17. Image Measure

In the following, $K$ denotes $\mathbf{R}$ or $\mathbf{C}$. We denote $\mathcal{M}_n(K)$, $n \geq 1$, the set of all $n \times n$-matrices with $K$-valued entries. We recall that for all $M = (m_{ij}) \in \mathcal{M}_n(K)$, $M$ is identified with the linear map $M : K^n \to K^n$ uniquely determined by:

$$\forall j = 1, \ldots, n, \; Me_j \triangleq \sum_{i=1}^n m_{ij}e_i$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $K^n$, i.e. $e_i \triangleq (0, \ldots, 1, \ldots, 0)$.

**Exercise 1.** For all $\alpha \in K$, let $H_\alpha \in \mathcal{M}_n(K)$ be defined by:

$$H_\alpha \triangleq \begin{pmatrix}
\alpha & 1 & 0 \\
1 & 0 & \ddots \\
0 & \ddots & 1
\end{pmatrix}$$

i.e. by $H_\alpha e_1 = \alpha e_1$, $H_\alpha e_j = e_j$, for all $j \geq 2$. Note that $H_\alpha$ is obtained from the identity matrix, by multiplying the top left entry by $\alpha$. For $k, l \in \{1, \ldots, n\}$, we define the matrix $\Sigma_{kl} \in \mathcal{M}_n(K)$ by $\Sigma_{kl}e_k = e_l$, $\Sigma_{kl}e_l = e_k$ and $\Sigma_{kl}e_j = e_j$, for all $j \in \{1, \ldots, n\} \setminus \{k, l\}$. Note that $\Sigma_{kl}$ is obtained from the identity matrix, by interchanging column $k$ and column $l$. If $n \geq 2$, we define the matrix $U \in \mathcal{M}_n(K)$ by:

$$U \triangleq \begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & \ddots \\
0 & \ddots & \ddots & 1
\end{pmatrix}$$

i.e. by $Ue_1 = e_1 + e_2$, $Ue_j = e_j$ for all $j \geq 2$. Note that the matrix $U$ is obtained from the identity matrix, by adding column 2 to column 1. If $n = 1$, we put $U = 1$. We define $\mathcal{N}_n(K) = \{H_\alpha : \alpha \in K\} \cup \{\Sigma_{kl} : k, l = 1, \ldots, n\} \cup \{U\}$, and $\mathcal{M}_n(K)$ to be the set of all finite products of elements of $\mathcal{N}_n(K)$:

$$\mathcal{M}_n(K) \triangleq \{M \in \mathcal{M}_n(K) : M = Q_1 \ldots Q_p, \; p \geq 1, \; Q_j \in \mathcal{N}_n(K), \; \forall j\}$$

We shall prove that $\mathcal{M}_n(K) = \mathcal{M}_n'(K)$.

1. Show that if $\alpha \in K \setminus \{0\}$, $H_\alpha$ is non-singular with $H_\alpha^{-1} = H_{1/\alpha}$
2. Show that if $k, l = 1, \ldots, n$, $\Sigma_{kl}$ is non-singular with $\Sigma_{kl}^{-1} = \Sigma_{kl}$.
3. Show that $U$ is non-singular, and that for $n \geq 2$:

$$U^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & \ddots & 1
\end{pmatrix}$$
4. Let $M = (m_{ij}) \in \mathcal{M}_n(K)$. Let $R_1, \ldots, R_n$ be the rows of $M$:

$$M \triangleq \begin{pmatrix}
R_1 \\
R_2 \\
\vdots \\
R_n
\end{pmatrix}$$

Show that for all $\alpha \in K$:

$$H_\alpha M = \begin{pmatrix}
\alpha R_1 \\
R_2 \\
\vdots \\
R_n
\end{pmatrix}$$

Conclude that multiplying $M$ by $H_\alpha$ from the left, amounts to multiplying the first row of $M$ by $\alpha$.

5. Show that multiplying $M$ by $H_\alpha$ from the right, amounts to multiplying the first column of $M$ by $\alpha$.

6. Show that multiplying $M$ by $\Sigma_{kl}$ from the left, amounts to interchanging the rows $R_l$ and $R_k$.

7. Show that multiplying $M$ by $\Sigma_{kl}$ from the right, amounts to interchanging the columns $C_l$ and $C_k$.

8. Show that multiplying $M$ by $U^{-1}$ from the left ($n \geq 2$), amounts to subtracting $R_1$ from $R_2$, i.e.:

$$U^{-1}. \begin{pmatrix}
R_1 \\
R_2 \\
\vdots \\
R_n
\end{pmatrix} = \begin{pmatrix}
R_1 \\
R_2 - R_1 \\
\vdots \\
R_n
\end{pmatrix}$$

9. Show that multiplying $M$ by $U^{-1}$ from the right (for $n \geq 2$), amounts to subtracting $C_2$ from $C_1$.

10. Define $U' = \Sigma_{12}.U^{-1}.\Sigma_{12}$, ($n \geq 2$). Show that multiplying $M$ by $U'$ from the right, amounts to subtracting $C_1$ from $C_2$.

11. Show that if $n = 1$, then indeed we have $\mathcal{M}_1(K) = \mathcal{M}_1'(K)$.

Exercise 2. Further to exercise (1), we now assume that $n \geq 2$, and make the induction hypothesis that $\mathcal{M}_{n-1}(K) = \mathcal{M}_{n-1}'(K)$.

1. Let $O_n \in \mathcal{M}_n(K)$ be the matrix with all entries equal to zero. Show the existence of $Q_1', \ldots, Q_p' \in \mathcal{N}_{n-1}(K)$, $p \geq 1$, such that:

$$O_{n-1} = Q_1' \cdots Q_p'$$
2. For $k = 1, \ldots, p$, we define $Q_k \in \mathcal{M}_n(\mathbf{K})$, by:

$$Q_k \triangleq \begin{pmatrix}
0 \\
Q_k' \\
0 \\
0 \\
0 \\
1
\end{pmatrix}$$

Show that $Q_k \in \mathcal{N}_n(\mathbf{K})$, and that we have:

$$\Sigma_1 Q_1 \ldots Q_p \Sigma_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \ddots & & \\
& & \ddots & 0 \\
& & & 0 & O_{n-1}
\end{pmatrix}$$

3. Conclude that $O_n \in \mathcal{M}'_n(\mathbf{K})$.

4. We now consider $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, $M \neq O_n$. We want to show that $M \in \mathcal{M}'_n(\mathbf{K})$. Show that for some $k, l \in \{1, \ldots, n\}$:

$$H^{-1}_{m_{kl}} \Sigma_{1k} M \Sigma_{1l} = \begin{pmatrix}
1 & * & \ldots & * \\
* & \ddots & & \\
& & \ddots & 0 \\
& & & 1
\end{pmatrix}$$

5. Show that if $H^{-1}_{m_{kl}} \Sigma_{1k} M \Sigma_{1l} \in \mathcal{M}'_n(\mathbf{K})$, then $M \in \mathcal{M}'_n(\mathbf{K})$. Conclude that without loss of generality, in order to prove that $M$ lies in $\mathcal{M}'_n(\mathbf{K})$ we can assume that $m_{11} = 1$.

6. Let $i = 2, \ldots, n$. Show that if $m_{i1} \neq 0$, we have:

$$H^{-1}_{m_{i1}} \Sigma_{2i} U^{-1}, \Sigma_{2i} H^{-1}_{1/m_{i1}} M = \begin{pmatrix}
1 & * & \ldots & * \\
0 & \ddots & & \\
& & \ddots & 0 \\
& & & 1
\end{pmatrix}$$

7. Conclude that without loss of generality, we can assume that $m_{i1} = 0$ for all $i \geq 2$, i.e. that $M$ is of the form:

$$M = \begin{pmatrix}
1 & * & \ldots & * \\
0 & \ddots & & \\
& & \ddots & * \\
& & & 0
\end{pmatrix}$$

8. Show that in order to prove that $M \in \mathcal{M}'_n(\mathbf{K})$, without loss of generality, we can assume that $M$ is of the form:

$$M = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \ddots & & \\
& & \ddots & M' \\
& & & 0
\end{pmatrix}$$

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9. Prove that \( M \in \mathcal{M}_n'(K) \) and conclude with the following:

**Theorem 103**  Given \( n \geq 2 \), any \( n \times n \)-matrix with values in \( K \) is a finite product of matrices \( Q \) of the following types:

\[
\begin{align*}
(i) \quad & Qe_1 = \alpha e_1, Qe_j = e_j, \forall j = 2, \ldots, n, (\alpha \in K) \\
(ii) \quad & Qe_i = e_k, Qe_k = e_i, Qe_j = e_j, \forall j \neq k, l, (k, l \in \mathbb{N}_n) \\
(iii) \quad & Qe_1 = e_1 + e_2, Qe_j = e_j, \forall j = 2, \ldots, n
\end{align*}
\]

where \((e_1, \ldots, e_n)\) is the canonical basis of \( K^n \).

**Definition 123**  Let \( X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}') \) be a measurable map, where \((\Omega, \mathcal{F})\) and \((\Omega', \mathcal{F}')\) are two measurable spaces. Let \( \mu \) be a (possibly complex) measure on \((\Omega, \mathcal{F})\). Then, we call distribution of \( X \) under \( \mu \), or image measure of \( \mu \) by \( X \), or even law of \( X \) under \( \mu \), the (possibly complex) measure on \((\Omega', \mathcal{F}')\), denoted \( \mu^X \), \( X(\mu) \) or \( L_\mu(X) \), and defined by:

\[ \forall B \in \mathcal{F}', \mu^X(B) = \mu((X \in B)) = \mu(X^{-1}(B)) \]

**Exercise 3.** Let \( X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}') \) be a measurable map, where \((\Omega, \mathcal{F})\) and \((\Omega', \mathcal{F}')\) are two measurable spaces.

1. Let \( B \in \mathcal{F}' \). Show that if \((B_n)_{n \geq 1}\) is a measurable partition of \( B \), then \((X^{-1}(B_n))_{n \geq 1}\) is a measurable partition of \( X^{-1}(B) \).
2. Show that if \( \mu \) is a measure on \((\Omega, \mathcal{F})\), \( \mu^X \) is a well-defined measure on \((\Omega', \mathcal{F}')\).
3. Show that if \( \mu \) is a complex measure on \((\Omega, \mathcal{F})\), \( \mu^X \) is a well-defined complex measure on \((\Omega', \mathcal{F}')\).
4. Show that if \( \mu \) is a complex measure on \((\Omega, \mathcal{F})\), then \( |\mu^X| \leq |\mu|^X \).
5. Let \( Y : (\Omega', \mathcal{F}') \to (\Omega'', \mathcal{F}'') \) be a measurable map, where \((\Omega'', \mathcal{F}'')\) is another measurable space. Show that for all (possibly complex) measure \( \mu \) on \((\Omega, \mathcal{F})\), we have:

\[ Y(X(\mu)) = (Y \circ X)(\mu) = (\mu^X)^Y = \mu^{(Y \circ X)} \]

**Definition 124**  Let \( \mu \) be a (possibly complex) measure on \( \mathbb{R}^n \), \( n \geq 1 \). We say that \( \mu \) is invariant by translation, if and only if \( \tau_a(\mu) = \mu \) for all \( a \in \mathbb{R}^n \), where \( \tau_a : \mathbb{R}^n \to \mathbb{R}^n \) is the translation mapping defined by \( \tau_a(x) = a + x \), for all \( x \in \mathbb{R}^n \).

**Exercise 4.** Let \( \mu \) be a (possibly complex) measure on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\).

1. Show that \( \tau_a : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \) is measurable.
2. Show \( \tau_a(\mu) \) is therefore a well-defined (possibly complex) measure on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\), for all \( a \in \mathbb{R}^n \).

3. Show that \( \tau_a(dx) = dx \) for all \( a \in \mathbb{R}^n \).

4. Show the Lebesgue measure on \( \mathbb{R}^n \) is invariant by translation.

**Exercise 5.** Let \( k_\alpha : \mathbb{R}^n \to \mathbb{R}^n \) be defined by \( k_\alpha(x) = \alpha x \), \( \alpha > 0 \).

1. Show that \( k_\alpha : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \) is measurable.

2. Show that \( k_\alpha(dx) = \alpha^{-n}dx \).

**Exercise 6.** Show the following:

**Theorem 104 (Integral Projection 1)** Let \( X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}') \) be a measurable map, where \((\Omega, \mathcal{F}), (\Omega', \mathcal{F}')\) are measurable spaces. Let \( \mu \) be a measure on \((\Omega, \mathcal{F})\). Then, for all \( f : (\Omega', \mathcal{F}') \to [0, +\infty] \) non-negative and measurable, we have:

\[
\int_{\Omega} f \circ X \, d\mu = \int_{\Omega'} f \, dX(\mu)
\]

**Exercise 7.** Show the following:

**Theorem 105 (Integral Projection 2)** Let \( X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}') \) be a measurable map, where \((\Omega, \mathcal{F}), (\Omega', \mathcal{F}')\) are measurable spaces. Let \( \mu \) be a measure on \((\Omega, \mathcal{F})\). Then, for all \( f : (\Omega', \mathcal{F}') \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) measurable, we have the equivalence:

\[
f \circ X \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu) \iff f \in L^1_{\mathbb{C}}(\Omega', \mathcal{F}', X(\mu))
\]

in which case, we have:

\[
\int_{\Omega} f \circ X \, d\mu = \int_{\Omega'} f \, dX(\mu)
\]

**Exercise 8.** Further to theorem (105), suppose \( \mu \) is in fact a complex measure on \((\Omega, \mathcal{F})\). Show that:

\[
\int_{\Omega'} |f|d|X(\mu)| \leq \int_{\Omega} |f \circ X|d|\mu|
\]

Conclude with the following:

**Theorem 106 (Integral Projection 3)** Let \( X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}') \) be a measurable map, where \((\Omega, \mathcal{F}), (\Omega', \mathcal{F}')\) are measurable spaces. Let \( \mu \) be a complex measure on \((\Omega, \mathcal{F})\). Then, for all measurable maps \( f : (\Omega', \mathcal{F}') \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \), we have:

\[
f \circ X \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu) \Rightarrow f \in L^1_{\mathbb{C}}(\Omega', \mathcal{F}', X(\mu))
\]

and when the left-hand side of this implication is satisfied:

\[
\int_{\Omega} f \circ X \, d\mu = \int_{\Omega'} f \, dX(\mu)
\]
Exercise 9. Let \( X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) be a measurable map with distribution \( \mu = X(P) \), where \( (\Omega, \mathcal{F}, P) \) is a probability space.

1. Show that \( X \) is integrable, i.e. \( \int |X| dP < +\infty \), if and only if:
   \[
   \int_{-\infty}^{+\infty} |x| d\mu(x) < +\infty
   \]

2. Show that if \( X \) is integrable, then:
   \[
   E[X] = \int_{-\infty}^{+\infty} x d\mu(x)
   \]

3. Show that:
   \[
   E[X^2] = \int_{-\infty}^{+\infty} x^2 d\mu(x)
   \]

Exercise 10. Let \( \mu \) be a locally finite measure on \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \), which is invariant by translation. For all \( a = (a_1, \ldots, a_n) \in (\mathbb{R}^+)^n \), we define \( Q_a = [0, a_1] \times \ldots \times [0, a_n] \), and in particular \( Q = Q(1, \ldots, 1) = [0, 1]^n \).

1. Show that \( \mu(Q_a) < +\infty \) for all \( a \in (\mathbb{R}^+)^n \), and \( \mu(Q) < +\infty \).

2. Let \( p = (p_1, \ldots, p_n) \) where \( p_i \geq 1 \) is an integer for all \( i \)'s. Show:
   \[
   Q_p = \bigcup_{k \in \mathbb{N}^n} \left[ \begin{array}{c} k_1 + 1 \\ k_1 \\ 0 \leq k_i < p_i \end{array} \right] \times \ldots \times \left[ \begin{array}{c} k_n + 1 \\ k_n \\ 0 \leq k_i < p_i \end{array} \right]
   \]

3. Show that \( \mu(Q_p) = p_1 \ldots p_n \mu(Q) \).

4. Let \( q_1, \ldots, q_n \geq 1 \) be \( n \) positive integers. Show that:
   \[
   Q_p = \bigcup_{k \in \mathbb{N}^n} \left[ \begin{array}{c} k_1 p_1 \\ q_1 \\ 0 \leq k_i < q_i \end{array} \right] \times \ldots \times \left[ \begin{array}{c} k_n p_n \\ q_n \\ 0 \leq k_i < q_i \end{array} \right] \times \left[ \begin{array}{c} (k_1 + 1) p_1 \\ q_1 \\ 0 \leq k_i < q_i \end{array} \right] \times \ldots \times \left[ \begin{array}{c} (k_n + 1) p_n \\ q_n \\ 0 \leq k_i < q_i \end{array} \right]
   \]

5. Show that \( \mu(Q_p) = q_1 \ldots q_n \mu(Q(q_1/q_1, \ldots, q_n/q_n)) \)

6. Show that \( \mu(Q_r) = r_1 \ldots r_n \mu(Q) \), for all \( r \in (\mathbb{Q}^+)^n \).

7. Show that \( \mu(Q_a) = a_1 \ldots a_n \mu(Q) \), for all \( a \in (\mathbb{R}^+)^n \).

8. Show that \( \mu(B) = \mu(Q) dx(B) \), for all \( B \in \mathcal{C} \), where:
   \[
   \mathcal{C} \triangleq \{ [a_1, b_1] \times \ldots \times [a_n, b_n] : a_i, b_i \in \mathbb{R}, a_i \leq b_i, \forall i \in \mathbb{N}^n \}
   \]

9. Show that \( B(\mathbb{R}^n) = \sigma(\mathcal{C}) \).

10. Show that \( \mu = \mu(Q) dx \), and conclude with the following:
Theorem 107  Let \( \mu \) be a locally finite measure on \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \). If \( \mu \) is invariant by translation, then there exists \( \alpha \in \mathbb{R}^+ \) such that:

\[
\mu = \alpha dx
\]

Exercise 11. Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear bijection.

1. Show that \( T \) and \( T^{-1} \) are continuous.
2. Show that for all \( B \subseteq \mathbb{R}^n \), the inverse image \( T^{-1}(B) = \{ T \in B \} \) coincides with the direct image:

\[
T^{-1}(B) \triangleq \{ y : y = T^{-1}(x) \text{ for some } x \in B \}
\]
3. Show that for all \( B \subseteq \mathbb{R}^n \), the direct image \( T(B) \) coincides with the inverse image \((T^{-1})^{-1}(B) = \{ T^{-1} \in B \} \).
4. Let \( K \subseteq \mathbb{R}^n \) be compact. Show that \( \{ T \in K \} \) is compact.
5. Show that \( T(dx) \) is a locally finite measure on \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \).
6. Let \( \tau_a \) be the translation of vector \( a \in \mathbb{R}^n \). Show that:

\[
T \circ \tau_{T^{-1}(a)} = \tau_a \circ T
\]
7. Show that \( T(dx) \) is invariant by translation.
8. Show the existence of \( \alpha \in \mathbb{R}^+ \), such that \( T(dx) = \alpha dx \). Show that such constant is unique, and denote it by \( \Delta(T) \).
9. Show that \( Q = T([0,1]^n) \in \mathcal{B}(\mathbb{R}^n) \) and that we have:

\[
\Delta(T)dx(Q) = T(dx)(Q) = 1
\]
10. Show that \( \Delta(T) \neq 0 \).
11. Let \( T_1, T_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be two linear bijections. Show that:

\[
(T_1 \circ T_2)(dx) = \Delta(T_1)\Delta(T_2)dx
\]
and conclude that \( \Delta(T_1 \circ T_2) = \Delta(T_1)\Delta(T_2) \).

Exercise 12. Let \( \alpha \in \mathbb{R} \setminus \{0\} \). Let \( H_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the linear bijection uniquely defined by \( H_\alpha(e_1) = \alpha e_1, H_\alpha(e_j) = e_j \) for \( j \geq 2 \).

1. Show that \( H_\alpha(dx)([0,1]^n) = |\alpha|^{-1} \).
2. Conclude that \( \Delta(H_\alpha) = |\det H_\alpha|^{-1} \).

Exercise 13. Let \( k, l \in \mathbb{N}_n \) and \( \Sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the linear bijection uniquely defined by \( \Sigma(e_k) = e_l, \Sigma(e_l) = e_k, \Sigma(e_j) = e_j \), for \( j \neq k, l \).

1. Show that \( \Sigma(dx)([0,1]^n) = 1 \).
2. Show that $\Sigma \Sigma = I_n$. (Identity mapping on $\mathbb{R}^n$).

3. Show that $|\det \Sigma| = 1$.

4. Conclude that $\Delta(\Sigma) = |\det \Sigma|^{-1}$.

Exercise 14. Let $n \geq 2$ and $U : \mathbb{R}^n \to \mathbb{R}^n$ be the linear bijection uniquely defined by $U(e_1) = e_1 + e_2$ and $U(e_j) = e_j$ for $j \geq 2$. Let $Q = [0, 1]^n$.

1. Show that:
   
   $U^{-1}(Q) = \{x \in \mathbb{R}^n : 0 \leq x_1 + x_2 < 1, \ 0 \leq x_i < 1, \ \forall i \neq 2\}$

2. Define:
   
   $\Omega_1 = U^{-1}(Q) \cap \{x \in \mathbb{R}^n : x_2 \geq 0\}$
   
   $\Omega_2 = U^{-1}(Q) \cap \{x \in \mathbb{R}^n : x_2 < 0\}$

   Show that $\Omega_1, \Omega_2 \in \mathcal{B}(\mathbb{R}^n)$.

3. Let $\tau_{e_2}$ be the translation of vector $e_2$. Draw a picture of $Q, \Omega_1, \Omega_2$ and $\tau_{e_2}(\Omega_2)$ in the case when $n = 2$.

4. Show that if $x \in \Omega_1$, then $0 \leq x_2 < 1$.

5. Show that $\Omega_1 \subseteq Q$.

6. Show that if $x \in \tau_{e_2}(\Omega_2)$, then $0 \leq x_2 < 1$.

7. Show that $\tau_{e_2}(\Omega_2) \subseteq Q$.

8. Show that if $x \in Q$ and $x_1 + x_2 < 1$ then $x \in \Omega_1$.

9. Show that if $x \in Q$ and $x_1 + x_2 \geq 1$ then $x \in \tau_{e_2}(\Omega_2)$.

10. Show that if $x \in \tau_{e_2}(\Omega_2)$ then $x_1 + x_2 \geq 1$.

11. Show that $\tau_{e_2}(\Omega_2) \cap \Omega_1 = \emptyset$.

12. Show that $Q = \Omega_1 \uplus \tau_{e_2}(\Omega_2)$.

13. Show that $\int_{\Omega} dx = \int_{U^{-1}(Q)} dx$.

14. Show that $\Delta(U) = 1$.

15. Show that $\Delta(U) = |\det U|^{-1}$.

Exercise 15. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection, ($n \geq 1$).

1. Show the existence of linear bijections $Q_1, \ldots, Q_p : \mathbb{R}^n \to \mathbb{R}^n$, $p \geq 1$, with $T = Q_1 \circ \cdots \circ Q_p$, $\Delta(Q_i) = |\det Q_i|^{-1}$ for all $i \in \mathbb{N}_p$.

2. Show that $\Delta(T) = |\det T|^{-1}$. 
3. Conclude with the following:

**Theorem 108** Let \( n \geq 1 \) and \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear bijection. Then, the image measure \( T(dx) \) of the Lebesgue measure on \( \mathbb{R}^n \) is:

\[
T(dx) = |\det T|^{-1}dx
\]

**Exercise 16.** Let \( f : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \rightarrow [0, +\infty] \) be a non-negative and measurable map. Let \( a, b, c, d \in \mathbb{R} \) such that \( ad - bc \neq 0 \). Show that:

\[
\int_{\mathbb{R}^2} f(ax + by, cx + dy)dxdy = |ad - bc|^{-1}\int_{\mathbb{R}^2} f(x, y)dxdy
\]

**Exercise 17.** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear bijection. Show that for all \( B \in \mathcal{B}(\mathbb{R}^n) \), we have \( T(B) \in \mathcal{B}(\mathbb{R}^n) \) and:

\[
dx(T(B)) = |\det T|\dx(B)
\]

**Exercise 18.** Let \( V \) be a linear subspace of \( \mathbb{R}^n \) and \( p = \dim V \). We assume that \( 1 \leq p \leq n - 1 \). Let \( u_1, \ldots, u_p \) be an orthonormal basis of \( V \), and \( u_{p+1}, \ldots, u_n \) be such that \( u_1, \ldots, u_n \) is an orthonormal basis of \( \mathbb{R}^n \). For \( i \in \mathbb{N}_n \), let \( \phi_i : \mathbb{R}^n \rightarrow \mathbb{R} \) be defined by \( \phi_i(x) = \langle u_i, x \rangle \).

1. Show that all \( \phi_i \)'s are continuous.
2. Show that \( V = \bigcap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) \).
3. Show that \( V \) is a closed subset of \( \mathbb{R}^n \).
4. Let \( Q = (q_{ij}) \in M_n(\mathbb{R}) \) be the matrix uniquely defined by \( Qe_j = u_j \) for all \( j \in \mathbb{N}_n \), where \( (e_1, \ldots, e_n) \) is the canonical basis of \( \mathbb{R}^n \). Show that for all \( i, j \in \mathbb{N}_n \):

\[
\langle u_i, u_j \rangle = \sum_{k=1}^{n} q_{ki}q_{kj}
\]

5. Show that \( Q^tQ = I_n \) and conclude that \(|\det Q| = 1\).
6. Show that \( dx(\{Q \in V\}) = dx(V) \).
7. Show that \( \{Q \in V\} = \text{span}(e_1, \ldots, e_p) \).
8. For all \( m \geq 1 \), we define:

\[
E_m \overset{\Delta}{=} [-m, m] \times \cdots \times [-m, m] \times \{0\}
\]

Show that \( dx(E_m) = 0 \) for all \( m \geq 1 \).
9. Show that \( dx(\text{span}(e_1, \ldots, e_{n-1})) = 0 \).

\(^{1}\text{i.e. the linear subspace of } \mathbb{R}^n \text{ generated by } e_1, \ldots, e_p.\)
10. Conclude with the following:

**Theorem 109** Let $n \geq 1$. Any linear subspace $V$ of $\mathbb{R}^n$ is a closed subset of $\mathbb{R}^n$. Moreover, if $\dim V \leq n - 1$, then $dx(V) = 0$. 
Solutions to Exercises

Exercise 1.

1. Let $\alpha \in K \setminus \{0\}$. Then, we have:

$$H_{1/\alpha} \circ H_{\alpha} e_1 = H_{1/\alpha} (\alpha e_1) = \alpha H_{1/\alpha} e_1 = \alpha (1/\alpha) e_1 = e_1$$

and for all $j \geq 2$, $H_{1/\alpha} \circ H_{\alpha} e_j = H_{1/\alpha} e_j = e_j$. If $I_n$ denotes the identity matrix of $M_n(K)$, then $I_n$ and $H_{1/\alpha} \circ H_{\alpha}$ coincide on the basis $(e_1, \ldots, e_n)$ of $K^n$. It follows that $I_n$ and $H_{1/\alpha} \circ H_{\alpha}$ are in fact equal. So $H_\alpha$ is non-singular and $H_{1/\alpha} = H_{1/\alpha}^{-1}$.

2. The linear map $\Sigma_{kl} : K^n \to K^n$ is defined by $\Sigma_{kl} e_k = e_l$, $\Sigma_{kl} e_l = e_k$ and $\Sigma_{kl} e_j = e_j$ for all $j \notin \{k, l\}$. Hence, it is clear that $\Sigma_{kl} \circ \Sigma_{kl} e_j = e_j$ for all $j \in N_n$, and consequently $\Sigma_{kl} \circ \Sigma_{kl} = I_n$. So $\Sigma_{kl}$ is non-singular and $\Sigma_{kl}^{-1} = \Sigma_{kl}$.

3. If $n = 1$, then $U = 1$ and it is indeed non-singular. We assume that $n \geq 2$. Then $U$ is defined by $U e_1 = e_1 + e_2$ and $U e_j = e_j$ for all $j \geq 2$. Consider the linear map $U' : K^n \to K^n$ defined by $U' e_1 = e_1 - e_2$ and $U' e_j = e_j$ for all $j \geq 2$. Then, we have:

$$U' \circ U e_1 = U'(e_1 + e_2) = U' e_1 + U' e_2 = e_1 - e_2 + e_2 = e_1$$

and it is clear that $U' \circ U e_j = e_j$ for all $j \geq 2$. It follows that $U' \circ U e_j = e_j$ for all $j \in N_n$ and consequently $U' \circ U = I_n$. We have proved that $U$ is invertible and $U^{-1} = U'$, i.e.:

$$U^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & \ddots & 1 \\
\end{pmatrix}$$

4. Let $M = (m_{ij}) \in M_n(K)$, and $R_1, \ldots, R_n$ be the rows of $M$, i.e.

$$M \triangleq \begin{pmatrix}
R_1 \\
R_2 \\
\vdots \\
R_n
\end{pmatrix}$$

Specifically, for all $i \in N_n$, each $R_i$ is the row vector:

$$R_i = (m_{i1}, m_{i2}, \ldots, m_{in})$$

Let $\alpha \in K$, and consider the matrix $M' \in M_n(K)$ defined by:

$$M' \triangleq \begin{pmatrix}
\alpha R_1 \\
R_2 \\
\vdots \\
R_n
\end{pmatrix}$$
i.e. \( M'e_j = \alpha m_{1j}e_1 + \sum_{i=2}^{n} m_{ij}e_i \) for all \( j \in \mathbb{N}_n \). Then:

\[
H_\alpha \circ M e_j = H_\alpha \left( \sum_{i=1}^{n} m_{ij}e_i \right) \\
= \sum_{i=1}^{n} m_{ij}H_\alpha e_i \\
= m_{1j}H_\alpha e_1 + \sum_{i=2}^{n} m_{ij}H_\alpha e_i \\
= \alpha m_{1j}e_1 + \sum_{i=2}^{n} m_{ij}e_i \\
= M'e_j
\]

This being true for all \( j \in \mathbb{N}_n \), we have proved that \( H_\alpha M = M' \), i.e.

\[
H_\alpha M = \begin{pmatrix}
\alpha R_1 \\
R_2 \\
\vdots \\
R_n
\end{pmatrix}
\]

We conclude that multiplying \( M \) by \( H_\alpha \) from the left, amounts to multiplying the first row of \( M \) by \( \alpha \).

5. Let \( M = (m_{ij}) \in \mathcal{M}_n(\mathbb{K}) \), and \( C_1, \ldots, C_n \) be the columns of \( M \):

\[
M \triangleq (C_1, C_2, \ldots, C_n)
\]

Specifically, for all \( j \in \mathbb{N}_n \), each \( C_j \) is the column vector:

\[
C_j = \begin{pmatrix}
m_{1j} \\
m_{2j} \\
\vdots \\
m_{nj}
\end{pmatrix}
\]

Let \( \alpha \in \mathbb{K} \), and consider the matrix \( M' \) defined by:

\[
M' = (\alpha C_1, C_2, \ldots, C_n)
\]

i.e. \( M'e_1 = \sum_{i=1}^{n} \alpha m_{i1}e_i \) and \( M'e_j = \sum_{i=1}^{n} m_{ij}e_i \) for \( j \geq 2 \):

\[
M \circ H_\alpha e_1 = M(\alpha e_1) = \alpha Me_1 = \sum_{i=1}^{n} \alpha m_{i1}e_i = M'e_1
\]

and furthermore, for all \( j \geq 2 \):

\[
M \circ H_\alpha e_j = Me_j = \sum_{i=1}^{n} m_{ij}e_i = M'e_j
\]
So $M \circ H_\alpha e_j = M' e_j$ for all $j \in \mathbb{N}_n$, i.e. $MH_\alpha = M'$. Hence:

$$MH_\alpha = (\alpha C_1, C_2, \ldots, C_n)$$

We conclude that multiplying $M$ by $H_\alpha$ from the right, amounts to multiplying the first column of $M$ by $\alpha$.

6. Let $M = (m_{ij}) \in \mathcal{M}_n(K)$ and $R_1, \ldots, R_n$ be the rows of $M$, i.e.

$$M \triangleq \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Specifically, for all $i \in \mathbb{N}_n$, $R_i$ is the row vector:

$$R_i = (m_{i1}, m_{i2}, \ldots, m_{in})$$

Let $M' = (m'_{ij}) \in \mathcal{M}_n(K)$ be the matrix defined by:

$$M' \triangleq \begin{pmatrix} R'_1 \\ R'_2 \\ \vdots \\ R'_n \end{pmatrix}$$

where $R'_k = R_l$, $R'_l = R_k$ and $R'_i = R_i$ for all $i \notin \{k, l\}$. In other words, the matrix $M'$ is nothing but the matrix $M$, where the rows $R_k$ and $R_l$ have been interchanged. Note that for all $i, j \in \mathbb{N}_n$, $m'_{kj} = m_{lj}$, $m'_{lj} = m_{kj}$ and $m'_{ij} = m_{ij}$ for all $i \notin \{k, l\}$. Now, given $j \in \mathbb{N}_n$, we have:

$$\Sigma_{kl} \circ M e_j = \Sigma_{kl} \left( \sum_{i=1}^{n} m_{ij} e_i \right) = \sum_{i=1}^{n} m_{ij} \Sigma_{kl} e_i = \sum_{i \neq k, l} m_{ij} e_i + m_{kj} e_l + m_{lj} e_k = \sum_{i \neq k, l} m'_{ij} e_i + m'_{lj} e_l + m'_{kj} e_k = \sum_{i=1}^{n} m'_{ij} e_i = M' e_j$$

This being true for all $j \in \mathbb{N}_n$, $\Sigma_{kl} M = M'$. We conclude that multiplying $M$ by $\Sigma_{kl}$ from the left, amounts to interchanging the rows $R_l$ and $R_k$ of $M$.

7. Let $M = (m_{ij}) \in \mathcal{M}_n(K)$, and $C_1, \ldots, C_n$ be the columns of $M$:

$$M \triangleq (C_1, C_2, \ldots, C_n)$$

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Specifically, for all \( j \in \mathbb{N}_n \), each \( C_j \) is the column vector:

\[
C_j = \begin{pmatrix}
m_{1j} \\
m_{2j} \\
\vdots \\
m_{nj}
\end{pmatrix}
\]

Let \( M' = (m'_{ij}) \in \mathcal{M}_n(\mathbb{K}) \) be the matrix defined by:

\[
M' \triangleq (C'_1, C'_2, \ldots, C'_n)
\]

where \( C'_k = C_k \), \( C'_l = C_l \) and \( C'_j = C_j \) for all \( j \not\in \{k, l\} \). In other words, the matrix \( M' \) is nothing but the matrix \( M \), where the columns \( C_k \) and \( C_l \) have been interchanged. For all \( i, j \in \mathbb{N}_n \), \( m'_{ik} = m_{il} \), \( m'_{il} = m_{ik} \) and \( m'_{ij} = m_{ij} \) for all \( j \not\in \{k, l\} \). Now:

\[
M \circ \Sigma_{kl} e_k = Me_l = \sum_{i=1}^{n} m_{il} e_i = \sum_{i=1}^{n} m'_{ik} e_i = M' e_k
\]

and similarly \( M \circ \Sigma_{kl} e_l = M' e_l \). Furthermore, if \( j \neq k, l \):

\[
M \circ \Sigma_{kl} e_j = Me_j = \sum_{i=1}^{n} m_{ij} e_i = \sum_{i=1}^{n} m'_{ij} e_i = M' e_j
\]

It follows that \( M \circ \Sigma_{kl} e_j = M' e_j \) for all \( j \in \mathbb{N}_n \). We conclude that \( M \Sigma_{kl} = M' \) and consequently, multiplying \( M \) by \( \Sigma_{kl} \) from the right, amounts to interchanging the columns \( C_l \) and \( C_k \) of \( M \).

8. Let \( M = (m_{ij}) \in \mathcal{M}_n(\mathbb{K}) \) and \( R_1, \ldots, R_n \) be the rows of \( M \), i.e.

\[
M \triangleq \begin{pmatrix}
R_1 \\
R_2 \\
\vdots \\
R_n
\end{pmatrix}
\]

Specifically, for all \( i \in \mathbb{N}_n \), \( R_i \) is the row vector:

\[
R_i = (m_{i1}, m_{i2}, \ldots, m_{in})
\]
Let $M' = (m'_{ij}) \in M_n(\mathbb{K})$ be the matrix defined by:

$$M' \triangleq \begin{pmatrix} R_1 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix}$$

Specifically, $M'$ is exactly the matrix $M$, where the second row $R_2$ has been replaced by $R_2 - R_1$, i.e. where the first row $R_1$ has been subtracted from the second row $R_2$. Recall from 3. that $U^{-1}$ is given by $U^{-1}e_1 = e_1 - e_2$ and $U^{-1}e_j = e_j$ for all $j \geq 2$. Note that for all $i, j \in \mathbb{N}_n$, we have $m'_{ij} = m_{ij}$ if $i \neq 2$, and $m'_{2j} = m_{2j} - m_{1j}$. Now for all $j \in \mathbb{N}_n$:

$$U^{-1}Me_j = U^{-1} \left( \sum_{i=1}^n m_{ij}e_i \right)$$

$$= \sum_{i=1}^n m_{ij}U^{-1}e_i$$

$$= m_{1j}(e_1 - e_2) + \sum_{i=2}^n m_{ij}e_i$$

$$= \sum_{i \neq 2} m_{ij}e_i + (m_{2j} - m_{1j})e_2$$

$$= \sum_{i=1}^n m'_{ij}e_i = M'e_j$$

It follows that $U^{-1}M = M'$, and we conclude that multiplying $M$ by $U^{-1}$ from the left, amounts to subtracting $R_1$ from $R_2$.

9. Let $M = (m_{ij}) \in M_n(\mathbb{K})$, and $C_1, \ldots, C_n$ be the columns of $M$:

$$M \triangleq (C_1, C_2, \ldots, C_n)$$

Specifically, for all $j \in \mathbb{N}_n$, each $C_j$ is the column vector:

$$C_j = \begin{pmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{pmatrix}$$

Let $M' = (m'_{ij}) \in M_n(\mathbb{K})$ be the matrix defined by:

$$M' \triangleq (C_1 - C_2, C_2, \ldots, C_n)$$

Specifically, $M'$ is exactly the matrix $M$, where the second column $C_2$ has been subtracted from the first column $C_1$. For all $i, j \in \mathbb{N}_n$, we have
\[ m'_{ij} = m_{ij} \text{ if } j \neq 1 \text{ and } m'_{i1} = m_{i1} - m_{i2}. \] Furthermore:

\[
MU^{-1}e_1 = M(e_1 - e_2) = Me_1 - Me_2 = \sum_{i=1}^{n} m_{i1}e_i - \sum_{i=1}^{n} m_{i2}e_i = \sum_{i=1}^{n} (m_{i1} - m_{i2})e_i = \sum_{i=1}^{n} m'_{i1}e_i = M'e_1
\]

and for all \( j \geq 2 \):

\[
MU^{-1}e_j = Me_j = \sum_{i=1}^{n} m_{ij}e_i = \sum_{i=1}^{n} m'_{ij}e_i = M'e_j
\]

Having proved that \( MU^{-1}e_j = M'e_j \) for all \( j \in \mathbb{N}_n \), we conclude that \( MU^{-1} = M' \), or equivalently that multiplying \( M \) by \( U^{-1} \) from the right, amounts to subtracting \( C_2 \) from \( C_1 \).

10. Let \( U' = \Sigma_{12}U^{-1}\Sigma_{12} \). Let \( C_1, \ldots, C_2 \) be the column vectors of \( M \in \mathcal{M}_n(\mathbb{K}) \). It follows from 7. and 9. that:

\[
MU' = M\Sigma_{12}U^{-1}\Sigma_{12} = (C_1, C_2, \ldots, C_n)\Sigma_{12}U^{-1}\Sigma_{12} = (C_2, C_1, \ldots, C_n)\Sigma_{12} = (C_1, C_2 - C_1, \ldots, C_n)
\]

We conclude that multiplying \( M \) by \( U' \) from the right, amounts to subtracting \( C_1 \) from \( C_2 \).

11. Suppose \( n = 1 \). It is clear that \( \mathcal{M}'_n(\mathbb{K}) \subseteq \mathcal{M}_n(\mathbb{K}) \) for all \( n \geq 1 \), and in particular \( \mathcal{M}'_1(\mathbb{K}) \subseteq \mathcal{M}_1(\mathbb{K}) \). Suppose \( M \in \mathcal{M}_1(\mathbb{K}) \). Then \( M = (\alpha) \) for some \( \alpha \in \mathbb{K} \). However, \( (\alpha) = H_\alpha \) (one-dimensional). Hence, defining \( Q_1 = H_\alpha \), we have \( Q_1 \in \mathcal{N}_1(\mathbb{K}) \) with \( M = Q_1 \). In particular, \( M \) is a finite product of elements of \( \mathcal{N}_1(\mathbb{K}) \). So \( M \in \mathcal{M}'_1(\mathbb{K}) \) and we have proved the equality \( \mathcal{M}_1(\mathbb{K}) = \mathcal{M}'_1(\mathbb{K}) \).

Exercise 1

Exercise 2.
1. Our induction hypothesis is $M_{n-1}(K) = M_{n-1}'(K)$, $n \geq 2$. For all $n \geq 1$, $O_n \in M_n(K)$ denotes the matrix with all entries equal to 0 in $K$. Since $O_{n-1} \in M_{n-1}(K) = M_{n-1}'(K)$, $O_{n-1}$ is a finite product of elements of $N_{n-1}(K)$. Hence, there exist $p \geq 1$ and $Q'_1, \ldots, Q'_p$ elements of $N_{n-1}(K)$ such that:

$$O_{n-1} = Q'_1 \ldots Q'_p$$

2. Given $k \in \{1, \ldots, p\} = N_p$, we define $Q_k \in M_n(K)$ by:

$$Q_k \triangleq \begin{pmatrix} 0 & & & & \\ Q'_k & & & & \\ & 0 & & & \\ & & \ldots & & \\ & & & 0 & 1 \end{pmatrix}$$

Since $Q'_k \in N_{n-1}(K)$, $Q'_k$ can be of three different forms: If $Q'_k$ is of the form $H_{\alpha}$ (of dimension $n-1$) for some $\alpha \in K$, it is clear that $Q_k = H_{\alpha}$, (of dimension $n$). If $Q'_k$ is of the form $S_{lm}$ for some $l, m \in N_{n-1}$, then $Q'_k e_l = e_m$, $Q'_k e_m = e_l$ and $Q'_k e_j = e_j$ for all $j \in N_{n-1} \setminus \{l, m\}$. Hence, it is clear that $Q_k e_l = e_m$, $Q_k e_m = e_l$ and $Q_k e_j = e_j$ for all $j \in N_n \setminus \{l, m\}$.

So $Q_k$ is of the form $S_{lm}$ (of dimension $n$) for some $l, m \in N_n$ (in fact, for some $l, m \in N_{n-1}$). Note that we have used the same notation $e_1, \ldots, e_{n-1}$ and $e_1, \ldots, e_n$ to denote successively the canonical basis of $K^{n-1}$ and $K^n$.

Now, if $Q'_k = U$ (of dimension $n-1$), it is clear that $Q_k = U$ (of dimension $n$) in the case when $n-1 \geq 2$. In the case when $n-1 = 1$, we have $Q'_k = (1)$ and consequently $Q_k = I_2 = H_1$ (of dimension 2). In any case, we see that $Q_k$ is an element of $N_{n-1}(K)$.

Now, using 6. and 7. together with block matrix multiplication, we obtain:

$$\Sigma_{1n} Q_1 \ldots Q_p \Sigma_{1n} = \Sigma_{1n} \cdot \begin{pmatrix} Q'_1 \ldots Q'_p & & & & \\ & 0 & & & \\ & & \ldots & & \\ & & & 0 & 1 \end{pmatrix} \cdot \Sigma_{1n}$$

$$= \Sigma_{1n} \cdot \begin{pmatrix} O_{n-1} & & & & \\ & 0 & & & \\ & & \ldots & & \\ & & & 0 & 1 \end{pmatrix} \cdot \Sigma_{1n}$$

$$= \Sigma_{1n} \cdot \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & O_{n-1} & & \\ & & & 0 & \ddots \\ & & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & 0 \end{pmatrix}$$

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which is exactly what we intended to prove.

3. Having proved that:

\[
\Sigma_{1n} \cdot Q_1 \cdots Q_p \cdot \Sigma_{1n} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & & & \\
\vdots & & \ddots & O_{n-1} \\
0 & & & 
\end{pmatrix}
\]

since \( H_0 \) can be written as:

\[
H_0 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & \cdots \\
0 & \cdots & 1 
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & & & \\
\vdots & & \ddots & I_{n-1} \\
0 & & & 
\end{pmatrix}
\]

we obtain:

\[
H_0 \cdot \Sigma_{1n} \cdot Q_1 \cdots Q_p \cdot \Sigma_{1n} = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & & & \\
\vdots & & \ddots & O_{n-1} \\
0 & & & 
\end{pmatrix} = O_n
\]

We have been able to express \( O_n \) as a finite product of elements of \( N_n(K) \). We conclude that \( O_n \in M'_n(K) \).

4. Let \( M = (m_{ij}) \in M_n(K) \). We assume that \( M \neq O_n \). Then, there exist \( k, l \in \mathbb{N}_n \) such that \( m_{kl} \neq 0 \). From 7. of exercise (1), multiplying \( M \) by \( \Sigma_{1l} \) from the right, amounts to interchanging column \( l \) with column 1. So \( m_{kl} \) appears in the matrix \( M \Sigma_{1l} \) as the \( k \)-th element of the first column. Multiplying \( M \Sigma_{1l} \) by \( \Sigma_{1k} \) from the left, amounts to interchanging row \( k \) with row 1. So \( m_{kl} \) now appears in the matrix \( \Sigma_{1k} M \Sigma_{1l} \) at the intersection of the first row and the first column, i.e. at the top left position. In other words, \( \Sigma_{1k} M \Sigma_{1l} \) is of the form:

\[
\Sigma_{1k} M \Sigma_{1l} = \begin{pmatrix}
m_{kl} & * & \cdots & * \\
* & & & \\
\vdots & & \ddots & * \\
* & & & 
\end{pmatrix}
\]

Multiplying by \( H^{-1}_{m_{kl}} = H_{1/m_{kl}} \) from the left, amounts to multiplying the first row by \( 1/m_{kl} \). We conclude that:

\[
H^{-1}_{m_{kl}} \Sigma_{1k} M \Sigma_{1l} = \begin{pmatrix}
1 & * & \cdots & * \\
* & & & \\
\vdots & & \ddots & * \\
* & & & 
\end{pmatrix}
\]
5. Suppose we have proved $H_{m_{ki}}^{-1} \Sigma_{lk} M \Sigma_{lt} \in \mathcal{M}_n'(K)$. Then this matrix is a finite product of elements of $\mathcal{N}_n(K)$. In other words, there exist $p \geq 1$ and $Q_1, \ldots, Q_p$ elements of $\mathcal{N}_n(K)$ with:

$$H_{m_{ki}}^{-1} \Sigma_{lk} M \Sigma_{lt} = Q_1 \ldots Q_p$$

Since $\Sigma_{lk} = \Sigma_{lk}$ and $\Sigma_{lt}^{-1} = \Sigma_{lt}$, we obtain:

$$M = \Sigma_{lk} H_{m_{ki}} Q_1 \ldots Q_p \Sigma_{lt}$$

So $M$ is therefore also a finite product of elements of $\mathcal{N}_n(K)$, i.e. $M \in \mathcal{M}_n'(K)$. Hence, in order to prove that $M \in \mathcal{M}_n'(K)$ it is sufficient to prove that $H_{m_{ki}}^{-1} \Sigma_{lk} M \Sigma_{lt}$ is an element of $\mathcal{M}_n'(K)$. It follows from 4. that without loss of generality, we may assume that $m_{i1} = 1$.

6. Let $i \in \{2, \ldots, n\}$ and suppose $m_{i1} \neq 0$. So $M$ is of the form:

$$M = \begin{pmatrix} 1 & * & \cdots & * \\ * & \ddots & \cdots & * \\ m_{i1} & \cdots & \star \\ * & \ddots & \cdots & * \end{pmatrix}$$

with $m_{i1} \neq 0$. Since $H_{1/m_{i1}}^{-1} = H_{m_{i1}}$, multiplying $M$ by $H_{1/m_{i1}}^{-1}$ from the left amounts to multiplying the first row of $M$ by $m_{i1}$. So $H_{1/m_{i1}}^{-1} M$ is of the form:

$$H_{1/m_{i1}}^{-1} M = \begin{pmatrix} m_{i1} & * & \cdots & * \\ * & \ddots & \cdots & * \\ m_{i1} & \cdots & \star \\ * & \ddots & \cdots & * \end{pmatrix}$$

Multiplying by $\Sigma_{2i}$ from the left amounts to interchanging row 2 with row $i$. Multiplying by $U^{-1}$ from the left amounts to subtracting row 1 from row 2. Multiplying once more by $\Sigma_{2i}$ from the left amounts to switching back row 2 and row $i$. It follows that $\Sigma_{2i} U^{-1} \Sigma_{2i} H_{1/m_{i1}}^{-1} M$ is of the form:

$$\Sigma_{2i} U^{-1} \Sigma_{2i} H_{1/m_{i1}}^{-1} M = \begin{pmatrix} m_{i1} & * & \cdots & * \\ * & \ddots & \cdots & * \\ 0 & \cdots & \star \\ * & \ddots & \cdots & * \end{pmatrix}$$

Multiplying once more by $H_{m_{i1}}^{-1} = H_{1/m_{i1}}$ from the left amounts to multiplying the first row by $1/m_{i1}$. We conclude that:

$$H_{m_{i1}}^{-1} \Sigma_{2i} U^{-1} \Sigma_{2i} H_{1/m_{i1}}^{-1} M = \begin{pmatrix} 1 & * & \cdots & * \\ * & \ddots & \cdots & * \\ 0 & \cdots & \star \\ * & \ddots & \cdots & * \end{pmatrix}$$
7. If we prove that the matrix:

\[ H_{m1}^{-1} \Sigma_2 U^{-1} \Sigma_2 H_{1/m1}^{-1} M = \begin{pmatrix}
1 & * & \cdots & * \\
* & & & \\
0 & & \leftarrow i & *
\end{pmatrix} \]

is a finite product of elements of \( N_n(K) \), then clearly \( M \) is also a finite product of elements of \( N_n(K) \). Hence in order to show that \( M \in M_n'(K) \), without loss of generality we may assume that \( m_{11} = 0 \). This being true of all \( i \in \{2, \ldots, n\} \), without loss of generality we may assume that \( M \) is of the form:

\[ M = \begin{pmatrix}
1 & * & \cdots & * \\
0 & & & \\
\vdots & & \ddots & * \\
0 & & & 
\end{pmatrix} \]

8. So we now want to prove that \( M \in M_n'(K) \), where:

\[ M = \begin{pmatrix}
1 & * & \cdots & * \\
0 & & & \\
\vdots & & \ddots & * \\
0 & & & 
\end{pmatrix} \]

Let \( j \in \{2, \ldots, n\} \) and suppose that \( m_{1j} \neq 0 \). From 5. of exercise (1), multiplying \( M \) by \( H_{1/m1j}^{-1} = H_{m_{1j}} \) from the right, amounts to multiplying the first column of \( M \) by \( m_{1j} \). So \( MH_{1/m1j}^{-1} \) is of the form:

\[ MH_{1/m1j}^{-1} = \begin{pmatrix}
m_{1j} & * & m_{1j} & * \\
0 & & j \uparrow & \\
\vdots & & & * \\
0 & & & 
\end{pmatrix} \]

Multiplying by \( \Sigma_{2j} \) from the right amounts to interchanging column 2 with column \( j \). From 10. of exercise (1), multiplying by \( U' = \Sigma_{12} U^{-1} \Sigma_{12} \) from the right amounts to subtracting column 1 from column 2. Multiplying by \( \Sigma_{2j} \) once more from the right, amounts to switching back column 2 and column \( j \). It follows that \( MH_{1/m1j}^{-1} \Sigma_{2j} U' \Sigma_{2j} \) is of the form:

\[ MH_{1/m1j}^{-1} \Sigma_{2j} U' \Sigma_{2j} = \begin{pmatrix}
m_{1j} & * & 0 & * \\
0 & & j \uparrow & \\
\vdots & & & * \\
0 & & & 
\end{pmatrix} \]
Multiplying once more by \( H^{-1}_{m_{1j}} = H_{1/m_{1j}} \) from the right:

\[
MH^{-1}_{i j} \Sigma_{2j} U' \Sigma_{2j} H^{-1}_{m_{1j}} = \begin{pmatrix}
1 & * & 0 & * \\
0 & j & \uparrow & \\
\vdots & & * & \\
0 & & & \\
\end{pmatrix}
\]

Since \( U' = \Sigma_{12} U^{-1} \Sigma_{12} \), it is clear that in order to prove that \( M \) is a finite product of elements of \( N_n(K) \), it is sufficient to prove that the above matrix is itself a finite product of elements of \( N_n(K) \). Hence, in order to prove that \( M \in M_n(K) \), without loss of generality we may assume that \( m_{1j} = 0 \). This being true for all \( j \in \{2, \ldots, n\} \), without loss of generality we may assume that \( M \) is of the form:

\[
M = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & & & \\
\vdots & & M' & \\
0 & & & \\
\end{pmatrix}
\]

where \( M' \in M_{n-1}(K) \).

9. So we now assume that \( M \in M_n(K) \) is of the form:

\[
M = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & & & \\
\vdots & & M' & \\
0 & & & \\
\end{pmatrix}
\]

and we shall prove that \( M \in M'_n(K) \), i.e. that \( M \) can be expressed as a finite product of elements of \( N'_n(K) \). Now since \( M' \in M_{n-1}(K) \), and \( M_{n-1}(K) = M'_{n-1}(K) \) being true from our induction hypothesis, \( M' \) can be expressed as a finite product of elements of \( N_{n-1}(K) \). Hence, there exist \( p \geq 1 \) and \( Q'_1, \ldots, Q'_p \) elements of \( N_{n-1}(K) \) such that:

\[
M' = Q'_1 \ldots Q'_p
\]

For all \( k \in N_p \), we define:

\[
Q_k \triangleq \begin{pmatrix}
Q'_k & 0 \\
\vdots & \uparrow & \\
0 & \ldots & 0 & 1 \\
\end{pmatrix}
\]

Following an argument identical to that contained in 2., each \( Q_k \) is an element of \( N_n(K) \). Furthermore, we have:

\[
Q_1 \ldots Q_p = \begin{pmatrix}
Q'_1 \ldots Q'_p & 0 \\
\vdots & \uparrow & \\
0 & \ldots & 0 & 1 \\
\end{pmatrix}
\]
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\[ = \begin{pmatrix} 0 \\ M' \\ \vdots \\ 0 \end{pmatrix} \]

and consequently:

\[ \Sigma_{1n}Q_1 \ldots Q_p \Sigma_{1n} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & 0 \\ \vdots & \ddots & \ddots & M' \\ 0 & \ldots & 0 & 1 \end{pmatrix} = M \]

It follows that \( M \) is indeed a finite product of elements of \( N_n(\mathbb{K}) \), and we have proved that \( M \in \mathcal{M}_n(\mathbb{K}) \). In 11. of exercise (1), we have proved that \( \mathcal{M}_1(\mathbb{K}) = \mathcal{M}_1'(\mathbb{K}) \). Having assumed that \( n \geq 2 \) and \( \mathcal{M}_{n-1}(\mathbb{K}) = \mathcal{M}_n'(\mathbb{K}) \), we have shown that \( O_n \in \mathcal{M}_n'(\mathbb{K}) \), and furthermore that if \( M \neq O_n \), then \( M \) is also an element of \( \mathcal{M}_n'(\mathbb{K}) \). This shows that the equality \( \mathcal{M}_n(\mathbb{K}) = \mathcal{M}_n'(\mathbb{K}) \) holds, and completes our induction argument.

We conclude that \( \mathcal{M}_n(\mathbb{K}) = \mathcal{M}_n'(\mathbb{K}) \) is true for all \( n \geq 1 \). In particular, it is true for all \( n \geq 2 \), which is the statement of theorem (103).

Exercise 2

1. Let \( B \in \mathcal{F}' \) and \( (B_n)_{n \geq 1} \) be a measurable partition of \( B \), i.e from definition (91), a sequence of pairwise disjoint elements of \( \mathcal{F}' \) such that \( B = \cup_{n \geq 1} B_n \). Then, we claim that \( (X^{-1}(B_n))_{n \geq 1} \) is a measurable partition of \( X^{-1}(B) \). Since \( X \) is measurable, \( X^{-1}(B) \) and each \( X^{-1}(B_n) \) is an element of \( \mathcal{F} \). So we only need to prove that:

\[ X^{-1}(B) = \bigcup_{n=1}^{+\infty} X^{-1}(B_n) \]

Since \( B_n \subseteq B \) for all \( n \geq 1 \), it is clear that \( X^{-1}(B_n) \subseteq X^{-1}(B) \), which establishes the inclusion \( \supseteq \). Let \( \omega \in X^{-1}(B) \). Then \( X(\omega) \in B = \cup_{n \geq 1} B_n \). There exists \( n \geq 1 \) such that \( X(\omega) \in B_n \), i.e. \( \omega \in X^{-1}(B_n) \). This proves the inclusion \( \subseteq \). In order to show that the \( X^{-1}(B_n) \)'s are pairwise disjoint, suppose we have \( \omega \in X^{-1}(B_n) \cap X^{-1}(B_m) \). Then \( X(\omega) \in B_n \cap B_m \), and since the \( B_n \)'s are pairwise disjoint, we conclude that \( n = m \).

2. Let \( \mu \) be a measure on \( (\Omega, \mathcal{F}) \). Then \( \mu : \mathcal{F} \to [0, +\infty] \) is a map such that \( \mu(\emptyset) = 0 \), and which is countably additive. Since \( X \) is measurable, for all \( B \in \mathcal{F}' \), \( X^{-1}(B) \) is an element of \( \mathcal{F} \), and:

\[ \mu^X(B) \overset{\triangle}{=} \mu(X^{-1}(B)) \]

is therefore well-defined. So \( \mu^X : \mathcal{F}' \to [0, +\infty] \) is a well-defined map. Since \( X^{-1}(\emptyset) = \emptyset \), it is clear that \( \mu^X(\emptyset) = 0 \). To show that \( \mu^X \) is a
measure on \((\Omega', \mathcal{F}')\), we only need to show that \(\mu^X\) is countably additive. Let \((B_n)_{n \geq 1}\) be a sequence of pairwise disjoint elements of \(\mathcal{F}'\), and \(B = \bigcup_{n \geq 1} B_n\). Then:

\[
X^{-1}(B) = \bigcup_{n=1}^{+\infty} X^{-1}(B_n)
\]
and consequently, \(\mu^X\) being countable additive:

\[
\mu^X(B) = \mu(X^{-1}(B)) = \sum_{n=1}^{+\infty} \mu(X^{-1}(B_n)) = \sum_{n=1}^{+\infty} \mu^X(B_n)
\]

So \(\mu^X\) is countably additive, and we have proved that \(\mu^X\) is indeed a well-defined measure on \((\Omega', \mathcal{F}')\).

3. Suppose that \(\mu\) is a complex measure on \((\Omega, \mathcal{F})\). Then from definition (92), \(\mu : \mathcal{F} \to \mathbb{C}\) is a map such that for any \(B \in \mathcal{F}\) and \((B_n)_{n \geq 1}\) measurable partition of \(B\), the series \(\sum_{n \geq 1} \mu(B_n)\) converges to \(\mu(B)\). Since \(X\) is measurable, for all \(B \in \mathcal{F}'\), \(X^{-1}(B) \in \mathcal{F}\) and consequently:

\[
\mu^X(B) \triangleq \mu(X^{-1}(B))
\]
is well-defined. So \(\mu^X : \mathcal{F}' \to \mathbb{C}\) is a well-defined map. Let \(B \in \mathcal{F}'\) and \((B_n)_{n \geq 1}\) be a measurable partition of \(B\). Then \((X^{-1}(B_n))_{n \geq 1}\) is a measurable partition of \(X^{-1}(B)\), and so:

\[
\mu^X(B) = \mu(X^{-1}(B)) = \lim_{N \to +\infty} \sum_{n=1}^{N} \mu(X^{-1}(B_n)) = \lim_{N \to +\infty} \sum_{n=1}^{N} \mu^X(B_n)
\]

Hence, the series \(\sum_{n \geq 1} \mu^X(B_n)\) converges to \(\mu^X(B)\), and \(\mu^X\) is indeed a well-defined complex measure on \((\Omega', \mathcal{F}')\).

4. Suppose \(\mu\) is a complex measure on \((\Omega, \mathcal{F})\). Let \(B \in \mathcal{F}'\) and \((B_n)_{n \geq 1}\) be a measurable partition of \(B\). Then, \((X^{-1}(B_n))_{n \geq 1}\) is a measurable partition of \(X^{-1}(B)\). From definition (94), since \(|\mu([X^{-1}(B)])|\) is an upper-bound of all sums \(\sum_{n \geq 1} |\mu(E_n)|\), as \((E_n)_{n \geq 1}\) ranges through all measurable partitions of \(X^{-1}(B)\):

\[
\sum_{n=1}^{+\infty} |\mu^X(B_n)| = \sum_{n=1}^{+\infty} |\mu(X^{-1}(B_n))| \leq |\mu([X^{-1}(B)])| = |\mu^X(B)|
\]
So $|\mu^X(B)|$ is an upper-bound of all sums $\sum_{n \geq 1} |\mu^X(B_n)|$, as $(B_n)_{n \geq 1}$ ranges through all measurable partitions of $B$. Since $|\mu^X|(B)$ is the smallest of such upper-bounds, we obtain:

$$|\mu^X|(B) \leq |\mu^X(B)|$$

This being true for all $B \in \mathcal{F}$, we have $|\mu^X| \leq |\mu|^X$.

5. Let $Y : (\Omega', \mathcal{F}') \to (\Omega'', \mathcal{F}'')$ be a measurable map, where $(\Omega'', \mathcal{F}'')$ is another measurable space. Let $\mu$ be a (possibly complex) measure on $(\Omega, \mathcal{F})$. Then $X(\mu)$ is a well-defined (possibly complex) measure on $(\Omega', \mathcal{F}')$. So $Y(X(\mu))$ is a well-defined (possibly complex) measure on $(\Omega'', \mathcal{F}'')$. For all $B \in \mathcal{F}''$:

$$Y(X(\mu))(B) = X(\mu)(Y^{-1}(B))$$

$$= \mu(X^{-1}(Y^{-1}(B)))$$

$$= \mu(Y \circ X)^{-1}(B))$$

$$= (Y \circ X)(\mu)(B)$$

This being true for all $B \in \mathcal{F}'$, $Y(X(\mu)) = (Y \circ X)(\mu)$. From definition (123), we obtain immediately:

$$(\mu^X)^Y = Y(\mu^X) = Y(X(\mu)) = (Y \circ X)(\mu) = \mu^{(Y \circ X)}$$

Exercise 3

1. Let $a \in \mathbb{R}^n$ and $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ be the associated translation mapping. Since $\|\tau_a(x) - \tau_a(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^n$, it is clear that $\tau_a$ is continuous. It is therefore Borel measurable.

2. Let $\mu$ be a (possibly complex) measure on $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$. Since $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ is measurable, $\tau_a(\mu)$ is a well-defined (possibly complex) measure on $\mathbb{R}^n$.

3. Let $a \in \mathbb{R}^n$ and $u, v \in \mathbb{R}^n$ with $u_i \leq v_i$ for all $i \in \mathbb{N}_n$. Then:

$$\tau_a(dx) \left( \prod_{i=1}^{n} [u_i, v_i] \right) = dx \left( \tau_a^{-1} \left( \prod_{i=1}^{n} [u_i, v_i] \right) \right)$$

$$= dx \left( \prod_{i=1}^{n} [u_i - a_i, v_i - a_i] \right)$$

$$= \prod_{i=1}^{n} (v_i - u_i)$$

$$= dx \left( \prod_{i=1}^{n} [u_i, v_i] \right)$$

From the uniqueness property of definition (63), $\tau_a(dx) = dx$. 

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Exercise 4

4. Having proved that $\tau_a(dx) = dx$ for all $a \in \mathbb{R}^n$, we conclude from definition (124) that the Lebesgue measure $dx$ on $\mathbb{R}^n$ is invariant by translation.

Exercise 5

1. Let $\alpha > 0$, and $k_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $k_\alpha(x) = \alpha x$. Since $\|k_\alpha(x) - k_\alpha(y)\| = \alpha \|x - y\|$ for all $x, y \in \mathbb{R}^n$, it is clear that $k_\alpha$ is continuous and consequently Borel measurable.

2. Since $k_\alpha$ is measurable, $k_\alpha(dx)$ is a well-defined measure on $\mathbb{R}^n$, and so is $\alpha^n k_\alpha(dx)$. Let $u, v \in \mathbb{R}^n$ with $u_i \leq v_i$ for all $i \in \mathbb{N}_n$:

$$\begin{align*}
\alpha^n k_\alpha(dx) \left( \prod_{i=1}^{n} [u_i, v_i] \right) & = \alpha^n dx \left( k_\alpha^{-1} \left( \prod_{i=1}^{n} [u_i, v_i] \right) \right) \\
& = \alpha^n dx \left( \prod_{i=1}^{n} \left[ \frac{u_i}{\alpha}, \frac{v_i}{\alpha} \right] \right) \\
& = \alpha^n \prod_{i=1}^{n} \frac{v_i - u_i}{\alpha} \\
& = \prod_{i=1}^{n} (v_i - u_i) \\
& = dx \left( \prod_{i=1}^{n} [u_i, v_i] \right)
\end{align*}$$

From the uniqueness property of definition (63), $\alpha^n k_\alpha(dx) = dx$. It follows that $k_\alpha(dx) = \alpha^{-n} dx$.

Exercise 6

Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are measurable spaces. Let $\mu$ be a measure on $(\Omega, \mathcal{F})$. Let $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ be a non-negative and measurable map. We claim that:

$$\int_{\Omega} f \circ X d\mu = \int_{\Omega'} f dX(\mu) \quad \text{(2)}$$

Note that $X$ being measurable, $X(\mu)$ is a well-defined measure on $(\Omega', \mathcal{F}')$ and consequently the right-hand-side of (2) is perfectly meaningful. Furthermore, $f \circ X$ is a non-negative and measurable map on $(\Omega, \mathcal{F})$ and the left-hand-side of (2) is also perfectly meaningful. In the case when $f = 1_A$ for some $A \in \mathcal{F}'$, equation (2) reduces to:

$$\begin{align*}
\int_{\Omega} f \circ X d\mu & = \int_{\Omega} 1_A \circ X d\mu \\
& = \int_{\Omega} 1_{X^{-1}(A)} d\mu
\end{align*}$$
\[ \int_{\Omega} X \circ d\mu = \int_{\Omega} \left( \sum_{i=1}^{n} \alpha_i 1_{A_i} \right) \circ X \circ d\mu \]
\[ = \int_{\Omega} \left( \sum_{i=1}^{n} \alpha_i 1_{A_i} \circ X \right) d\mu \]
\[ = \sum_{i=1}^{n} \alpha_i \int_{\Omega} 1_{A_i} \circ X d\mu \]
\[ = \sum_{i=1}^{n} \alpha_i \int_{\Omega'} 1_{A_i} dX(\mu) \]
\[ = \int_{\Omega'} \left( \sum_{i=1}^{n} \alpha_i 1_{A_i} \right) dX(\mu) \]
\[ = \int_{\Omega'} f dX(\mu) \]

Hence equation (2) is also true in the case when \( f \) is a simple function on \((\Omega', \mathcal{F}')\). We now assume that \( f \) is an arbitrary non-negative and measurable function on \((\Omega', \mathcal{F}')\). From theorem (18), there exists a sequence \((s_n)_{n \geq 1}\) of simple functions on \((\Omega', \mathcal{F}')\) such that \( s_n \uparrow f \), i.e. \( s_n \leq s_{n+1} \leq f \) for all \( n \geq 1 \) and \( s_n(\omega) \to f(\omega) \) for all \( \omega \in \Omega' \). Then it is clear that \( s_n \circ X \uparrow f \circ X \), and from the monotone convergence theorem (19), we obtain:

\[ \int_{\Omega} f \circ X d\mu = \lim_{n \to +\infty} \int_{\Omega} s_n \circ X d\mu \]
\[ = \lim_{n \to +\infty} \int_{\Omega'} s_n dX(\mu) \]
\[ = \int_{\Omega'} f dX(\mu) \]

This completes the proof of theorem (104).

Exercise 6

Exercise 7. Let \( X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}') \) be a measurable map, where \((\Omega, \mathcal{F})\) and \((\Omega', \mathcal{F}')\) are measurable spaces. Let \( \mu \) be a measure on \((\Omega, \mathcal{F})\). Let \( f : (\Omega', \mathcal{F}') \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) be a measurable map. Then, the map \( f \circ X : (\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) is
also measurable. Applying theorem (104) to the non-negative and measurable map $|f|$, we obtain:

$$
\int_{\Omega} |f \circ X| d\mu = \int_{\Omega} |f| \circ X d\mu = \int_{\Omega'} |f| d\mu(\mu)
$$

It follows that $\int_{\Omega} |f \circ X| d\mu < +\infty \Leftrightarrow \int_{\Omega'} |f| d\mu(\mu) < +\infty$, or equivalently, all maps involved being measurable:

$$
f \circ X \in L^1_C(\Omega, \mathcal{F}, \mu) \Leftrightarrow f \in L^1_C(\Omega', \mathcal{F}', X(\mu))
$$

We now assume that $f \in L^1_C(\Omega', \mathcal{F}', X(\mu))$. Let $u = Re(f)$ and $v = Im(f)$. Then $f = u^+ - u^- + iv^+ - iv^-$, and applying theorem (104) to each non-negative and measurable map $u^+, v^\pm$, we obtain:

$$
\int_{\Omega} f \circ X d\mu = \int_{\Omega} [u^+ - u^- + iv^+ - iv^-] \circ X d\mu = \int_{\Omega} u^+ \circ X d\mu - \int_{\Omega} u^- \circ X d\mu + i \left( \int_{\Omega} v^+ \circ X d\mu - \int_{\Omega} v^- \circ X d\mu \right) = \int_{\Omega'} u^+ d\mu(\mu) - \int_{\Omega'} u^- d\mu(\mu) + i \left( \int_{\Omega'} v^+ d\mu(\mu) - \int_{\Omega'} v^- d\mu(\mu) \right) = \int_{\Omega'} [u^+ - u^- + iv^+ - iv^-] d\mu(\mu) = \int_{\Omega'} f d\mu(\mu)
$$

Note that this derivation is perfectly legitimate, as all the integrals involved are finite. This completes the proof of theorem (105).

Exercise 7

**Exercise 8.** Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are measurable spaces. Let $\mu$ be a complex measure on $(\Omega, \mathcal{F})$. Let $f : (\Omega', \mathcal{F}') \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ be measurable. From 4. of exercise (3), we have $|\mu^X| \leq |\mu|^X$, or equivalently $|X(\mu)| \leq X(|\mu|)$. Using exercise (18) of Tutorial 12 together with theorem (104):

$$
\int_{\Omega'} |f| d\mu(\mu) \leq \int_{\Omega'} |f| d\mu(\mu) = \int_{\Omega} |f \circ X| d\mu = \int_{\Omega} |f \circ X| d\mu
$$
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So \( f \circ X \in L^1_C(\Omega, \mathcal{F}, \mu) \Rightarrow f \in L^1_C(\Omega', \mathcal{F}', X(\mu)) \)

We now assume that \( f \circ X \in L^1_C(\Omega, \mathcal{F}, \mu) \). Let \( \mu_1 = Re(\mu) \) and \( \mu_2 = Im(\mu) \).

Then, we have \( \mu = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-) \), and from exercise (19) of Tutorial 12, \( f \circ X \in L^1_C(\Omega, \mathcal{F}, \mu^\pm_k) \), \( k = 1, 2 \), with:

\[
\int f \circ X d\mu = \int f \circ X d\mu_1^+ - \int f \circ X d\mu_1^- + i \left( \int f \circ X d\mu_2^+ - \int f \circ X d\mu_2^- \right)
\]

(3)

Applying theorem (105) to each measure \( \mu^\pm_k \), we obtain:

\[
\int f \circ X d\mu^\pm_k = \int f dX(\mu^\pm_k), \; k = 1, 2
\]

(4)

Moreover, for all \( B \in \mathcal{F}' \), we have:

\[
X(\mu)(B) = \mu(X^{-1}(B)) = \mu_1^+(X^{-1}(B)) - \mu_1^-(X^{-1}(B)) + i(\mu_2^+(X^{-1}(B)) - \mu_2^-(X^{-1}(B))) = X(\mu_1^+)(B) - X(\mu_1^-)(B) + i(X(\mu_2^+)(B) - X(\mu_2^-)(B))
\]

and consequently:

\[
X(\mu) = X(\mu_1^+) - X(\mu_1^-) + i(X(\mu_2^+) - X(\mu_2^-))
\]

Since \( f \in L^1_C(\Omega', \mathcal{F}', X(\mu^\pm_k)) \), from exercise (17) of Tutorial 12:

\[
\int f dX(\mu) = \int f dX(\mu_1^+) - \int f dX(\mu_1^-) + i \left( \int f dX(\mu_2^+) - \int f dX(\mu_2^-) \right)
\]

(5)

From (3), (4) and (5), we conclude that:

\[
\int f \circ X d\mu = \int f dX(\mu)
\]

which completes the proof of theorem (106).

Exercise 8

Exercise 9.

1. Let \( X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) be a random variable with distribution \( \mu = X(P) \) under \( P \), where \( (\Omega, \mathcal{F}, P) \) is a probability space. Recall that the notions of probability space, random variable and expectation are defined
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in (70), (71) and (72) respectively. Let \( i : \mathbb{R} \to \mathbb{R} \) be the identity mapping. Applying theorem (104), we have:

\[
\int_{\Omega} |X| dP = \int_{\Omega} |i \circ X| dP = \int_{\Omega} |i| \circ X dP = \int_{\mathbb{R}} |i| dX(P) = \int_{-\infty}^{+\infty} |x| d\mu(x)
\]

So \( X \) is integrable, if and only if \( \int_{\mathbb{R}} |x| d\mu(x) < +\infty \).

2. If \( \int |X| dP < +\infty \), applying theorem (105) we obtain:

\[
E[X] = \int_{\Omega} X dP = \int_{\Omega} i \circ X dP = \int_{\mathbb{R}} idX(P) = \int_{-\infty}^{+\infty} x d\mu(x)
\]

3. Let \( f : x \to x^2 \). From theorem (104), we have:

\[
E[X^2] = \int_{\Omega} X^2 dP = \int_{\Omega} f \circ X dP = \int_{\mathbb{R}} f dX(P) = \int_{-\infty}^{+\infty} x^2 d\mu(x)
\]

Exercise 9

Exercise 10.

1. Let \( \mu \) be a locally finite measure on \( \mathbb{R}^n \), which is invariant by translation. Given \( a \in \mathbb{R}^n \), let \( Q_a = [0, a_1] \times \ldots \times [0, a_n] \). Let \( K_a = [0, a_1] \times \ldots \times [0, a_n] \). Then \( K_a \) is a closed subset of \( \mathbb{R}^n \). Indeed, it can be written as \( K_a = \cap_{i=1}^n p_i^{-1}([0, a_i]) \), where \( p_i : \mathbb{R}^n \to \mathbb{R} \) denotes the \( i \)-th canonical projection, which is a continuous map. Since \([0, a_i]\) is closed in \( \mathbb{R} \), each \( p_i^{-1}([0, a_i]) \) is closed in \( \mathbb{R}^n \), and \( K_a \) is closed. Moreover, for all \( x, y \in K_a \):

\[
\|x - y\| \leq \|x\| + \|y\| \leq 2\|a\|
\]

Taking the supremum as \( x, y \in K_a \), we obtain \( \delta(K_a) \leq 2\|a\| \), and in particular \( \delta(K_a) < +\infty \), where \( \delta(K_a) \) is the diameter of \( K_a \), as defined in (68). So \( K_a \) is a closed and bounded subset of \( \mathbb{R}^n \). From theorem (48), \( K_a \) is a compact subset of \( \mathbb{R}^n \). Hence, from exercise (10) of Tutorial 13, since \( \mu \) is locally finite, we have \( \mu(K_a) < +\infty \). We conclude from \( Q_a \subseteq K_a \) that:

\[
\mu(Q_a) \leq \mu(K_a) < +\infty
\]

In particular, if \( Q = Q_{(1, \ldots, 1)} \) then \( \mu(Q) < +\infty \).
2. Let \( p = (p_1, \ldots, p_n) \) where \( p_i \in \mathbb{N}^* \) for all \( i \in \mathbb{N}_n \). We claim:

\[
Q_p = \bigcup_{k \in \mathbb{N}^n} [k_1, k_1 + 1] \times \ldots \times [k_n, k_n + 1]
\]

Let \( k \in \mathbb{N}^n \) with \( 0 \leq k_i < p_i \) for all \( i \in \mathbb{N}_n \). Let \( x \in \mathbb{R}^n \) and suppose that \( k_i \leq x_i < k_i + 1 \) for all \( i \in \mathbb{N}_n \). Then, we have:

\[
0 \leq k_i \leq x_i < k_i + 1 \leq p_i , \forall i \in \mathbb{N}_n
\]

So in particular \( x \in Q_p \). This shows the inclusion \( \supseteq \). To show the reverse inclusion, suppose \( x \in Q_p \). Given \( i \in \mathbb{N}_n \), consider the set \( X_i = \{ k \in \mathbb{N} : 0 \leq x_i < k + 1 \} \). Since \( 0 \leq x_i < p_i \) and \( p_i \geq 1 \), it is clear that \( p_i - 1 \in X_i \). So \( X_i \) is a non-empty subset of \( \mathbb{N} \) which therefore has a smallest element \( k_i \leq p_i - 1 \). Defining \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \), we have \( 0 \leq k_i < p_i \) for all \( i \in \mathbb{N}_n \), and furthermore:

\[
k_i \leq x_i < k_i + 1 , \forall i \in \mathbb{N}_n
\]

This shows the inclusion \( \subseteq \). It remains to show that the above union is indeed a union of pairwise disjoint sets. Let \( k, k' \in \mathbb{N}^n \) and suppose that \( x \in \mathbb{R}^n \) is such that:

\[
x \in \left( \prod_{i=1}^{n} [k_i, k_i + 1] \right) \cap \left( \prod_{i=1}^{n} [k'_i, k'_i + 1] \right)
\]

Then for all \( i \in \mathbb{N}_n \), \( x_i \in [k_i, k_i + 1] \cap [k'_i, k'_i + 1] \) and consequently \( k_i = k'_i \). So \( k = k' \).

3. For all \( k \in \mathbb{N}^n \) with \( 0 \leq k_i < p_i \), define:

\[
A_k = [k_1, k_1 + 1] \times \ldots \times [k_n, k_n + 1]
\]

Let \( \tau_k : \mathbb{R}^n \to \mathbb{R}^n \) be the translation mapping of vector \( k \), defined by \( \tau_k(x) = k + x \) for all \( x \in \mathbb{R}^n \). Since \( \mu \) is invariant by translation, \( \tau_k(\mu) = \mu \) and consequently:

\[
\mu(A_k) = \tau_k(\mu)(A_k) = \mu(\tau_k^{-1}(A_k)) = \mu(\{ \tau_k \in A_k \}) = \mu(\{ x : k_i \leq k_i + x_i < k_i + 1, \forall i \in \mathbb{N}_n \}) = \mu(\{ x : 0 \leq x_i < 1, \forall i \in \mathbb{N}_n \}) = \mu(Q)
\]

Having proved in 2 that \( Q_p = \bigcup_k A_k \), we obtain:

\[
\mu(Q_p) = \sum_k \mu(A_k) = \sum_k \mu(Q) = p_1 \ldots p_n \mu(Q)
\]
4. Let $q_1, \ldots, q_n \geq 1$ be positive integers. We claim that:

$$Q_p = \bigcup_{k \in \mathbb{N}^n} \left[ \frac{k_1 p_1}{q_1}, \frac{(k_1 + 1) p_1}{q_1} \right] \times \cdots \times \left[ \frac{k_n p_n}{q_n}, \frac{(k_n + 1) p_n}{q_n} \right]$$

Let $k \in \mathbb{N}^n$ with $0 \leq k_i < q_i$ for all $i \in \mathbb{N}_n$. Let $x \in \mathbb{R}^n$ with:

$$\frac{k_i p_i}{q_i} \leq x_i < \frac{(k_i + 1) p_i}{q_i}, \quad \forall i \in \mathbb{N}_n$$

Then in particular $0 \leq x_i < p_i$ for all $i$'s and consequently $x \in Q_p$. This shows the inclusion $\subseteq$. To show the reverse inclusion, suppose $x \in Q_p$. Given $i \in \mathbb{N}_n$, consider the set:

$$X_i = \left\{ k \in \mathbb{N} : x_i < \frac{(k + 1) p_i}{q_i} \right\}$$

Since $0 \leq x_i < p_i$ and $q_i \geq 1$, it is clear that $q_i - 1 \in X_i$. So $X_i$ is a non-empty subset of $\mathbb{N}$, which therefore has a smallest element $k_i \leq q_i - 1$. Defining $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, it is clear that $0 \leq k_i < q_i$ for all $i \in \mathbb{N}_n$ and furthermore:

$$\frac{k_i p_i}{q_i} \leq x_i < \frac{(k_i + 1) p_i}{q_i}, \quad \forall i \in \mathbb{N}_n$$

This shows the inclusion $\subseteq$. It remains to show that the above union is indeed a union a pairwise disjoint sets. But if $k, k' \in \mathbb{N}^n$ are such that there exists $x \in \mathbb{R}^n$ with:

$$x_i \in \left[ \frac{k_i p_i}{q_i}, \frac{(k_i + 1) p_i}{q_i} \right] \cap \left[ \frac{k'_i p_i}{q_i}, \frac{(k'_i + 1) p_i}{q_i} \right]$$

for all $i \in \mathbb{N}_n$, then $k_i = k'_i$ for all $i$'s and consequently $k = k'$.

5. Given $i \in \mathbb{N}_n$, define $r_i = p_i/q_i$. Let $r = (r_1, \ldots, r_n)$. Given $k \in \mathbb{N}^n$ with $0 \leq k_i < q_i$ for all $i \in \mathbb{N}_n$, define:

$$A_k = [k_1 r_1, (k_1 + 1) r_1] \times \cdots \times [k_n r_n, (k_n + 1) r_n]$$

Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ be the translation mapping associated with the vector $u = (k_1 r_1, \ldots, k_n r_n)$, and defined by $\tau(x) = u + x$ for all $x \in \mathbb{R}^n$. Since $\mu$ is invariant by translation, we have $\tau(\mu) = \mu$, and consequently:

$$\mu(A_k) = \tau(\mu)(A_k) = \mu(\tau^{-1}(A_k)) = \mu(\{\tau \in A_k\}) = \mu(\{x : k_i r_i \leq k_i r_i + x_i < (k_i + 1) r_i, \forall i \in \mathbb{N}_n\})$$
Having proved in 4. that \( Q_p = \cup_k A_k \), we obtain:

\[
\mu(Q_p) = \sum_k \mu(A_k) = \sum_k \mu(Q_r) = q_1 \ldots q_n \mu(Q_r)
\]

where we have used the fact that:

\[
\text{card}\{k \in \mathbb{N}^n : 0 \leq k_i < q_i, \; \forall i \in \mathbb{N}_n\} = q_1 \ldots q_n
\]

Hence, we have proved that:

\[
\mu(Q_p) = q_1 \ldots q_n \mu(Q_{(p_1/q_1, \ldots, p_n/q_n)})
\]

6. Let \( r \in (\mathbb{Q}^+)^n \). We claim that:

\[
\mu(Q_r) = r_1 \ldots r_n \mu(Q) \tag{6}
\]

If \( r_i = 0 \) for some \( i \in \mathbb{N}_n \), then it is clear that \( Q_r = \emptyset \) and (6) is satisfied. So we assume that \( r_i > 0 \) for all \( i \in \mathbb{N}_n \). There exist integers \( p_1, \ldots, p_n \geq 1 \) and \( q_1, \ldots, q_n \geq 1 \) such that \( r_i = p_i/q_i \) for all \( i \in \mathbb{N}_n \). Using 5. and 3. we obtain:

\[
\mu(Q_r) = \frac{\mu(Q_{a})}{q_1 \ldots q_n} = \frac{p_1 \ldots p_n}{q_1 \ldots q_n} \mu(Q) = r_1 \ldots r_n \mu(Q)
\]

which establishes our claim of equation (6).

7. Let \( a \in (\mathbb{R}^+)^n \). We claim that:

\[
\mu(Q_a) = a_1 \ldots a_n \mu(Q) \tag{7}
\]

If \( a_i = 0 \) for some \( i \in \mathbb{N}_n \), then (7) is obviously true. So we assume that \( a_i > 0 \) for all \( i \in \mathbb{N}_n \). Let \( (r^p)_{p \geq 1} \) be a sequence in \((\mathbb{Q}^+)^n\) such that \( r^p_i \uparrow a_i \) for all \( i \in \mathbb{N}_n \), i.e. \( r^p_i \leq r^{p+1}_i < a_i \) for all \( p \geq 1 \) and \( r^p_i \to a_i \) as \( p \to +\infty \). The map \( \phi : \mathbb{R}^n \to \mathbb{R} \) defined by \( \phi(x) = x_1 \ldots x_n \) can be written as \( \phi = p_1 \ldots p_n \) where \( p_i : \mathbb{R}^n \to \mathbb{R} \) is the \( i \)-th canonical projection. Since each \( p_i \) is continuous, \( \phi \) is itself continuous. Furthermore, since \( r^p_i \to a_i \) for all \( i \in \mathbb{N}_n \), we have \( r^p \to a \) with respect to the product topology of \( \mathbb{R}^n \) (which is also the usual topology of \( \mathbb{R}^n \)). Hence:

\[
\lim_{p \to +\infty} r^p_1 \ldots r^p_n = \lim_{p \to +\infty} \phi(r^p) = \phi(a) = a_1 \ldots a_n \tag{8}
\]

We now claim that \( Q_{r^p} \uparrow Q_a \). Since \( r^p_i \leq r^{p+1}_i \) for all \( i \in \mathbb{N}_n \) and \( p \geq 1 \), it is clear that \( Q_{r^p} \subseteq Q_{r^{p+1}} \) for all \( p \geq 1 \). So we only need to prove that \( Q_a = \cup_{p \geq 1} Q_{r^p} \). From \( r^p_i < a_i \) (and in particular \( r^p_i \leq a_i \)) for all \( i \in \mathbb{N}_n \) and \( p \geq 1 \), we obtain \( Q_{r^p} \subseteq Q_a \) for all \( p \geq 1 \). This shows the inclusion \( \supseteq \). To show the reverse inclusion, let \( x \in Q_a \). Given \( i \in \mathbb{N}_n \), we have \( 0 \leq x_i < a_i \). Since \( r^p_i \to a_i \) as \( p \to +\infty \), there exist \( N_i \geq 1 \) such that:

\[
p \geq N_i \Rightarrow x_i < r^p_i < a_i
\]
Taking \( p = \max(N_1, \ldots, N_n) \) we obtain \( 0 \leq x_i < r^p_i \) for all \( i \in \mathbb{N}_n \), and consequently \( x \in Q_{r^p} \). This shows the inclusion \( \subseteq \). Having proved that \( Q_{r^p} \uparrow Q_a \), from theorem (7) we have:

\[
\lim_{p \to +\infty} \mu(Q_{r^p}) = \mu(Q_a) \tag{9}
\]

Using 6. together with (8) and (9) we obtain:

\[
\mu(Q_a) = \lim_{p \to +\infty} \mu(Q_{r^p}) = \lim_{p \to +\infty} r^p_1 \ldots r^p_n \mu(Q) = a_1 \ldots a_n \mu(Q)
\]

which establishes our claim of equation (7). Note that the third equality is legitimate from \( \mu(Q) < +\infty \) and the continuity of the map \( \psi : \mathbb{R}^+ \to \mathbb{R} \) defined by \( \psi(x) = x\mu(Q) \). If we had \( \mu(Q) = +\infty \), the conclusion would remain valid (the sequence \( r^p_1 \ldots r^p_n \) is non-decreasing), but it would no longer be true that \( \psi \) (with values in \([0, +\infty]\)) is continuous, (recall that \((1/p) \cdot (+\infty)\) does not converge to \(0 \cdot (+\infty)\) as \( p \to +\infty \)).

8. We define the set of subsets of \( \mathbb{R}^n \):

\[
C = \{[a_1, b_1] \times \ldots \times [a_n, b_n] : a_i, b_i \in \mathbb{R}, a_i \leq b_i, \forall i \in \mathbb{N}_n\}
\]

Let \( B = [a_1, b_1] \times \ldots \times [a_n, b_n] \in C \). Let \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) and \( b = (b_1, \ldots, b_n) \in \mathbb{R}^n \). Let \( c = b - a \in (\mathbb{R}^+)^n \). Let \( \tau_a : \mathbb{R}^n \to \mathbb{R}^n \) be the translation mapping of vector \( a \), defined by \( \tau_a(x) = a + x \) for all \( x \in \mathbb{R}^n \). Since \( \mu \) is invariant by translation, we have \( \tau_a(\mu) = \mu \). Using 7. we obtain:

\[
\mu(B) = \tau_a(\mu)(B) = \mu(\tau^{-1}_a(B)) = \mu(\{\tau_a \in B\}) = \mu(\{x : a_i \leq x_i < b_i, \forall i \in \mathbb{N}_n\}) = \mu(\{x : 0 \leq x_i < c_i, \forall i \in \mathbb{N}_n\}) = \mu(Q) = c_1 \ldots c_n \mu(Q) = \mu(Q) \prod_{i=1}^n (b_i - a_i) = \mu(Q) \prod_{i=1}^n dx^i([a_i, b_i]) = \mu(Q) \prod_{i=1}^n dx^i([a_i, b_i]) = \mu(Q) dx^1 \otimes \ldots \otimes dx^n(B) = \mu(Q) dx(B)
\]

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Solutions to Exercises

So we have proved that $\mu(B) = \mu(Q)dx(B)$ for all $B \in \mathcal{C}$. Note that in obtaining this equality, we have refrained from writing directly:

$$\prod_{i=1}^{n}(b_i - a_i) = dx\left(\prod_{i=1}^{n}[a_i, b_i]\right) = dx(B) \quad (10)$$

as this equality has not been proved anywhere in the Tutorials. Indeed, definition (63) of the Lebesgue measure on $\mathbb{R}^n$, defines it as the unique measure with the property (given $a, b \ldots)$:

$$\prod_{i=1}^{n}(b_i - a_i) = dx\left(\prod_{i=1}^{n}[a_i, b_i]\right)$$

which is not quite the same as (10). However, if $dx^i$ denotes the Lebesgue measure on $\mathbb{R}$, then it is clear that:

$$dx^i([a_i, b_i]) = dx^i([a_i, b_i]) = dx^i([a_i, b_i])$$

and furthermore, it is not difficult from the uniqueness property of definition (63) to establish the fact that the Lebesgue measure $dx$ on $\mathbb{R}^n$ is the product measure $dx = dx^1 \otimes \ldots \otimes dx^n$.

9. Let $C_1 = \{[a, b]; a, b \in \mathbb{R}\}$. It is by now a standard exercise to show that $\mathcal{B}(\mathbb{R}) = \sigma(C_1)$. Let $C_1^{ln}$ be the $n$-fold product $C_1 \times \ldots \times C_1$, i.e. the set of rectangles, as per definition (52):

$$C_1^{ln} = \{A_1 \times \ldots \times A_n : A_i \in C_1 \cup \{\mathbb{R}\}, \forall i \in \mathbb{N}_n\}$$

Since $\mathbb{R}$ is separable (has a countable base), from exercise (18) of Tutorial 6, we have $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R})^\otimes n$ and consequently from theorem (26):

$$\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R})^\otimes n = \sigma(C_1)^\otimes n = \sigma(C_1^{ln})$$

Hence, in order to prove that $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{C})$, we only need to show that $\sigma(\mathcal{C}) = \sigma(C_1^{ln})$. It is clear that $\mathcal{C} \subseteq C_1^{ln}$ which establishes the inclusion $\subseteq$. To show the reverse inclusion, it is sufficient to prove that $C_1^{ln} \subseteq \sigma(\mathcal{C})$. Let $B = A_1 \times \ldots \times A_n$ be a rectangle of $C_1^{ln}$. Suppose $A_1 = \mathbb{R}$. Then, we have:

$$B = \bigcup_{p=1}^{+\infty}[-p, p][A_2 \times \ldots \times A_n]$$

and in order to prove that $B \in \sigma(\mathcal{C})$, it is sufficient to prove that each $[-p, p][A_2 \times \ldots \times A_n]$ is an element of $\sigma(\mathcal{C})$. Hence, without loss of generality, we may assume that $A_1 \in C_1$. Likewise, we may assume that $A_2 \in C_1$, and in fact we may assume without loss of generality that $A_i \in C_1$ for all $i \in \mathbb{N}_n$, in which case $B \in \mathcal{C} \subseteq \sigma(\mathcal{C})$. This completes our proof, and $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{C})$.

10. Given $p \geq 1$ we define:

$$D_p = \{B \in \mathcal{B}(\mathbb{R}^n) : \mu(B \cap [-p, p]^n) = \mu(Q)dx(B \cap [-p, p]^n)\}$$

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Having proved in 8. that $\mu(B) = \mu(Q)dx(B)$ for all $B \in \mathcal{C}$, since $\mathcal{C}$ is closed under finite intersection and $[-p, p^n] \in \mathcal{C}$, it is clear that $\mathcal{C} \subseteq \mathcal{D}_p$ and $\mathbb{R}^n \in \mathcal{D}_p$. Furthermore, if $A, B \in \mathcal{D}_p$ are such that $A \subseteq B$, then:

$$
\mu((B \setminus A) \cap [-p, p^n]) = \mu(B \cap [-p, p^n]) - \mu(A \cap [-p, p^n])
$$

$$
= \mu(Q)dx(B \cap [-p, p^n]) - \mu(Q)dx(A \cap [-p, p^n])
$$

$$
= \mu(Q)dx((B \setminus A) \cap [-p, p^n])
$$

So $B \setminus A \in \mathcal{D}_p$. Note that the above derivation is legitimate, as all the quantities involved are finite since $\mu(Q) < +\infty$. This is a very important point, and is in fact the very reason why we have localized the problem on $[-p, p^n]$ by defining $\mathcal{D}_p$, rather than considering directly:

$$
\mathcal{D} = \{ B \in \mathcal{B}(\mathbb{R}^n) : \mu(B) = \mu(Q)dx(B) \}
$$

for which the property $B \setminus A \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$, $A \subseteq B$, may not be easy to establish, if at all true. Let $(B_k)_{k \geq 1}$ be a sequence of elements of $\mathcal{D}_p$ such that $B_k \uparrow B$. From theorem (7):

$$
\mu(B \cap [-p, p^n]) = \lim_{k \to +\infty} \mu(B_k \cap [-p, p^n])
$$

$$
= \lim_{k \to +\infty} \mu(Q)dx(B_k \cap [-p, p^n])
$$

$$
= \mu(Q) \lim_{k \to +\infty} dx(B_k \cap [-p, p^n])
$$

$$
= \mu(Q)dx(B \cap [-p, p^n])
$$

So $B \in \mathcal{D}_p$, and we have proved that $\mathcal{D}_p$ is a Dynkin system on $\mathbb{R}^n$. Since $\mathcal{C} \subseteq \mathcal{D}_p$ and $\mathcal{C}$ is closed under finite intersection, from the Dynkin system theorem (1), we obtain $\sigma(\mathcal{C}) \subseteq \mathcal{D}_p$. Having proved in 9. that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^n)$, it follows that $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{D}_p$ for all $p \geq 1$. Hence, given $B \in \mathcal{B}(\mathbb{R}^n)$, using theorem (7):

$$
\mu(B) = \lim_{p \to +\infty} \mu(B \cap [-p, p^n])
$$

$$
= \lim_{p \to +\infty} \mu(Q)dx(B \cap [-p, p^n])
$$

$$
= \mu(Q) \lim_{p \to +\infty} dx(B \cap [-p, p^n])
$$

$$
= \mu(Q)dx(B)
$$

So $\mu = \mu(Q)dx$. Given a locally finite measure $\mu$ on $\mathbb{R}^n$, which is invariant by translation, we have found $\alpha = \mu(Q) \in \mathbb{R}^+$, such that $\mu = \alpha dx$. This completes the proof of theorem (107).

**Exercise 11.**

1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection. In particular, $T$ is a linear map defined on a finite dimensional normed space. So $T$ is continuous.
Likewise, $T^{-1}$ is a linear map defined on a finite dimensional normed space, so $T^{-1}$ is continuous. The fact that a linear map defined on a finite dimensional normed space is continuous, has not yet been proved in these Tutorials (we have not even defined what a normed space is, see Tutorial 18). For those not familiar with the result, the proof in the case $\mathbb{R}^n$ (together with its usual inner-product) goes as follows: Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$ and $x, y \in \mathbb{R}$. We have:

\[
\|T(x) - T(y)\| = \left\| T\left( \sum_{i=1}^{n} x_i e_i \right) - T\left( \sum_{i=1}^{n} y_i e_i \right) \right\| \\
= \left\| \sum_{i=1}^{n} (x_i - y_i) T(e_i) \right\| \\
\leq \sum_{i=1}^{n} |x_i - y_i| \cdot \|T(e_i)\| \\
\leq \left( \sum_{i=1}^{n} \|T(e_i)\|^2 \right)^{1/2} \left( \sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \\
= M \|x - y\|
\]

where $M = (\sum_{i=1}^{n} \|T(e_i)\|^2)^{1/2}$, and we have used the Cauchy-Schwarz inequality (50). Having proved the existence of $M \in \mathbb{R}^+$ such that $\|T(x) - T(y)\| \leq M \|x - y\|$ for all $x, y \in \mathbb{R}^n$, it is clear that $T$ is continuous. Similarly, there exists $M' \in \mathbb{R}^+$ such that $\|T^{-1}(x) - T^{-1}(y)\| \leq M' \|x - y\|$ for all $x, y \in \mathbb{R}^n$. So $T^{-1}$ is continuous.

2. Let $B \subseteq \mathbb{R}^n$. The notation $T^{-1}(B)$ is potentially ambiguous, as it may refer to the inverse image of $B$ by $T$ as defined in (26), or the direct image of $B$ by $T^{-1}$ as defined in (25). Let $S = T^{-1}$, and let $S(B)$ denote the direct image, whereas $T^{-1}(B)$ denotes the inverse image. We claim that $T^{-1}(B) = S(B)$. Indeed, suppose that $x \in T^{-1}(B)$. Then $T(x) \in B$. Let $y = T(x)$. Then $y \in B$ and $S(y) = T^{-1}(T(x)) = x$. So $x \in S(B)$. This shows that $T^{-1}(B) \subseteq S(B)$. To show the reverse inclusion, suppose $x \in S(B)$. There exists $y \in B$ such that $x = S(y)$. So $T(x) = T(S(y)) = y$. So $T(x) \in B$, and $x \in T^{-1}(B)$. This shows that $S(B) \subseteq T^{-1}(B)$. We have proved that $T^{-1}(B) = S(B)$, and it follows that whether we view $T^{-1}(B)$ as an inverse image (that of $B$ by $T$) or a direct image (that of $B$ by $T^{-1}$) makes no difference, as the two sets are in fact equal. The notation $T^{-1}(B)$ is no longer ambiguous.

3. Let $B \subseteq \mathbb{R}^n$. Since $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear bijection, $T^{-1}$ is also a linear bijection. Applying 2. to $T^{-1}$, it follows that the direct image $T(B)$ of $B$ by $T = (T^{-1})^{-1}$ coincides with the inverse image $(T^{-1})^{-1}(B)$ of $B$ by $T^{-1}$, i.e. $T(B) = (T^{-1})^{-1}(B)$.

4. Let $K \subseteq \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$. $\{T \in K\} = T^{-1}(K)$ denotes the inverse image of $K$ by $T$. However from 2. it can also be viewed as
the direct image of \( K \) by \( T^{-1} \). Having proved that \( T^{-1} : \mathbb{R}^n \to \mathbb{R}^n \) is continuous and \( K \) being compact, it follows from exercise (8) of Tutorial 8 that \( T^{-1}(K) \) is a compact subset of \( \mathbb{R}^n \). We conclude that \( \{ T \in K \} \) is a compact subset of \( \mathbb{R}^n \).

5. The Lebesgue measure \( dx \) on \( \mathbb{R}^n \) is clearly locally finite, as can be seen from definition (102). Indeed, given \( x \in \mathbb{R}^n \), the set \( U = \prod_{i=1}^{n} [x_i - 1, x_i + 1] \) is an open neighborhood of \( x \) with finite Lebesgue measure \( (dx)(U) = 2^n < +\infty \). From exercise (10) of Tutorial 13, if \( K' \) is a compact subset of \( \mathbb{R}^n \), then we have \( dx(K') < +\infty \). Furthermore, \( \mathbb{R}^n \) is locally compact, as can be seen from definition (105). Indeed, given \( x \in \mathbb{R}^n \), \( x \) has an open neighborhood with compact closure: taking \( U \) as above, the closure \( K = \overline{U} \) is closed and bounded, and therefore compact from theorem (48). Having proved in 4. that \( K' = \{ T \in K \} \) is itself compact, it follows that:

\[
T(dx)(U) \leq T(dx)(K) = dx(\{ T \in K \}) = dx(K') < +\infty
\]

Given \( x \in \mathbb{R}^n \), we have shown the existence of \( U \) open, such that \( x \in U \) and \( T(dx)(U) < +\infty \). We conclude from definition (102) that \( T(dx) \) (which is well-defined since \( T \) is continuous, hence Borel measurable) is a locally finite measure on \( \mathbb{R}^n \).

6. Given \( a \in \mathbb{R}^n \), let \( \tau_a : \mathbb{R}^n \to \mathbb{R}^n \) be the translation mapping of vector \( a \), defined by \( \tau_a(x) = a + x \) for all \( x \in \mathbb{R}^n \). We have:

\[
T \circ \tau_{T^{-1}(a)}(x) = T(T^{-1}(a) + x) = T(T^{-1}(a)) + T(x) = a + T(x) = \tau_a(T(x)) = \tau_a \circ T(x)
\]

This being true for all \( x \in \mathbb{R}^n \), \( T \circ \tau_{T^{-1}(a)} = \tau_a \circ T \).

7. Using 6. together with 5. of exercise (3), we have:

\[
\tau_a(T(dx)) = (\tau_a \circ T)(dx) = (T \circ \tau_{T^{-1}(a)})(dx) = T(\tau_{T^{-1}(a)}(dx)) = T(dx)
\]

where the last equality stems from the fact that the Lebesgue measure \( dx \) is invariant by translation. Having proved that \( \tau_a(T(dx)) = T(dx) \) for all \( a \in \mathbb{R}^n \), we conclude that \( T(dx) \) is itself invariant by translation.

8. From 5. \( T(dx) \) is a locally finite measure on \( \mathbb{R}^n \). From 7. it is invariant by translation. It follows from theorem (107) that there exists \( \alpha \in \mathbb{R}^+ \) such that \( T(dx) = \alpha dx \). Suppose \( \beta \) is another element of \( \mathbb{R}^+ \) such that \( T(dx) = \beta dx \). Then:

\[
\alpha = \alpha dx([0,1]^n) = \beta dx([0,1]^n) = \beta
\]

Hence, \( \alpha \) is unique and we denote it \( \Delta(T) \), so that \( \Delta(T) \) is the unique element of \( \mathbb{R}^+ \) such that \( T(dx) = \Delta(T)dx \).
9. Let \( Q = T([0,1]^n) \). Then \( Q \) is the direct image of \([0,1]^n\) by \( T \). However from 3. it can also be viewed as the inverse image \((T^{-1})^{-1}([0,1]^n)\) of \([0,1]^n\) by \( T^{-1} \). Since \( T^{-1} \) is continuous, in particular it is Borel measurable. It follows from \([0,1]^n \in \mathcal{B}(\mathbb{R}^n)\) that \((T^{-1})^{-1}([0,1]^n) \in \mathcal{B}(\mathbb{R}^n)\). So \( Q \in \mathcal{B}(\mathbb{R}^n) \). Furthermore, denoting \( S = T^{-1} \), we have:

\[
\Delta(T)dx(Q) = T(dx)(Q) = dx(T^{-1}(Q)) = dx(T^{-1}(T([0,1]^n))) = dx(S(T([0,1]^n))) = dx((S \circ T)([0,1]^n)) = dx([0,1]^n) = 1
\]

10. Since \( \Delta(T)dx(Q) = 1 \) for some \( Q \in \mathcal{B}(\mathbb{R}^n) \), \( \Delta(T) \neq 0 \).

11. Let \( T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n \) be two linear bijections. If \( B \in \mathcal{B}(\mathbb{R}^n) \):

\[
(T_1 \circ T_2)(dx)(B) = T_1(T_2(dx))(B) = T_1(\Delta(T_2)dx)(B) = (\Delta(T_2)dx)(T_1^{-1}(B)) = \Delta(T_2)dx(T_1^{-1}(B)) = \Delta(T_2)T_1(dx)(B) = \Delta(T_2)(\Delta(T_1)dx(B)) = \Delta(T_1)\Delta(T_2)dx(B)
\]

This being true for all \( B \in \mathcal{B}(\mathbb{R}^n) \), we have:

\[
(T_1 \circ T_2)(dx) = \Delta(T_1)\Delta(T_2)dx
\]

Since \( \Delta(T_1 \circ T_2) \) is the unique element of \( \mathbb{R}^+ \) with the property \((T_1 \circ T_2)(dx) = \Delta(T_1 \circ T_2)dx\), we conclude that:

\[
\Delta(T_1 \circ T_2) = \Delta(T_1)\Delta(T_2)
\]

Exercise 11.

Exercise 12.

1. Let \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( H_\alpha : \mathbb{R}^n \to \mathbb{R}^n \) be the linear bijection defined by \( H_\alpha e_1 = \alpha e_1 \) and \( H_\alpha e_j = e_j \) for \( j \geq 2 \), where \( e_1, \ldots, e_n \) is the canonical basis of \( \mathbb{R}^n \). If \( \alpha > 0 \), we have:

\[
H_\alpha(dx)([0,1]^n) = dx(H_\alpha^{-1}([0,1]^n)) = dx(\{x : H_\alpha x \in [0,1]^n\}) = dx(\left\{x : \sum_{j=1}^n x_j H_\alpha e_j \in [0,1]^n\right\})
\]
Solutions to Exercises

\[ = \int \{ x : (\alpha x_1, x_2, \ldots, x_n) \in [0,1]^n \} \]
\[ = \int (\alpha^{-1}] \times [0,1]^{n-1}) = \alpha^{-1} \]

If \( \alpha < 0 \), we have similarly:
\[ H_\alpha(dx)([0,1]^n) = dx([-1,0] \times [0,1]^{n-1}) = -\alpha^{-1} \]
In any case we obtain \( H_\alpha(dx)([0,1]^n) = |\alpha|^{-1} \).

2. The determinant \( \det H_\alpha \) of \( H_\alpha \) has not been defined in these Tutorials. Until we do so, we will have to accept that:
\[ \det H_\alpha = \det \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} = \alpha \]
This being granted, using 1. we have:
\[ \Delta(H_\alpha) = \Delta(H_\alpha)dx([0,1]^n) \]
\[ = H_\alpha(dx)([0,1]^n) \]
\[ = |\alpha|^{-1} = |\det H_\alpha|^{-1} \]

Exercise 12

Exercise 13.

1. Let \( k, l \in \mathbb{N}_n \) and \( \Sigma : \mathbb{R}^n \to \mathbb{R}^n \) be the linear bijection defined by \( \Sigma e_k = e_l \), \( \Sigma e_j = e_k \) and \( \Sigma e_j = e_j \) for \( j \neq k, l \), where \( e_1, \ldots, e_n \) is the canonical basis of \( \mathbb{R}^n \). Let \( \sigma : \mathbb{N}_n \to \mathbb{N}_n \) be the permutation of \( \mathbb{N}_n \) defined by \( \sigma(k) = l \), \( \sigma(l) = k \) and \( \sigma(j) = j \) for \( j \neq k, l \). Then \( \Sigma e_j = e_{\sigma(j)} \) for all \( j \in \mathbb{N}_n \). We have:
\[ \Sigma(dx)([0,1]^n) = dx(\Sigma^{-1}([0,1]^n)) \]
\[ = dx(\{ x : \Sigma x \in [0,1]^n \}) \]
\[ = dx(\left\{ x : \sum_{j=1}^n x_j \Sigma e_j \in [0,1]^n \right\}) \]
\[ = dx(\left\{ x : \sum_{j=1}^n x_{\sigma^{-1}(j)} e_{\sigma^{-1}(j)} \in [0,1]^n \right\}) \]
\[ = dx(\{ x : (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \in [0,1]^n \}) \]
\[ = dx([0,1]^n) = 1 \]

2. Since \( \Sigma \cdot \Sigma e_j = e_j \) for all \( j \in \mathbb{N}_n \), we have \( \Sigma \cdot \Sigma = I_n \).

3. Until we have a Tutorial on the determinant, we shall have to accept that given \( A, B \in \mathcal{M}_n(\mathbb{K}) \), we have:
\[ \det AB = \det A \det B \]
This being granted, using 2, we obtain:
\[ 1 = \det I_n = \det \Sigma \Sigma = (\det \Sigma)^2 \]
from which we conclude that \(|\det \Sigma| = 1\).

4. Using 1. we have:
\[
\Delta(\Sigma) = \Delta(\Sigma) dx([0, 1]^n)
\]
\[
= \Sigma(dx)([0, 1]^n)
\]
\[= 1 = |\det \Sigma|^{-1} \]

Exercise 14.

1. Let \( n \geq 2 \) and \( U : \mathbb{R}^n \to \mathbb{R}^n \) be the linear bijection defined by \( U e_1 = e_1 + e_2 \) and \( U e_j = e_j \) for \( j \geq 2 \), where \( e_1, \ldots, e_n \) is the canonical basis of \( \mathbb{R}^n \). Let \( Q = [0, 1]^n \). Given \( x \in \mathbb{R}^n \), we have:
\[
U x = U \left( \sum_{j=1}^{n} x_j e_j \right)
\]
\[
= \sum_{j=1}^{n} x_j U e_j
\]
\[
= x_1 (e_1 + e_2) + \sum_{j=2}^{n} x_j e_j
\]
\[
= (x_1, x_1 + x_2, x_3, \ldots, x_n)
\]
Since \( U^{-1}(Q) = \{ x \in \mathbb{R}^n : U x \in [0, 1]^n \} \) we conclude that:
\[
U^{-1}(Q) = \{ x \in \mathbb{R}^n : 0 \leq x_1 + x_2 < 1, 0 \leq x_i < 1, \forall i \neq 2 \}
\]

2. We define:
\[
\Omega_1 \triangleq U^{-1}(Q) \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \}
\]
\[
\Omega_2 \triangleq U^{-1}(Q) \cap \{ x \in \mathbb{R}^n : x_2 < 0 \}
\]
Given \( i \in \mathbb{N}_n \), let \( p_i : \mathbb{R}^n \to \mathbb{R} \) be the \( i \)-th canonical projection. Then each \( p_i \) is continuous and therefore Borel measurable. From 1. we obtain:
\[
U^{-1}(Q) = (p_1 + p_2)^{-1}([0, 1]) \cap \left( \bigcap_{i \neq 2} p_i^{-1}([0, 1]) \right)
\]
So it is clear that \( U^{-1}(Q) \in \mathcal{B}(\mathbb{R}^n) \). From:
\[
\Omega_1 = U^{-1}(Q) \cap p_2^{-1}([0, +\infty[)
\]
\[
\Omega_2 = U^{-1}(Q) \cap p_2^{-1}([-\infty, 0[)
\]
we conclude that \( \Omega_1, \Omega_2 \in \mathcal{B}(\mathbb{R}^n) \).
3. It is impossible for me to draw a picture with Latex. Some people can do it, but I can’t. A picture is not a proof of anything, and is therefore not essential. However, if you have spent the time drawing it, it should be clear to you that \( \{\Omega_1, \tau_{c_2}(\Omega_2)\} \) forms a partition of \( Q \), which we shall prove formally in this exercise.

4. Suppose \( x \in \Omega_1 \). Then \( x_2 \geq 0 \) and furthermore \( x \in U^{-1}(Q) \). So \( 0 \leq x_1 + x_2 < 1 \) while \( 0 \leq x_1 < 1 \). Hence, we have:

\[
0 \leq x_2 \leq x_1 + x_2 < 1
\]

We have proved that \( x \in \Omega_1 \Rightarrow 0 \leq x_2 < 1 \).

5. If \( x \in \Omega_1 \) then in particular \( x \in U^{-1}(Q) \). So \( 0 \leq x_i < 1 \) for all \( i \in \mathbb{N}_n \), \( i \neq 2 \). However from 4. we have \( 0 \leq x_2 < 1 \). It follows that \( 0 \leq x_i < 1 \) for all \( i \in \mathbb{N}_n \). So \( x \in Q = [0,1]^n \). We have proved that \( \Omega_1 \subseteq Q \).

6. Suppose \( x \in \tau_{c_2}(\Omega_2) \). There exists \( y \in \Omega_2 \) such that \( x = \tau_{c_2}(y) = e_2 + y \).
   In particular, \( x_1 = y_1 \) and \( x_2 = 1 + y_2 \) for some \( y \in \Omega_2 \). The fact that \( y \in \Omega_2 \) implies in particular that \( y_2 < 0 \) and \( y \in U^{-1}(Q) \). So \( 0 \leq y_1 < 1 \) and \( 0 \leq y_1 + y_2 < 2 \). Hence:

\[
0 \leq y_1 + y_2 < 1 + y_2 = x_2 < 1 + 0 = 1
\]

We have proved that \( x \in \tau_{c_2}(\Omega_2) \Rightarrow 0 \leq x_2 < 1 \). In fact, we have proved the stronger inequality \( 0 < x_2 < 1 \), but we shall not need it.

7. Suppose \( x \in \tau_{c_2}(\Omega_2) \). There exists \( y \in \Omega_2 \) such that \( x = \tau_{c_2}(y) = e_2 + y \).
   So \( x_2 = 1 + y_2 \) and \( x_i = y_i \) for all \( i \neq 2 \). The fact that \( y \in \Omega_2 \) implies in particular that \( y \in U^{-1}(Q) \). So \( 0 \leq y_i < 1 \) for all \( i \neq 2 \) and consequently \( 0 \leq x_i < 1 \) for all \( i \neq 2 \). However, we have proved in 6. that \( 0 \leq x_2 < 1 \). So \( 0 \leq x_i < 1 \) for all \( i \in \mathbb{N}_n \), i.e. \( x \in Q = [0,1]^n \). We have proved that \( \tau_{c_2}(\Omega_2) \subseteq Q \).

8. Suppose \( x \in Q \) and \( x_1 + x_2 < 1 \). Then for all \( i \in \mathbb{N}_n \), we have \( 0 \leq x_i < 1 \) and furthermore \( x_1 + x_2 < 1 \). In particular, we have \( x_2 \geq 0 \) and \( 0 \leq x_1 + x_2 < 1 \), while \( 0 \leq x_i < 1 \) for all \( i \neq 2 \). So \( x \in U^{-1}(Q) \) while \( x_2 \geq 0 \), i.e. \( x \in \Omega_1 \). We have proved that \( x \in Q \) and \( x_1 + x_2 < 1 \) implies that \( x \in \Omega_1 \).

9. Suppose \( x \in Q \) and \( x_1 + x_2 \geq 1 \). Then for all \( i \in \mathbb{N}_n \) we have \( 0 \leq x_i < 1 \) and furthermore \( x_1 + x_2 \geq 1 \). Define \( y = (x_1, -1 + x_2, x_3, \ldots, x_n) \). Then it is clear that \( e_2 + y = x \). So \( x = \tau_{c_2}(y) \). We claim that \( y \in \Omega_2 \). From \( x_2 < 1 \) we obtain \( y_2 = -1 + x_2 < 0 \). Furthermore, for all \( i \neq 2 \) we have \( x_i = y_i \) and consequently \( 0 \leq y_i < 1 \). Finally, from \( x_1 + x_2 \geq 1 \), we obtain:

\[
0 \leq x_1 + x_2 - 1 = y_1 + y_2 < 1 + 0 = 1
\]

Hence, we see that \( y \in U^{-1}(Q) \) while \( y_2 < 0 \). So \( y \in \Omega_2 \) and since \( x = \tau_{c_2}(y) \), we have \( x \in \tau_{c_2}(\Omega_2) \). We have proved that \( x \in Q \) and \( x_1 + x_2 \geq 1 \) implies that \( x \in \tau_{c_2}(\Omega_2) \).
10. Suppose \( x \in \tau_{e_{2}}(\Omega_{2}) \). There exists \( y \in \Omega_{2} \) such that \( x = \tau_{e_{2}}(y) = e_{2} + y \).
   In particular, \( x_{1} = y_{1} \) and \( x_{2} = 1 + y_{2} \) for some \( y \in \Omega_{2} \). The fact that \( y \in \Omega_{2} \) implies that \( y \in U^{-1}(Q) \) and \( 0 \leq y_{1} + y_{2} < 1 \). Hence, we have:
   \[
   1 \leq 1 + y_{1} + y_{2} = x_{1} + x_{2}
   \]
   We have proved that \( x \in \tau_{e_{2}}(\Omega_{2}) \Rightarrow x_{1} + x_{2} \geq 1 \).

11. Suppose \( x \in \tau_{e_{2}}(\Omega_{2}) \cap \Omega_{1} \). From \( x \in \Omega_{1} \) we have in particular \( x \in U^{-1}(Q) \) and consequently \( x_{1} + x_{2} < 1 \). From \( x \in \tau_{e_{2}}(\Omega_{2}) \) using 10. we have \( x_{1} + x_{2} \geq 1 \). This is a contradiction. We have proved that \( \tau_{e_{2}}(\Omega_{2}) \cap \Omega_{1} = \emptyset \).

12. From 5. we have \( \Omega_{1} \subseteq Q \) while from 7. we have \( \tau_{e_{2}}(\Omega_{2}) \subseteq Q \). This shows that \( \Omega_{1} \cup \tau_{e_{2}}(\Omega_{2}) \subseteq Q \). To show the reverse inclusion, suppose \( x \in Q \). If \( x_{1} + x_{2} < 1 \) from 8. we have \( x \in \Omega_{1} \). If \( x_{1} + x_{2} \geq 1 \) from 9. we have \( x \in \tau_{e_{2}}(\Omega_{2}) \). In any case, we have \( x \in \Omega_{1} \cup \tau_{e_{2}}(\Omega_{2}) \). This shows that \( Q \subseteq \Omega_{1} \cup \tau_{e_{2}}(\Omega_{2}) \), and we have proved that \( Q = \Omega_{1} \cup \tau_{e_{2}}(\Omega_{2}) \). Having proved that \( \Omega_{1} \) and \( \tau_{e_{2}}(\Omega_{2}) \) are disjoint, we conclude that \( Q = \Omega_{1} \cup \tau_{e_{2}}(\Omega_{2}) \).

13. Noting that \( \tau_{e_{2}}(\Omega_{2}) = \tau_{-e_{2}}^{-1}(\Omega_{2}) \in B(\mathbb{R}^{n}) \), we have:
   \[
   dx(Q) = dx(\Omega_{1} \cup \tau_{e_{2}}(\Omega_{2})) = dx(\Omega_{1}) + dx(\tau_{e_{2}}(\Omega_{2})) = dx(\Omega_{1}) + dx(\Omega_{2}) = dx(U^{-1}(Q) \cap \{x_{2} \geq 0\}) + dx(U^{-1}(Q) \cap \{x_{2} < 0\}) = dx(U^{-1}(Q))
   \]
   where the third equality stems from the fact that the Lebesgue measure \( dx \) is invariant by translation.

14. It follows from 13. that:
   \[
   \Delta(U) = \Delta(U)dx(Q) = U(dx)(Q) = dx(U^{-1}(Q)) = dx(Q) = 1
   \]

15. Until we have a Tutorial on determinants, we shall accept:
   \[
   \det U = \det \begin{pmatrix}
   1 & 0 & \\
   0 & 1 & 0 \\
   \vdots & & 1
   \end{pmatrix} = 1
   \]
   This being granted, we conclude from 14. that:
   \[
   \Delta(U) = 1 = |\det U|^{-1}
   \]

Exercise 14

Exercise 15.
1. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear bijection where \( n \geq 1 \). If \( n = 1 \) then \( T \) is of the form \( T = H_\alpha \) as defined in exercise (12), where \( \alpha \neq 0 \). In particular, we have \( \Delta(T) = |\det T|^{-1} \). We now assume that \( n \geq 2 \). From theorem (103), there exist \( p \geq 1 \) and \( Q_1, \ldots, Q_p \in \mathcal{M}_n(\mathbb{R}) \) such that:

\[
T = Q_1 \circ \ldots \circ Q_p
\]

and each \( Q_i \) is of the form \( H_\alpha \) of exercise (12), or of the form of exercise (13), or is equal to \( U \) as defined in exercise (14). From (11) we obtain

\[
\det T = \det Q_1 \cdots \det Q_p
\]

and since \( T \) is a bijection, \( \det T \neq 0 \). It follows that \( \det Q_i \neq 0 \) for all \( i \in \mathbb{N}_p \), and in particular that \( \alpha \neq 0 \) whenever \( Q_i \) is of the form \( Q_i = H_\alpha \) of exercise (12). This shows that exercise (12) can be applied as much as exercise (13) and exercise (14), from which we see that \( \Delta(Q_i) = |\det Q_i|^{-1} \) for all \( i \in \mathbb{N}_p \).

2. Using (11), we obtain:

\[
\Delta(T) = \Delta(Q_1 \circ \ldots \circ Q_p) = \Delta(Q_1) \cdots \Delta(Q_p) = |\det Q_1|^{-1} \cdots |\det Q_p|^{-1} = |\det Q_1 \cdots Q_p|^{-1} = |\det T|^{-1}
\]

3. Given \( n \geq 1 \) and a linear bijection \( T : \mathbb{R}^n \to \mathbb{R}^n \), we have proved in exercise (11) that \( T(dx) = \Delta(T)dx \) for a unique constant \( \Delta(T) \in \mathbb{R}^+ \). However, it follows from 2. that \( \Delta(T) = |\det T|^{-1} \). So \( T(dx) = |\det T|^{-1}dx \), which completes the proof of theorem (108).

Exercise 15

Exercise 16. Let \( f : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to [0, +\infty] \) be a non-negative and measurable map. Let \( a, b, c, d \in \mathbb{R} \) be such that \( ad - bc \neq 0 \). Let \( T \in \mathcal{M}_2(\mathbb{R}) \) be defined by:

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

Then \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear map, and \( \det T = ad - bc \neq 0 \). So \( T \) is a linear bijection. Using theorem (104) with theorem (108):

\[
\int_{\mathbb{R}^2} f(ax + by, cx + dy)dxdy = \int_{\mathbb{R}^2} f \circ T(x, y)dxdy = \int_{\mathbb{R}^2} f \circ Tdx = \int_{\mathbb{R}^2} fT(dx)
\]
\[
\begin{align*}
\int_{\mathbb{R}^2} f(|\det T|^{-1}dx) &= |\det T|^{-1} \int_{\mathbb{R}^2} f dx \\
&= |ad - bc|^{-1} \int_{\mathbb{R}^2} f(x,y) dxdy
\end{align*}
\]

where the fifth equality stems from exercise (18) of Tutorial 12.

Exercise 16

Exercise 17. Let \( B \in \mathcal{B}(\mathbb{R}^n) \) and \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear bijection. From 3. of exercise (11), the direct image \( T(B) \) is also the inverse image \((T^{-1})^{-1}(B)\) of \( B \) by \( T^{-1} \). Since \( T^{-1} \) is continuous, in particular it is Borel measurable, and consequently \( T(B) \in \mathcal{B}(\mathbb{R}^n) \). From \( TT^{-1} = I_n \), we obtain \( \det T \det T^{-1} = 1 \), and it follows that \( \det T^{-1} = (\det T)^{-1} \). Applying theorem (108) to \( T^{-1} \), we obtain:

\[
dx(T(B)) = dx((T^{-1})^{-1}(B)) = T^{-1}(dx)(B) = |\det T^{-1}|^{-1} dx(B) = |(\det T)^{-1}|^{-1} dx(B) = |\det T| dx(B)
\]

Exercise 18.

1. Let \( V \) be a linear subspace of \( \mathbb{R}^n \), and \( p = \dim V \). We assume that \( 1 \leq p \leq n - 1 \). Let \( u_1, \ldots, u_p \) be an orthonormal basis of \( V \), and \( u_{p+1}, \ldots, u_n \) be such that \( u_1, \ldots, u_n \) is an orthonormal basis of \( \mathbb{R}^n \). Note that the existence of an orthonormal basis of \( V \), and the fact that such basis can be extended to an orthonormal basis of \( \mathbb{R}^n \), has not been proved in these Tutorials. So we shall have to accept it for the time being. Given \( i \in \mathbf{N}_n \), we define \( \phi_i : \mathbb{R}^n \to \mathbb{R} \) by \( \phi_i(x) = \langle u_i, x \rangle \) for all \( x \in \mathbb{R}^n \), where \( \langle \cdot, \cdot \rangle \) denotes the usual inner-product of \( \mathbb{R}^n \). From the Cauchy-Schwarz inequality (50), for all \( x, y \in \mathbb{R}^n \), we have:

\[
|\phi_i(x) - \phi_i(y)| = |\langle u_i, x \rangle - \langle u_i, y \rangle| = |\langle u_i, x - y \rangle| \leq ||u_i|| \cdot ||x - y||
\]

So it is clear that \( \phi_i : \mathbb{R}^n \to \mathbb{R} \) is continuous.

2. Let \( x \in \mathbb{R}^n \). Since \( u_1, \ldots, u_n \) is a basis of \( \mathbb{R}^n \), there exists a unique \((\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \) such that:

\[
x = \alpha_1 u_1 + \ldots + \alpha_n u_n
\]
Now suppose that \( x \in \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) \). Then for all \( j \geq p + 1 \) we have \( \phi_j(x) = 0 \), i.e.:

\[
0 = \phi_j(x) = \langle u_j, x \rangle = \langle u_j, \alpha_1 u_1 + \ldots + \alpha_n u_n \rangle = \sum_{i=1}^{n} \alpha_i \langle u_j, u_i \rangle = \alpha_j \langle u_j, u_j \rangle = \alpha_j
\]

where we have used the fact that \( u_1, \ldots, u_n \) is an orthonormal basis of \( \mathbb{R}^n \).

Since \( \alpha_j = 0 \) for all \( j \geq p + 1 \), we obtain \( x = \alpha_1 u_1 + \ldots + \alpha_p u_p \in V \). This shows that \( \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) \subseteq V \). To show the reverse inclusion, suppose \( x \in V \). Since \( u_1, \ldots, u_p \) is a basis of \( V \), there exists \( \alpha_1, \ldots, \alpha_p \in \mathbb{R} \) such that \( x = \alpha_1 u_1 + \ldots + \alpha_p u_p \), and since \( u_1, \ldots, u_n \) is orthogonal, it is clear that \( \langle u_j, x \rangle = 0 \) for all \( j \geq p + 1 \). Hence, we have \( x \in \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) \) and we have proved that \( V \subseteq \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) \). We conclude that \( V = \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) \).

3. Since \( \phi_j \) is continuous for all \( j \in \mathbb{N}_n \), in particular \( \phi_j^{-1}(\{0\}) \) is a closed subset of \( \mathbb{R}^n \) for all \( j \in \mathbb{N}_n \). It follows from 2. that \( V = \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) \) is a closed subset of \( \mathbb{R}^n \).

4. Let \( Q = (q_{ij}) \in M_n(\mathbb{R}) \) be the matrix defined by \( Q e_j = u_j \) for all \( j \in \mathbb{N}_n \), where \( e_1, \ldots, e_n \) is the canonical basis of \( \mathbb{R}^n \). For all \( i, j \in \mathbb{N}_n \), we have:

\[
\langle u_i, u_j \rangle = \langle Q e_i, Q e_j \rangle = \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ki} q_{lj}<e_k, e_l> = \sum_{k=1}^{n} q_{ki} q_{kj} = \sum_{k=1}^{n} q_{ki} q_{kj}
\]

5. Using 4. for all \( i, j \in \mathbb{N}_n \), we obtain:

\[
(Q^t Q)_{ij} = \sum_{k=1}^{n} (Q^t)_{ik} (Q)_{kj} = \sum_{k=1}^{n} q_{ki} q_{kj}
\]
\[ \langle u_i, u_j \rangle = (I_n)_{ij} \]

This being true for all \( i, j \in \mathbb{N}_n \), \( Q^t \cdot Q = I_n \). Accepting the fact that \( \det Q^t = \det Q \), we obtain:

\[ 1 = \det I_n = \det Q^t \cdot Q = \det Q^t \det Q = (\det Q)^2 \]

We conclude that \( |\det Q| = 1 \).

6. Applying theorem (108) to \( Q \), we obtain:

\[
\begin{align*}
\text{dx}( \{Q \in V\} ) &= Q(\text{dx}(V)) \\
&= |\det Q|^{-1} \text{dx}(V) = \text{dx}(V)
\end{align*}
\]

7. Let \( \text{span}(e_1, \ldots, e_p) \) denote the linear subspace of \( \mathbb{R}^n \) generated by \( e_1, \ldots, e_p \), i.e. the set:

\[ \text{span}(e_1, \ldots, e_p) = \{ \alpha_1 e_1 + \ldots + \alpha_p e_p : \alpha_i \in \mathbb{R}, \forall i \in \mathbb{N}_p \} \]

We claim that \( \{Q \in V\} = \text{span}(e_1, \ldots, e_p) \). Let \( x \in \{Q \in V\} \). Then \( Q(x) \in V \). Given \( j \in \{p+1, \ldots, n\} \), it follows from 2. that \( \phi_j(Q(x)) = 0 \), i.e.:

\[
\begin{align*}
0 &= \phi_j(Q(x)) \\
&= \langle u_j, x_1 e_1 + \ldots + x_n e_n \rangle \\
&= \langle u_j, x_1 u_1 + \ldots + x_n u_n \rangle \\
&= x_j \langle u_j, u_j \rangle = x_j
\end{align*}
\]

So \( x_j = 0 \) for all \( j \geq p + 1 \) and consequently:

\[ x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{p} x_i e_i \in \text{span}(e_1, \ldots, e_p) \]

This shows the inclusion \( \subseteq \). To show the reverse inclusion, suppose \( x \in \text{span}(e_1, \ldots, e_p) \). Then \( x_j = 0 \) for all \( j \geq p + 1 \), and going back through the preceding calculation, it is clear that \( \phi_j(Q(x)) = 0 \) for all \( j \geq p + 1 \). So \( Q(x) \in \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) = V \), i.e. \( x \in \{Q \in V\} \). This shows the inclusion \( \supseteq \), and we have proved that \( \{Q \in V\} = \text{span}(e_1, \ldots, e_p) \).

8. Let \( m \geq 1 \) be an integer. We define:

\[ E_m \triangleq \left[ -m, m \right] \times \ldots \times \left[ -m, m \right] \times \{0\} \]

It is clear from definition (63) that \( \text{dx}(E_m) = 0 \) for all \( m \geq 1 \).

9. Since \( E_m \uparrow \text{span}(e_1, \ldots, e_{n-1}) \), i.e. \( E_m \subseteq E_{m+1} \) for all \( m \geq 1 \) and \( \cup_{m \geq 1} E_m = \text{span}(e_1, \ldots, e_{n-1}) \), from theorem (7) we obtain:

\[ \text{dx}(\text{span}(e_1, \ldots, e_{n-1})) = \lim_{m \to +\infty} \text{dx}(E_m) = 0 \]
10. Using 6. and 7. together with 9. we have:

\[
\begin{align*}
dx(V) &= dx(\{Q \in V\}) = dx(\text{span}(e_1, \ldots, e_p)) \\
&\leq dx(\text{span}(e_1, \ldots, e_{n-1})) = 0
\end{align*}
\]

This completes the proof of theorem (109) in the case when \(1 \leq \dim V \leq n - 1\). The case \(\dim V = 0\), i.e. \(V = \{0\}\) is clear.