilon(1+r)))$$

Since $dx_{|\Omega'}(B(a,\epsilon)) = dx(B(a,\epsilon)) > 0$, using 13. we obtain:

$$(1+r)^{-n} \leq \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} \leq (1+r)^n \tag{11}$$

15. Given $r > 0$, we have found $\epsilon_0 > 0$ such that (11) is true for all $\epsilon \in]0, \epsilon_0[$. Let $\eta > 0$. It is clear that $\lim_{r\to 0}(1+r)^n = 1$. It follows that $(1+r)^n \leq 1+\eta$ for $r > 0$ sufficiently small. Likewise, since $\lim_{r\to 0}(1+r)^{-n} = 1$, we have $1-\eta \leq (1+r)^{-n}$ for $r > 0$ sufficiently small. Hence, given $\eta > 0$, it is possible to find $r > 0$ sufficiently small such that:

$$1-\eta \leq (1+r)^{-n} \leq (1+r)^n \leq 1+\eta$$

It follows that given $\eta > 0$, there exists $\epsilon_0 > 0$ such that:

$$1-\eta \leq \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} \leq 1+\eta$$

for all $\epsilon \in]0, \epsilon_0[$. This shows that:

$$\lim_{\epsilon\downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1$$

Exercise 25

Exercise 26.

1. Let Ω, Ω' be open in $\mathbf{R}^n$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Let $\psi = \phi^{-1}$ and $a \in \Omega'$. Let $A = d\psi(a)$. Then A is a linear map $A : \mathbf{R}^n \to \mathbf{R}^n$. Furthermore, from 2. of exercise (23):

$$d\phi(\psi(a)) \circ d\psi(a) = I_n = d\phi(\psi(a)) \circ A$$

It follows that $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear bijection.

2. Let $\Omega'' = A^{-1}(\Omega)$. From exercise (11) (part 2.) of Tutorial 17, the inverse image $A^{-1}(\Omega)$ of Ω by A coincides with the direct image $A^{-1}(\Omega)$ of Ω by A^{-1}. It follows that the definition of Ω'' does not depend on whether $A^{-1}(\Omega)$ is viewed as an inverse or a direct image.

3. Since $A : \mathbf{R}^n \to \mathbf{R}^n$ is linear and defined on a finite dimensional space, it is continuous. This general statement has not been proved yet, but the particular case at hand can be found in exercise (11) (part 1.) of Tutorial 17. Since Ω is open in $\mathbf{R}^n$, the inverse image $\Omega'' = A^{-1}(\Omega)$ is open in $\mathbf{R}^n$.

4. Let $\tilde{\phi} : \Omega'' \to \Omega'$ be defined by $\tilde{\phi}(x) = \phi \circ A(x)$ for all $x \in \Omega''$. Then $\tilde{\phi} = \phi \circ A_{|\Omega''}$ where $A_{|\Omega''} : \Omega'' \to \mathbf{R}^n$ is the restriction of A to Ω''. Note that for all $x \in \Omega'' = A^{-1}(\Omega)$, we have $A(x) \in \Omega$ and consequently $A_{|\Omega''}(\Omega'') \subseteq \Omega$. This shows that $\tilde{\phi} = \phi \circ A_{|\Omega''}$ is well-defined on Ω'', (and it has indeed values in Ω'). From exercise (17), $A_{|\Omega''}$ is of class C^1 on Ω''. Since $\phi : \Omega \to \Omega'$ is a C^1-diffeomorphism, in particular $\phi : \Omega \to \mathbf{R}^n$ is of class C^1 on Ω. Since $A_{|\Omega''}(\Omega'') \subseteq \Omega$, it follows from theorem (111) that $\tilde{\phi} = \phi \circ A_{|\Omega''}$ is of class C^1 on Ω''. Let $\tilde{\psi} : \Omega' \to \Omega''$ be defined by $\tilde{\psi} = A^{-1} \circ \psi$. Note that for all $x \in \Omega'$, we have $\psi(x) \in \Omega$ and consequently:

$$\tilde{\psi}(x) = A^{-1}(\psi(x)) \in A^{-1}(\Omega) = \Omega''$$

So $\tilde{\psi}$ has indeed values in Ω'' (and it is well-defined on Ω'). For all $x \in \Omega'$, we have:

$$\begin{aligned}(\tilde{\phi} \circ \tilde{\psi})(x) &= \phi \circ A_{|\Omega''} \circ A^{-1} \circ \psi(x)\\ &= \phi \circ A \circ A^{-1} \circ \psi(x)\\ &= \phi \circ \psi(x) = x\end{aligned}$$

and for all $x \in \Omega''$:

$$\begin{aligned}(\tilde{\psi} \circ \tilde{\phi})(x) &= A^{-1} \circ \psi \circ \phi \circ A_{|\Omega''}(x)\\ &= A^{-1} \circ \psi \circ \phi \circ A(x)\\ &= A^{-1} \circ A(x) = x\end{aligned}$$

Hence, we have $\tilde{\phi} \circ \tilde{\psi} = id_{\Omega'}$ and $\tilde{\psi} \circ \tilde{\phi} = id_{\Omega''}$, and we have proved that $\tilde{\phi} : \Omega'' \to \Omega'$ is a bijection with $\tilde{\phi}^{-1} = \tilde{\psi}$. Having assumed $\phi : \Omega \to \Omega'$ to be a C^1-diffeomorphism, in particular $\psi : \Omega' \to \mathbf{R}^n$ is of class C^1 on Ω'. From exercise (17), $A^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ is of class C^1 on $\mathbf{R}^n$. It follows from theorem (111) that $\tilde{\psi} = A^{-1} \circ \psi$ is of class C^1 on Ω'. We have proved that

$\tilde{\phi} : \Omega'' \to \Omega'$ is a bijection, such that $\tilde{\phi} : \Omega'' \to \mathbf{R}^n$ and $\tilde{\phi}^{-1} : \Omega' \to \mathbf{R}^n$ are both of class C^1. From definition (133), we conclude that $\tilde{\phi} : \Omega'' \to \Omega'$ is a C^1-diffeomorphism.

5. Using theorem (110) and exercise (17), we obtain:

$$\begin{array}{rcl} d\tilde{\psi}(a) & = & d(A^{-1} \circ \psi)(a) \\ & = & d(A^{-1})(\psi(a)) \circ d\psi(a) \\ & = & A^{-1} \circ d\psi(a) \\ & = & A^{-1} \circ A = I_n \end{array}$$

6. Since $\tilde{\phi} : \Omega'' \to \Omega'$ is a C^1-diffeomorphism with $\tilde{\psi} = \tilde{\phi}^{-1}$, and $a \in \Omega'$ is such that $d\tilde{\psi}(a) = I_n$, applying 15. of exercise (25):

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1$$

7. Let $\epsilon > 0$ with $B(a,\epsilon) \subseteq \Omega'$. Then $B(a,\epsilon) \in \mathcal{B}(\Omega')$ and:

$$\begin{array}{rcl} \tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon)) & = & dx_{|\Omega''}(\{\tilde{\phi} \in B(a,\epsilon)\}) \\ \text{Definition (134)} \to & = & dx(\{\tilde{\phi} \in B(a,\epsilon)\}) \\ \tilde{\phi} = \phi \circ A_{|\Omega''} \;\; \to & = & dx(\{x \in \Omega'' :\ \phi \circ A(x) \in B(a,\epsilon)\}) \\ (*) & = & dx(\{x \in \Omega'' : A(x) \in \phi^{-1}(B(a,\epsilon))\}) \\ (**) & = & dx(\{x \in \mathbf{R}^n : A(x) \in \phi^{-1}(B(a,\epsilon))\}) \\ \text{Definition (123)} \to & = & A(dx)(\{\phi \in B(a,\epsilon)\}) \\ \text{Theorem (108)} \to & = & |\det A|^{-1} dx(\{\phi \in B(a,\epsilon)\}) \\ \text{Definition (134)} \to & = & |\det A|^{-1} dx_{|\Omega}(\{\phi \in B(a,\epsilon)\}) \\ \text{Definition (123)} \to & = & |\det A|^{-1} \phi(dx_{|\Omega})(B(a,\epsilon)) \end{array}$$

where the first equality stems from definition (123), and equality $(*)$ stems from the equivalence, given $y \in \Omega$:

$$\phi(y) \in B(a,\epsilon) \;\Leftrightarrow\; y \in \phi^{-1}(B(a,\epsilon))$$

As for equality $(**)$, it follows from the fact that for all $x \in \mathbf{R}^n$:

$$A(x) \in \phi^{-1}(B(a,\epsilon)) \;\Rightarrow\; A(x) \in \Omega \;\Rightarrow\; x \in \Omega''$$

8. For $\epsilon > 0$ sufficiently small, we have $B(a,\epsilon) \subseteq \Omega'$, and from 7.:

$$\phi(dx_{|\Omega})(B(a,\epsilon)) = |\det A| \tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon))$$

Hence, from 6. we conclude that:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |\det A|$$

9. From definition (132) we have $\det A = \det[d\psi(a)] = J(\psi)(a)$. Hence, given a C^1-diffeomorphism $\phi : \Omega \to \Omega'$, given $a \in \Omega'$ and $\psi = \phi^{-1}$, we have proved that:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |J(\psi)(a)|$$

This completes the proof of theorem (118).

Exercise 26

Exercise 27.

1. Let Ω, Ω' be open in $\mathbf{R}^n$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Let $\psi = \phi^{-1}$. Let $K \subseteq \Omega'$ be a non-empty compact subset of Ω' such that $dx_{|\Omega'}(K) = 0$. Let $x \in \Omega'$. Since the Lebesgue measure dx on $\mathbf{R}^n$ is locally finite, there exists U open in $\mathbf{R}^n$ such that $x \in U$ and $dx(U) < +\infty$. It follows that $U \cap \Omega'$ is open in Ω', $x \in U \cap \Omega'$ and furthermore:

$$dx_{|\Omega'}(U \cap \Omega') = dx(U \cap \Omega') \leq dx(U) < +\infty$$

Hence, the Lebesgue measure $dx_{|\Omega'}$ on Ω' is also locally finite. From theorem (74), $dx_{|\Omega'}$ is therefore a regular measure on $(\Omega', \mathcal{B}(\Omega'))$. From definition (103), we obtain:

$$dx_{|\Omega'}(K) = \inf\{dx_{|\Omega'}(V) : K \subseteq V,\, V \text{ open in } \Omega'\}$$

Let $\epsilon > 0$. Having assumed that $dx_{|\Omega'}(K) = 0$, in particular we have $dx_{|\Omega'}(K) < \epsilon$. Since $dx_{|\Omega'}(K)$ is the greatest lower-bound of all $dx_{|\Omega'}(V)$'s as V ranges through the set of all open subsets of Ω' with $K \subseteq V$, ϵ cannot be such an lower-bound. Hence, there exists V open in Ω' such that $K \subseteq V (\subseteq \Omega')$ and $dx_{|\Omega'}(V) < \epsilon$. In particular we have $dx_{|\Omega'}(V) \leq \epsilon$.

2. Since V is open in Ω', from definition (23) of the induced topology, there exists U open in $\mathbf{R}^n$ such that $V = U \cap \Omega'$. Since Ω' is open in $\mathbf{R}^n$, we conclude that V is also open in $\mathbf{R}^n$.

3. Let $M = \sup_{x \in K} \|d\psi(x)\|$. Having assumed that $\phi : \Omega \to \Omega'$ is a C^1-diffeomorphism, in particular $\psi : \Omega' \to \mathbf{R}^n$ is of class C^1 on Ω'. Hence, the differential $d\psi : \Omega' \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$ is continuous. Since for all $l, l' \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$ we have:

$$|\, \|l\| - \|l'\| \,| \leq \|l - l'\|$$

the norm $\|\cdot\| : \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}^+$ is also continuous. It follows that $\|d\psi(\cdot)\| : \Omega' \to \mathbf{R}^+$ is a continuous map, and its restriction $\|d\psi(\cdot)\|_{|K}$ is therefore a continuous map defined on the non-empty compact topological space K. From theorem (37), $\|d\psi(\cdot)\|_{|K}$ attains its maximum. In other words, there exists $x_M \in K$ such that:

$$M = \sup_{x \in K} \|d\psi(x)\| = \|d\psi(x_M)\|$$

We conclude that $M \in \mathbf{R}^+$.

4. Let $x \in K$. Since $K \subseteq V$, in particular $x \in V$. Since V is open in $\mathbf{R}^n$, there exists $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq V$. Furthermore, since $K \subseteq \Omega'$, $x \in \Omega'$ and ψ is therefore differentiable at x. Applying definition (128) to ψ and $\epsilon = 1$, there exists $\delta > 0$ such that for all $h \in \mathbf{R}^n$, the condition $\|h\| \leq \delta$ implies that $x + h \in \Omega'$, and:

$$\|\psi(x+h) - \psi(x) - d\psi(x)(h)\| \leq \|h\|$$

Defining $\epsilon_x = \min(\epsilon_1, \delta/3)$, we have $B(x, \epsilon_x) \subseteq V$ and for all $h \in \mathbf{R}^n$ with $\|h\| \leq 3\epsilon_x$, we obtain $x + h \in \Omega'$, and:

$$\begin{aligned} \|\psi(x+h) - \psi(x)\| &\leq \|d\psi(x)(h)\| + \|h\| \\ &\leq \|d\psi(x)\| \cdot \|h\| + \|h\| \\ &\leq \left(\sup_{u \in K} \|d\psi(u)\| \right) \cdot \|h\| + \|h\| \\ &= (M+1)\|h\| \end{aligned}$$

5. Let $x \in K$. Let $y \in B(x, 3\epsilon_x)$. Define $h = y - x$. Then $h \in \mathbf{R}^n$ satisfies the condition $\|h\| \leq 3\epsilon_x$. It follows from 4. that $y = x + h \in \Omega'$, and we have proved that $B(x, 3\epsilon_x) \subseteq \Omega'$. Moreover, applying 4. once more, we obtain:

$$\begin{aligned} \|\psi(y) - \psi(x)\| &= \|\psi(x+h) - \psi(x)\| \\ &\leq (M+1)\|h\| \\ &< 3(M+1)\epsilon_x \end{aligned}$$

and consequently $\psi(y) \in B(\psi(y), 3(M+1)\epsilon_x)$. This being true for all $y \in B(x, 3\epsilon_x)$, we have proved that:

$$\psi(B(x, 3\epsilon_x)) \subseteq B(\psi(x), 3(M+1)\epsilon_x)$$

6. Let $x \in K$. We claim that $\psi(B(x, 3\epsilon_x)) = \{\phi \in B(x, 3\epsilon_x)\}$. Suppose $z \in \psi(B(x, 3\epsilon_x))$. There exists $y \in B(x, 3\epsilon_x)$ such that $z = \psi(y)$. So $\phi(z) = \phi(\psi(y)) = y$ and consequently we have $\phi(z) \in B(x, 3\epsilon_x)$, i.e. $z \in \{\phi \in B(x, 3\epsilon_x)\}$. This shows the inclusion $\subseteq$. To show the reverse inclusion, suppose $\phi(z) \in B(x, 3\epsilon_x)$. Then $z = \psi(\phi(z)) \in \psi(B(x, 3\epsilon_x))$. This shows the inclusion $\supseteq$.

7. We claim the existence of a finite subset $\{x_1, \ldots, x_p\}$ of K with:

$$K \subseteq B(x_1, \epsilon_{x_1}) \cup \ldots \cup B(x_p, \epsilon_{x_p}) \tag{12}$$

Since K is compact and $K \subseteq \cup_{x \in K} B(x, \epsilon_x)$ where each $B(x, \epsilon_x)$ is open, from exercise (2) (part 5.) of Tutorial 8, there exists $\{x_1, \ldots, x_p\} \subseteq K$ such that the inclusion (12) holds. Note that since $K \neq \emptyset$, we must have $p \geq 1$.

8. Since $B(x_1, \epsilon_{x_1}), \ldots, B(x_p, \epsilon_{x_p})$ is a finite sequence of open balls in $\mathbf{R}^n$, from exercise (14) of Tutorial 16, there exists S finite subset of $\mathbf{N}_p$ such

that $(B(x_i, \epsilon_{x_i}))_{i \in S}$ is a family of pairwise disjoint open balls, and furthermore:

$$\bigcup_{i=1}^{p} B(x_i, \epsilon_{x_i}) \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_{x_i})$$

It follows from 7. that:

$$K \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_{x_i})$$

9. Using 5., 6. and 8. we obtain:

$$\begin{array}{rcl}
\{\phi \in K\} & = & \phi^{-1}(K) \\
\text{From 8.} \to & \subseteq & \phi^{-1}\left(\bigcup_{i \in S} B(x_i, 3\epsilon_{x_i})\right) \\
 & = & \bigcup_{i \in S} \phi^{-1}(B(x_i, 3\epsilon_{x_i})) \\
 & = & \bigcup_{i \in S} \{\phi \in B(x_i, 3\epsilon_{x_i})\} \\
\text{From 6.} \to & = & \bigcup_{i \in S} \psi(B(x_i, 3\epsilon_{x_i})) \\
\text{From 5.} \to & \subseteq & \bigcup_{i \in S} B(\psi(x_i), 3(M+1)\epsilon_{x_i})
\end{array}$$

10. From 9. and exercise (12) of Tutorial 16, we obtain:

$$\begin{array}{rcl}
\phi(dx_{|\Omega})(K) & \stackrel{\triangle}{=} & dx_{|\Omega}(\{\phi \in K\}) \\
 & \leq & dx_{|\Omega}\left(\bigcup_{i \in S} B(\psi(x_i), 3(M+1)\epsilon_{x_i})\right) \\
 & \leq & \sum_{i \in S} dx_{|\Omega}(B(\psi(x_i), 3(M+1)\epsilon_{x_i})) \\
 & = & \sum_{i \in S} dx(B(\psi(x_i), 3(M+1)\epsilon_{x_i})) \\
 & = & \sum_{i \in S} 3^n (M+1)^n \epsilon_{x_i}^n dx(B(0,1)) \\
 & = & \sum_{i \in S} 3^n (M+1)^n dx(B(x_i, \epsilon_{x_i}))
\end{array}$$

11. Since $(B(x_i, \epsilon_{x_i}))_{i \in S}$ is a family of pairwise disjoint (Borel) sets:

$$dx\left(\biguplus_{i \in S} B(x_i, \epsilon_{x_i})\right) = \sum_{i \in S} dx(B(x_i, \epsilon_{x_i}))$$

Hence, having proved in 4. that $B(x,\epsilon_x) \subseteq V$ for all $x \in K$, we obtain from 10.:

$$\begin{aligned}\phi(dx_{|\Omega})(K) &\leq \sum_{i\in S} 3^n(M+1)^n dx(B(x_i,\epsilon_{x_i}))\\ &= 3^n(M+1)^n dx\left(\biguplus_{i\in S} B(x_i,\epsilon_{x_i})\right)\\ &\leq 3^n(M+1)^n dx(V)\end{aligned}$$

12. Since $dx(V) = dx_{|\Omega'}(V) \leq \epsilon$, it follows from 11.:

$$\phi(dx_{|\Omega})(K) \leq 3^n(M+1)^n\epsilon \tag{13}$$

13. We have found $M \in \mathbf{R}^+$ for which inequality (13) holds for all $\epsilon > 0$. It follows that $\phi(dx_{|\Omega})(K) = 0$

14. Let $x \in \Omega'$. Then $\psi(x) \in \Omega$. Since $dx_{|\Omega}$ is a locally finite measure on Ω, there exists W open in Ω, such that $\psi(x) \in W$ and $dx_{|\Omega}(W) < +\infty$. Define $U = \psi^{-1}(W)$. Then U is open in Ω' and $x \in U$. Moreover:

$$\begin{aligned}\phi(dx_{|\Omega})(U) &= dx_{|\Omega}(\phi^{-1}(U))\\ &= dx_{|\Omega}(\phi^{-1}(\psi^{-1}(W)))\\ &= dx_{|\Omega}((\psi\circ\phi)^{-1}(W))\\ &= dx_{|\Omega}(W) < +\infty\end{aligned}$$

Hence, given $x \in \Omega'$ we have found U open in Ω' such that $x \in U$ and $\phi(dx_{|\Omega})(U) < +\infty$. From definition (102), we conclude that $\phi(dx_{|\Omega})$ is a locally finite measure on $(\Omega', \mathcal{B}(\Omega'))$.

15. Having proved in 14. that $\phi(dx_{|\Omega})$ is a locally finite measure, from theorem (74) it follows that $\phi(dx_{|\Omega})$ is a regular measure. Given $B \in \mathcal{B}(\Omega')$, from definition (103) we obtain:

$$\phi(dx_{|\Omega})(B) = \sup\{\phi(dx_{|\Omega})(K) :\ K \subseteq B\ ,\ K \text{ compact }\}$$

16. Let $B \in \mathcal{B}(\Omega')$ with $dx_{|\Omega'}(B) = 0$. Let K be a compact subset of B. Then in particular, K is a compact subset of Ω' with $dx_{|\Omega'}(K) = 0$. If $K \neq \emptyset$, it follows from 13. that $\phi(dx_{|\Omega})(K) = 0$. This is obviously still true if $K = \emptyset$. Hence we see that $\phi(dx_{|\Omega})(B)$ is the supremum of the set $\{0\}$, and consequently $\phi(dx_{|\Omega})(B) = 0$. We have proved that for all $B \in \mathcal{B}(\Omega')$:

$$dx_{|\Omega'}(B) = 0 \ \Rightarrow\ \phi(dx_{|\Omega})(B) = 0 \tag{14}$$

17. Given Ω, Ω' open in $\mathbf{R}^n$ and $\phi : \Omega \to \Omega'$ C^1-diffeomorphism, we have proved that for all $B \in \mathcal{B}(\Omega')$ the implication (14) holds. From definition (96), it follows that the image measure $\phi(dx_{|\Omega})$ is absolutely continuous with respect to $dx_{|\Omega'}$, i.e.:

$$\phi(dx_{|\Omega}) << dx_{|\Omega'}$$

This completes the proof of theorem (119).

Exercise 27

Exercise 28.

1. Let Ω, Ω' be open in $\mathbf{R}^n$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Let $\psi = \phi^{-1}$. Since $\mathbf{R}^n$ is metrizable and strongly σ-compact, since Ω' is open in $\mathbf{R}^n$, from theorem (76), Ω' is itself strongly σ-compact. From definition (104), there exists a sequence $(V_p)_{p\geq 1}$ of open subsets of Ω', such that $V_p \uparrow \Omega'$ and for all $p \geq 1$ the closure of V_p in Ω' (denoted $\bar{V}_p^{\Omega'}$) is compact.

2. Being open in Ω', each V_p can be written as $V_p = U_p \cap \Omega'$ where U_p is open in $\mathbf{R}^n$. Since Ω' is itself open in $\mathbf{R}^n$, it follows that V_p is also open in $\mathbf{R}^n$. Let $\bar{V}_p$ denote the closure of V_p in $\mathbf{R}^n$. We claim that $\bar{V}_p^{\Omega'} = \bar{V}_p$. Since Ω' is open in $\mathbf{R}^n$, from exercise (19) of Tutorial 13 we have $\bar{V}_p^{\Omega'} = \bar{V}_p \cap \Omega'$. However, having assumed that $\bar{V}_p^{\Omega'}$ is a compact subset of Ω', it is also a compact subset of $\mathbf{R}^n$, and $\mathbf{R}^n$ is Hausdorff. It follows from theorem (35) that $\bar{V}_p^{\Omega'}$ is a closed subset of $\mathbf{R}^n$, which furthermore contains V_p in the inclusion sense. From exercise (21) of Tutorial 4, $\bar{V}_p$ is the smallest closed subset of $\mathbf{R}^n$ containing V_p in the inclusion sense. Hence, we see that $\bar{V}_p \subseteq \bar{V}_p^{\Omega'}$ and in particular $\bar{V}_p \subseteq \Omega'$. We conclude from $\bar{V}_p^{\Omega'} = \bar{V}_p \cap \Omega'$ that $\bar{V}_p^{\Omega'} = \bar{V}_p$.

3. Let $p \geq 1$. Using 14. of exercise (27), the image measure $\phi(dx_{|\Omega})$ is a locally finite measure on $(\Omega', \mathcal{B}(\Omega'))$. From exercise (10) of Tutorial 13, since $\bar{V}_p^{\Omega'}$ is a compact subset of Ω' we have $\phi(dx_{|\Omega})(\bar{V}_p^{\Omega'}) < +\infty$. Since $\bar{V}_p^{\Omega'} = \bar{V}_p$ we conclude that:
$$\phi(dx_{|\Omega})(V_p) \leq \phi(dx_{|\Omega})(\bar{V}_p) < +\infty$$

4. It follows from 3. that $(V_p)_{p\geq 1}$ is a sequence of Borel subsets of Ω' such that $V_p \uparrow \Omega'$ and $\phi(dx_{|\Omega})(V_p) < +\infty$ for all $p \geq 1$. From definition (61), we conclude that $\phi(dx_{|\Omega})$ is a σ-finite measure on $(\Omega', \mathcal{B}(\Omega'))$. Similarly, since $dx_{|\Omega'}$ is a locally finite measure, from exercise (10) of Tutorial 13, $\bar{V}_p^{\Omega'}$ being compact:
$$dx_{|\Omega'}(V_p) \leq dx_{|\Omega'}(\bar{V}_p) = dx_{|\Omega'}(\bar{V}_p^{\Omega'}) < +\infty$$
It follows that $dx_{|\Omega'}$ is also a σ-finite measure on $(\Omega', \mathcal{B}(\Omega'))$.

5. From theorem (119), we have $\phi(dx_{|\Omega}) << dx_{|\Omega'}$. Furthermore from 4. $\phi(dx_{|\Omega})$ and $dx_{|\Omega'}$ are two σ-finite measures on $(\Omega', \mathcal{B}(\Omega'))$. From the Radon-Nikodym theorem (61), there is $h : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ measurable such that:
$$\forall B \in \mathcal{B}(\Omega') \ , \ \phi(dx_{|\Omega})(B) = \int_B h dx_{|\Omega'}$$

6. Given $p \geq 1$, we define $h_p = h1_{V_p}$, and we put:

$$\forall x \in \mathbf{R}^n \ , \ \tilde{h}_p(x) \stackrel{\triangle}{=} \begin{cases} h_p(x) & \text{if} \quad x \in \Omega' \\ 0 & \text{if} \quad x \notin \Omega' \end{cases}$$

Using exercise (19) of Tutorial 16, $\tilde{h}_p$ is measurable, and:

$$\begin{array}{rcl} \int_{\mathbf{R}^n} \tilde{h}_p dx & = & \int_{\Omega'} h_p dx_{|\Omega'} \\ & = & \int_{\Omega'} h1_{V_p} dx_{|\Omega'} \\ & = & \int_{V_p} h dx_{|\Omega'} \\ \text{From 5.} \ \rightarrow & = & \phi(dx_{|\Omega})(V_p) \\ \text{From 3.} \ \rightarrow & < & +\infty \end{array}$$

We conclude that $\tilde{h}_p \in L^1_{\mathbf{R}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$.

7. Applying theorem (101) to $\tilde{h}_p$, dx-almost every $x \in \mathbf{R}^n$ is a Lebesgue point of $\tilde{h}_p$. In other words, there exists $N_p \in \mathcal{B}(\mathbf{R}^n)$ with $dx(N_p) = 0$ such that for all $x \in N_p^c$, x is a Lebesgue point of $\tilde{h}_p$, and in particular from exercise (17) of Tutorial 16:

$$\tilde{h}_p(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} \tilde{h}_p dx \tag{15}$$

Defining $N = \cup_{p \geq 1} N_p$, we have $N \in \mathcal{B}(\mathbf{R}^n)$ and $dx(N) = 0$, and furthermore (15) holds for all $x \in N^c$ and $p \geq 1$.

8. Let $N' = N \cap \Omega'$. Then $N' \in \mathcal{B}(\mathbf{R}^n)_{|\Omega'} = \mathcal{B}(\Omega')$, and:

$$dx_{|\Omega'}(N') = dx(N') \leq dx(N) = 0$$

9. Let $x \in \Omega'$. Suppose $p \geq 1$ is such that $x \in V_p$. Let $\epsilon > 0$ be such that $B(x,\epsilon) \subseteq V_p$. Then in particular $B(x,\epsilon) \subseteq \Omega'$ and:

$$B(x,\epsilon) = B(x,\epsilon) \cap \Omega' \in \mathcal{B}(\mathbf{R}^n)_{|\Omega'} = \mathcal{B}(\Omega')$$

It follows that $dx_{|\Omega'}(B(x,\epsilon))$ is meaningful, and:

$$dx(B(x,\epsilon)) = dx_{|\Omega'}(B(x,\epsilon))$$

Furthermore, it is clear that:

$$\forall u \in \mathbf{R}^n \ , \ (1_{B(x,\epsilon)}\tilde{h}_p)(u) \stackrel{\triangle}{=} \begin{cases} (1_{B(x,\epsilon)}h_p)(u) & \text{if} \quad u \in \Omega' \\ 0 & \text{if} \quad u \notin \Omega' \end{cases}$$

where we have used that same notation $1_{B(x,\epsilon)}$ to denote successively the characteristic function of $B(x,\epsilon)$ on $\mathbf{R}^n$ and on Ω'. Applying exercise (19) of Tutorial 16, we obtain:

$$\begin{aligned}\int_{B(x,\epsilon)} \tilde{h}_p dx &= \int_{\mathbf{R}^n} 1_{B(x,\epsilon)} \tilde{h}_p dx \\ \text{Ex. (19) of T. 16 } \rightarrow \quad &= \int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'}\end{aligned}$$

10. Since $h_p = h1_{V_p}$ and $B(x,\epsilon) \subseteq V_p$, using 5. we have:

$$\begin{aligned}\int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'} &= \int_{\Omega'} 1_{B(x,\epsilon)} h 1_{V_p} dx_{|\Omega'} \\ &= \int_{\Omega'} 1_{B(x,\epsilon)} h dx_{|\Omega'} \\ &= \int_{B(x,\epsilon)} h dx_{|\Omega'} \\ \text{From 5. } \rightarrow \quad &= \phi(dx_{|\Omega})(B(x,\epsilon))\end{aligned}$$

11. Let $x \in \Omega' \setminus N'$. Since $N' = N \cap \Omega'$, we have:

$$\begin{aligned}\Omega' \setminus N' &= \Omega' \cap (N \cap \Omega')^c \\ &= \Omega' \cap (N^c \cup (\Omega')^c) = \Omega' \cap N^c\end{aligned}$$

So in particular $x \in N^c$. It follows from 7. that for all $p \geq 1$:

$$\tilde{h}_p(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} \tilde{h}_p dx \tag{16}$$

However, by assumption $V_p \uparrow \Omega'$. Since $x \in \Omega'$, there exists $p \geq 1$ such that $x \in V_p$. In particular we obtain:

$$\tilde{h}_p(x) = h_p(x) = h(x) 1_{V_p}(x) = h(x) \tag{17}$$

Furthermore, since $x \in V_p$ and V_p is open in $\mathbf{R}^n$, there exists $\eta > 0$ such that $B(x,\eta) \subseteq V_p$. For all $\epsilon > 0$ with $\epsilon < \eta$ we have $B(x,\epsilon) \subseteq V_p$ and consequently from 9. and 10. we obtain:

$$\int_{B(x,\epsilon)} \tilde{h}_p dx = \phi(dx_{|\Omega})(B(x,\epsilon)) \tag{18}$$

and furthermore:

$$dx(B(x,\epsilon)) = dx_{|\Omega'}(B(x,\epsilon)) \tag{19}$$

Having proved the equalities (18) and (19) for $\epsilon > 0$ sufficiently small, we conclude from (16) and (17) that:

$$h(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(x,\epsilon))}{dx_{|\Omega'}(B(x,\epsilon))} \tag{20}$$

Hence, we have proved (20) for all $x \in \Omega' \setminus N'$.

12. Applying theorem (118), for all $x \in \Omega'$ we have:

$$|J(\psi)(x)| = \lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(x,\epsilon))}{dx_{|\Omega'}(B(x,\epsilon))}$$

It follows from (20) that h and $|J(\psi)|$ coincide on $\Omega' \setminus N'$. Having proved in 8. that $dx_{|\Omega'}(N') = 0$, we conclude that $h = |J(\psi)|$, $dx_{|\Omega'}$-almost surely.

13. From 5. and 12. we see that for all $B \in \mathcal{B}(\Omega')$:

$$\begin{aligned} \phi(dx_{|\Omega})(B) &= \int_B h dx_{|\Omega'} \\ &= \int_B |J(\psi)| dx_{|\Omega'} \end{aligned}$$

This being true for all $B \in \mathcal{B}(\Omega')$, we conclude that the image measure $\phi(dx_{|\Omega})$ has density $|J(\psi)|$ with respect to the Lebesgue measure $dx_{|\Omega'}$ on Ω', i.e.:

$$\phi(dx_{|\Omega}) = \int |J(\psi)| dx_{|\Omega'}$$

This completes the proof of theorem (120).

Exercise 28

Exercise 29. Let Ω, Ω' be open in $\mathbf{R}^n$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Let $\psi = \phi^{-1}$ and $f : (\Omega', \mathcal{B}(\Omega')) \to [0, +\infty]$ be a non-negative and measurable map. Applying the integral projection theorem (104), we have:

$$\int_\Omega f \circ \phi dx_{|\Omega} = \int_{\Omega'} f \phi(dx_{|\Omega}) \tag{21}$$

and furthermore, from theorem (120):

$$\phi(dx_{|\Omega}) = \int |J(\psi)| dx_{|\Omega'}$$

So from the stack integral theorem (21), we obtain:

$$\int_{\Omega'} f \phi(dx_{|\Omega}) = \int_{\Omega'} f |J(\psi)| dx_{|\Omega'} \tag{22}$$

From equations (21) and (22) we conclude that:

$$\int_\Omega f \circ \phi dx_{|\Omega} = \int_{\Omega'} f |J(\psi)| dx_{|\Omega'} \tag{23}$$

Having proved in exercise (23) that $J(\psi)$ is continuous, and furthermore that $J(\psi)(x) \neq 0$ for all $x \in \Omega'$, the map $f/|J(\psi)|$ is well-defined, non-negative and measurable. Applying equation (23) to $f/|J(\psi)|$:

$$\begin{aligned} \int_{\Omega'} f dx_{|\Omega'} &= \int_{\Omega'} \left(\frac{f}{|J(\psi)|} \right) |J(\psi)| dx_{|\Omega'} \\ \text{Equation (23)} \to \quad &= \int_\Omega \frac{f \circ \phi}{|J(\psi) \circ \phi|} dx_{|\Omega} \end{aligned}$$

$$\text{Exercise (23)} \rightarrow \quad = \quad \int_{\Omega} (f \circ \phi)|J(\phi)| dx_{|\Omega}$$

This completes the proof of theorem (121).

Exercise 29

Exercise 30. Let Ω, Ω' be open in $\mathbf{R}^n$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Let $\psi = \phi^{-1}$ and $f : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map. Since ϕ and $|J(\psi)|$ are continuous, in particular they are Borel measurable and consequently $f \circ \phi$ and $f|J(\psi)|$ are Borel measurable. Furthermore, applying the Jacobian formula (121) to the non-negative and measurable map $|f|$, we obtain:

$$\begin{aligned} \int_{\Omega} |f \circ \phi| dx_{|\Omega} &= \int_{\Omega} |f| \circ \phi dx_{|\Omega} \\ \text{Theorem (121)} \rightarrow \quad &= \int_{\Omega'} |f| \cdot |J(\psi)| dx_{|\Omega'} \\ &= \int_{\Omega'} |f J(\psi)| dx_{|\Omega'} \end{aligned}$$

Hence, we have proved the equivalence:

$$f \circ \phi \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \;\Leftrightarrow\; f|J(\psi)| \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

Similarly, since ϕ and $|J(\phi)|$ are continuous, both $(f \circ \phi)|J(\phi)|$ and f are Borel measurable, and from theorem (121):

$$\begin{aligned} \int_{\Omega'} |f| dx_{|\Omega'} &= \int_{\Omega} (|f| \circ \phi)|J(\phi)| dx_{|\Omega} \\ &= \int_{\Omega} |(f \circ \phi) J(\phi)| dx_{|\Omega} \end{aligned}$$

Hence, we have proved the equivalence:

$$(f \circ \phi)|J(\phi)| \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \;\Leftrightarrow\; f \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

Now suppose that $f \circ \phi \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega})$. Let $u = Re(f)$ and $v = Im(f)$, so that $f = u^+ - u^- + i(v^+ - v^-)$. Since $u^+, u^- \leq |u| \leq |f|$ and $v^+, v^- \leq |v| \leq |f|$, each $u^\pm \circ \phi$ and $v^\pm \circ \phi$ is an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega})$. It follows that each $u^\pm|J(\psi)|$ and $v^\pm|J(\psi)|$ is an element of $L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$, and we have:

$$\begin{aligned} \int_{\Omega} f \circ \phi dx_{|\Omega} &= \int_{\Omega} (u^+ \circ \phi) dx_{|\Omega} - \int_{\Omega} (u^- \circ \phi) dx_{|\Omega} \\ &+ i\left(\int_{\Omega} (v^+ \circ \phi) dx_{|\Omega} - \int_{\Omega} (v^- \circ \phi) dx_{|\Omega}\right) \\ \text{Theorem (121)} \rightarrow \quad &= \int_{\Omega'} u^+|J(\psi)| dx_{|\Omega'} - \int_{\Omega'} u^-|J(\psi)| dx_{|\Omega'} \\ &+ i\left(\int_{\Omega'} v^+|J(\psi)| dx_{|\Omega'} - \int_{\Omega'} v^-|J(\psi)| dx_{|\Omega'}\right) \\ &= \int_{\Omega'} f|J(\psi)| dx_{|\Omega'} \end{aligned}$$

Suppose now that $f \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$. Then u^+, u^-, v^+ and v^- are all elements of $L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$, and furthermore:

$$\begin{aligned}
\int_{\Omega'} f dx_{|\Omega'} &= \int_{\Omega'} [u^+ - u^- + i(v^+ - v^-)] dx_{|\Omega'} \\
&= \int_{\Omega'} u^+ dx_{|\Omega'} - \int_{\Omega'} u^- dx_{|\Omega'} \\
&+ i\left(\int_{\Omega'} v^+ dx_{|\Omega'} - \int_{\Omega'} v^- dx_{|\Omega'}\right) \\
\text{Theorem (121)} \rightarrow \quad &= \int_{\Omega} (u^+ \circ \phi)|J(\phi)| dx_{|\Omega} \\
&- \int_{\Omega} (u^- \circ \phi)|J(\phi)| dx_{|\Omega} \\
&+ i\int_{\Omega} (v^+ \circ \phi)|J(\phi)| dx_{|\Omega} \\
&- i\int_{\Omega} (v^- \circ \phi)|J(\phi)| dx_{|\Omega} \\
&= \int_{\Omega} (f \circ \phi)|J(\phi)| dx_{|\Omega}
\end{aligned}$$

This completes the proof of theorem (122).

Exercise 30

Exercise 31.

1. Let $f : \mathbf{R}^2 \to [0, +\infty]$ be defined by:

$$\forall (x,y) \in \mathbf{R}^2 \ , \ f(x,y) = \exp(-(x^2+y^2)/2)$$

   Using Fubini's theorem (31) we obtain:

$$\begin{aligned}
\int_{\mathbf{R}^2} f(x,y) dx dy &= \int_{\mathbf{R}\times\mathbf{R}} \exp(-(x^2+y^2)/2) dx dy \\
\text{Theorem (31)} \rightarrow \quad &= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \exp(-x^2/2)\exp(-y^2/2) dx\right) dy \\
&= \int_{\mathbf{R}} \exp(-y^2/2)\left(\int_{\mathbf{R}} \exp(-x^2/2) dx\right) dy \\
&= \left(\int_{\mathbf{R}} \exp(-x^2/2) dx\right) \int_{\mathbf{R}} \exp(-y^2/2) dy \\
&= \left(\int_{-\infty}^{+\infty} e^{-u^2/2} du\right)^2
\end{aligned}$$

2. We define the following subsets of $\mathbf{R}^2$:

$$\begin{aligned}
\Delta_1 &\stackrel{\triangle}{=} \{(x,y) \in \mathbf{R}^2 : \ x > 0 \ , \ y > 0\} \\
\Delta_2 &\stackrel{\triangle}{=} \{(x,y) \in \mathbf{R}^2 : \ x < 0 \ , \ y > 0\}
\end{aligned}$$

$$\begin{aligned}\Delta_3 &\stackrel{\triangle}{=} \{(x,y)\in\mathbf{R}^2:\ x>0\ ,\ y<0\}\\ \Delta_4 &\stackrel{\triangle}{=} \{(x,y)\in\mathbf{R}^2:\ x<0\ ,\ y<0\}\end{aligned}$$

and:

$$\Delta_5 = \{(x,y)\in\mathbf{R}^2:\ x=0\}\cup\{(x,y)\in\mathbf{R}^2:\ y=0\}$$

Then $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ and Δ_5 are pairwise disjoint, and:

$$\mathbf{R}^2 = \Delta_1 \uplus \Delta_2 \uplus \Delta_3 \uplus \Delta_4 \uplus \Delta_5$$

Moreover, since $\{x = 0\}$ and $\{y = 0\}$ are one-dimensional subspaces of $\mathbf{R}^2$, from theorem (109) we have:

$$dxdy(\Delta_5) \leq dxdy(\{x=0\}) + dxdy(\{y=0\}) = 0$$

Hence, we have:

$$\begin{aligned}\int_{\mathbf{R}^2} f(x,y)dxdy &= \int_{\Delta_1\uplus\ldots\uplus\Delta_5} f(x,y)dxdy\\ &= \int_{\Delta_1\uplus\ldots\uplus\Delta_4} f(x,y)dxdy + \int_{\Delta_5} f(x,y)dxdy\\ &= \int_{\Delta_1\uplus\ldots\uplus\Delta_4} f(x,y)dxdy\end{aligned}$$

3. Let $Q : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $Q(x,y) = (-x,y)$. Then Q is a linear bijection and furthermore:

$$\det Q = \det\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix} = -1$$

From theorem (108), we have:

$$Q(dxdy) = |\det Q|^{-1}dxdy$$

and consequently:

$$\begin{aligned}\int_{\Delta_1} f(x,y)dxdy &= \int 1_{\Delta_1} f dxdy\\ &= \int (1_{\Delta_1}\circ Q^{-1}\circ Q)(f\circ Q^{-1}\circ Q)dxdy\\ \text{Theorem (104)} \to\quad &= \int (1_{\Delta_1}\circ Q^{-1})(f\circ Q^{-1})Q(dxdy)\\ &= |\det Q|^{-1}\int (1_{\Delta_1}\circ Q^{-1})(f\circ Q^{-1})dxdy\\ &= \int 1_{\Delta_2}(f\circ Q^{-1})dxdy\\ &= \int_{\Delta_2} f\circ Q^{-1}(x,y)dxdy\end{aligned}$$

4. Since $f(x,y) = \exp(-(x^2+y^2)/2)$, $f \circ Q^{-1} = f$. So, from 3.:
$$\int_{\Delta_1} f(x,y)dxdy = \int_{\Delta_2} f(x,y)dxdy$$
Similarly, using $Q'(x,y) = (x,-y)$ and $Q''(x,y) = (-x,-y)$:
$$\begin{aligned}\int_{\Delta_1} f(x,y)dxdy &= \int_{\Delta_3} f(x,y)dxdy \\ &= \int_{\Delta_4} f(x,y)dxdy\end{aligned}$$
We conclude from 2.:
$$\begin{aligned}\int_{\mathbf{R}^2} f(x,y)dxdy &= \int_{\Delta_1\uplus\ldots\uplus\Delta_4} f(x,y)dxdy \\ &= \sum_{i=1}^{4}\int_{\Delta_i} f(x,y)dxdy \\ &= 4\int_{\Delta_1} f(x,y)dxdy\end{aligned}$$

5. Let $D_1 =]0,+\infty[\times]0,\pi/2[$ and $\phi : D_1 \to \Delta_1$ be defined by:
$$\forall (r,\theta) \in D_1\ ,\ \phi(r,\theta) \stackrel{\triangle}{=} (r\cos\theta, r\sin\theta)$$
Let $\psi : \Delta_1 \to D_1$ be defined by:
$$\forall (x,y) \in \Delta_1\ ,\ \psi(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x))$$
Then for all $(r,\theta) \in D_1$, we have:
$$\begin{aligned}\psi\circ\phi(r,\theta) &= \psi(r\cos\theta, r\sin\theta) \\ &= [\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}, \arctan(\sin\theta/\cos\theta)] \\ &= (|r|\sqrt{\cos^2\theta+\sin^2\theta}, \arctan(\tan\theta)) \\ &= (r,\theta)\end{aligned}$$
So $\psi\circ\phi = id_{D_1}$. Furthermore, for all $\theta \in]0,\pi/2[$ we have:
$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta}$$
and consequently, since $\cos\theta > 0$, we obtain:
$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} \tag{24}$$
Similarly, from:
$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{1-\sin^2\theta}$$

and the fact $\sin\theta > 0$ and $\tan\theta > 0$, we obtain:

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \tag{25}$$

From (24) and (25) we see that for all $(x,y) \in \Delta_1$:

$$\cos(\arctan(y/x)) = \frac{1}{\sqrt{1+y^2/x^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

and:

$$\sin(\arctan(y/x))\frac{y/x}{\sqrt{1+y^2/x^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

It follows that for all $(x,y) \in \Delta_1$:

$$\begin{aligned}
\phi\circ\psi(x,y) &= \phi(\sqrt{x^2+y^2}, \arctan(y/x)) \\
&= \sqrt{x^2+y^2}[\cos(\arctan(y/x)), \sin(\arctan(y/x))] \\
&= \sqrt{x^2+y^2}\left[\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right] \\
&= (x,y)
\end{aligned}$$

and we have proved that $\phi\circ\psi = id_{\Delta_1}$. Having proved that $\psi\circ\phi = id_{D_1}$ and $\phi\circ\psi = id_{\Delta_1}$, we conclude that $\phi : D_1 \to \Delta_1$ is bijective and $\psi = \phi^{-1}$.

6. In order to show that $\phi : D_1 \to \Delta_1$ is a C^1-diffeomorphism, we need to show that both $\phi : D_1 \to \mathbf{R}^2$ and $\psi : \Delta_1 \to \mathbf{R}^2$ are of class C^1. Given $(r,\theta) \in D_1$, define $\phi_x(r,\theta) = r\cos\theta$ and $\phi_y(r,\theta) = r\sin\theta$. Then, we have:

$$\begin{array}{lcl}
\frac{\partial\phi_x}{\partial r}(r,\theta) = \cos\theta & , & \frac{\partial\phi_x}{\partial\theta}(r,\theta) = -r\sin\theta \\
\frac{\partial\phi_y}{\partial r}(r,\theta) = \sin\theta & , & \frac{\partial\phi_y}{\partial\theta}(r,\theta) = r\cos\theta
\end{array}$$

So it is clear that $\frac{\partial\phi_x}{\partial r}$, $\frac{\partial\phi_x}{\partial\theta}$, $\frac{\partial\phi_y}{\partial r}$ and $\frac{\partial\phi_y}{\partial\theta}$ exist and are continuous on D_1. From theorem (117), it follows that $\phi : D_1 \to \mathbf{R}^2$ is of class C^1 and for all $(r,\theta) \in D_1$, we have:

$$d\phi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Given $(x,y) \in \Delta_1$, define $\psi_r(x,y) = \sqrt{x^2+y^2}$ together with $\psi_\theta(x,y) = \arctan(y/x)$. As some of us may have forgotten, recall that the map $\tan :]-\pi/2, \pi/2[\to \mathbf{R}$ is differentiable, and:

$$\begin{aligned}
(\tan\theta)' = \left(\frac{\sin\theta}{\cos\theta}\right)' &= \frac{(\sin\theta)'\cos\theta - (\cos\theta)'\sin\theta}{\cos^2\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\
&= 1 + \tan^2\theta
\end{aligned}$$

Moreover, the map arctan : $\mathbf{R} \to]-\pi/2, \pi/2[$ is also differentiable, and one way to remember its derivative is to differentiate both sides of the identity $x = \tan(\arctan x)$, to obtain:

$$\begin{aligned} 1 &= \tan'(\arctan x) \cdot (\arctan x)' \\ &= (1 + \tan^2(\arctan x)) \cdot (\arctan x)' \\ &= (1 + x^2) \cdot (\arctan x)' \end{aligned}$$

and consequently for all $x \in \mathbf{R}$:

$$(\arctan x)' = \frac{1}{1+x^2}$$

It follows that given $(x, y) \in \Delta_1$, we have:

$$\frac{\partial \psi_r}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \ , \ \frac{\partial \psi_r}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

and furthermore:

$$\begin{aligned} \frac{\partial \psi_\theta}{\partial x}(x,y) &= \left(-\frac{y}{x^2}\right) \cdot \arctan'(y/x) \\ &= -\frac{y}{x^2} \cdot \frac{1}{1+y^2/x^2} \\ &= -\frac{y}{x^2+y^2} \end{aligned}$$

as well as:

$$\begin{aligned} \frac{\partial \psi_\theta}{\partial y}(x,y) &= \frac{1}{x} \cdot \arctan'(y/x) \\ &= \frac{1}{x} \cdot \frac{1}{1+y^2/x^2} \\ &= \frac{x}{x^2+y^2} \end{aligned}$$

Hence, we see that $\frac{\partial \psi_r}{\partial x}$, $\frac{\partial \psi_r}{\partial y}$, $\frac{\partial \psi_\theta}{\partial x}$ and $\frac{\partial \psi_\theta}{\partial y}$ exist and are continuous on Δ_1. From theorem (117), it follows that $\psi : \Delta_1 \to \mathbf{R}^2$ is of class C^1 and for all $(x, y) \in \Delta_1$, we have:

$$d\psi(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

We have proved that $\phi : D_1 \to \Delta_1$ is a C^1-diffeomorphism.

7. From 6. and definition (132), for all $(r, \theta) \in D_1$:

$$\begin{aligned} J(\phi)(r,\theta) &= \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \cos\theta \cdot (r\cos\theta) - \sin\theta(-r\sin\theta) \\ &= r(\cos^2\theta + \sin^2\theta) \\ &= r \end{aligned}$$

8. From 6. and definition (132), for all $(x, y) \in \Delta_1$:

$$\begin{aligned} J(\psi)(x,y) &= \det\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \\ &= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} \\ &= \frac{1}{\sqrt{x^2+y^2}} \end{aligned}$$

9. Applying the Jacobian formula (121) to $f_{|\Delta_1} : \Delta_1 \to [0, +\infty]$:

$$\begin{aligned} \int_{\Delta_1} f(x,y)dxdy &= \int_{\mathbf{R}^2} 1_{\Delta_1} f dxdy \\ \text{Definition (45)} \to &= \int_{\Delta_1} f_{|\Delta_1}(dxdy)_{|\Delta_1} \\ \text{Theorem (121)} \to &= \int_{D_1} (f_{|\Delta_1} \circ \phi)|J(\phi)|(drd\theta)_{|D_1} \\ f_{|\Delta_1} \circ \phi(r) = e^{-r^2/2} \to &= \int_{D_1} \exp(-r^2/2) r (drd\theta)_{|D_1} \\ \text{Definition (45)} \to &= \int_{\mathbf{R}^2} 1_{D_1} \exp(-r^2/2) r dr d\theta \\ \text{Fubini (31)} \to &= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} 1_{D_1} \exp(-r^2/2) r d\theta \right) dr \\ &= \int_{\mathbf{R}} 1_{]0,+\infty[} \left(\frac{\pi}{2}\right) \exp(-r^2/2) r dr \\ &= \frac{\pi}{2} \int_{\mathbf{R}} 1_{[0,+\infty[} \exp(-r^2/2) r dr \\ \text{MON (19)} \to &= \lim_{n\to+\infty} \frac{\pi}{2} \int_{\mathbf{R}} 1_{[0,n]} \exp(-r^2/2) r dr \\ \text{Theorem (99)} \to &= \lim_{n\to+\infty} \frac{\pi}{2}[1 - \exp(-n^2/2)] = \frac{\pi}{2} \end{aligned}$$

10. Using 1., we obtain:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-u^2/2} du &= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbf{R}^2} f(x,y)dxdy \right)^{1/2} \\ \text{From 4.} \to &= \frac{1}{\sqrt{2\pi}} \left(4 \int_{\Delta_1} f(x,y)dxdy \right)^{1/2} \\ \text{From 9.} \to &= \frac{1}{\sqrt{2\pi}} \left(4 \cdot \frac{\pi}{2} \right)^{1/2} = 1 \end{aligned}$$

This complete the proof of theorem (123).

Exercise 31